We reveal new stripe states in deformed hexagonal array of photonic wave guides when the array is terminated to have a ribbon-shaped geometry. Unlike the well-known zero energy edge modes of honeycomb ribbon, the new one-dimensional states are shown to originate from high-energy saddle-shaped photonic bands of the ribbon’s two-dimensional counterpart. We find that the strain field deforming the ribbon generates pseudo-electric fields in contrast to pseudo-magnetic fields in other hexagonal crystals. Thus, the stripe states experience Bloch oscillation without any actual electric field so that the spatial distributions of stripes have a singular dependence on the strength of the field. The resulting stripe states are located inside the bulk and their positions depend on their energies.

1. Introduction

Energy bands for electrons on a two-dimensional honeycomb lattice have two notable aspects in the energy-momentum space. One is a conical band of low-energy electrons at the corners of Brillouin zone (BZ) [1–7] and the other is a saddle-shaped higher energy band at the middle point between the two corners [7–11]. The former shows the vanishing density of states (DOS) while the latter has a logarithmic divergent van Hove singularity in the DOS [7]. This divergency occurs at much higher energy of about ±2 eV away from a charge neutral energy of the system [4,7].

Various remarkable phenomena mainly associated with the low-energy conical bands have been realized in graphene, a prototypical honeycomb lattice material of carbon atoms, since its discovery [5–7].

Recently, several theoretical [9–14] and experimental efforts [8,15] have also been put forward to understand
possible collective behaviours of electrons originated from the saddle-shaped higher energy bands and their peculiar constant energy surfaces.

Alongside many works regarding the two-dimensional properties of graphene, there have been extensive studies to understand finite size effects on its band structures [4,7]. Specifically, if graphene is cut along a zigzag-shaped crystallographic direction, the lowest energy states become the edge-localized states [16–18]. Similarly, one would expect equally interesting states from the saddle-shaped energy bands in honeycomb lattice nanostructures. However, the consequences of the termination of honeycomb lattices on the higher energy bands have not been studied. Experimentally, to probe such states, precise control of edge atomic structures [19] and extremely high electron or hole dopings [8,15] are required simultaneously although each of them would be very challenging, if not impossible.

Recent rapid progress in developing photonic analogues of graphene can provide alternative platforms to realize the special energy bands of honeycomb lattice and to overcome the aforementioned difficulties [20–25]. Photonic graphene can show various unique phenomena of light related to conical dispersions [20] such as conical diffraction [20], Klein tunnelling [26] and topologically protected edge modes [21,22,27]. Moreover, the analogues of low-energy edge states in graphene nanoribbons (GNRs) and pseudo-magnetic field effects in the strained graphene are also demonstrated in their photonic counterparts [23–25]. These systems have merits in realizing single-particle bands in the whole phase spaces not being obscured by the Coulomb interaction of electronic systems. Motivated by recent progress in theoretical [9–14] and experimental [20–25] studies, we have carried out analytic and numerical study to explore finite size and strain effects near the saddle point in the energy bands of the honeycomb lattice.

In this work, we reveal a bulk-regime stripe state which is a manifestation of the saddle-shaped bands of honeycomb lattices when the lattice is terminated to have a ribbon shape. To illuminate its signatures, we investigate an optical wave-guide array on a honeycomb lattice ribbon. The stripe states are massively degenerate and are easily split sensitively by applying weak strain. The effective Hamiltonian for the stripe states under strain is shown to correspond to that of one-dimensional atomic chains [28,29] in the strong electric field. Unlike the pseudo-magnetic field effects for the conical band states in strained graphene [30–32], the strain field here plays the role of the pseudo-electric field which is strong enough to induce Bloch oscillation [33]. As such, the widths of the stripes have a singular dependence on the field. While the conventional edge states exist only at the edge and zero energy in GNRs [16–18], our stripe states, depending on their energy, can be located inside the ribbon.

2. Hexagonal photonic ribbon

We first consider a HPR where light propagates along the z-axis (figure 1a). The width of the HPR follows the conventional notation for the GNR [16–18] such that the HPR having N zigzag chains is denoted by N-HPR. When the contrast of the dielectric constant is not significant, the electric field can be expressed as \( E = \Psi(x,y,z) e^{i k_z (p_x x + p_y y + p_z z)} \) using a scalar function \( \Psi(x,y,z) \), where \( p_x, p_y \) and \( p_z \) are constants. The Maxwell equation is written as a wave equation, \( \nabla^2 + \left( \omega/c \right)^2 \epsilon(r) E = 0 \), where \( \epsilon(r) \) is a position-dependent dielectric constant. In the case where \( |\nabla^2 \epsilon| \ll 2 k_z \partial_z \Psi \), the wave equation can be written in the same form as the (2 + 1)D Schrödinger equation, \( i \partial_t \Psi \left( r, z \right) = \mathcal{H}_{PC} \Psi \left( r, z \right) \). Here \( \mathcal{H}_{PC} \equiv -\left(1/2k_z\right) \left( \nabla^2 + \left( \omega/c \right)^2 \epsilon(r) - k_z^2 \right) \). \( r = (x,y) \) and \( z \) plays the role of time. With \( \Psi = \psi(r) e^{i \beta z} \), the eigenvalue \( \beta \) and the eigenfunction \( \psi(r) \) can be obtained from an eigenvalue equation, \( \mathcal{H}_{PC} \psi(r) = -\beta \psi(r) \) (now we drop the subscript \( t \)). Then, the positions of the waveguides on two sublattices A and B of HPR are written as \( R^A_{mn} = ma_1 + na_2 \) and \( R^B_{mn} = R^A_{mn} + \delta_2 \), respectively, where \( m(n) \) is integer and \( a_1(2) \) and \( \delta_2 \) are defined in figure 1b. The \( \epsilon(r) \) for the waveguide array can be written as \( \epsilon(r) = \epsilon_{cl} + \sum_{mn} \left[ \delta \epsilon(r - R^A_{mn}) + \delta \epsilon(r - R^B_{mn}) \right] \), where \( \delta \epsilon(r < \eta) = \epsilon_{co} - \epsilon_{cl}, \delta \epsilon(r > \eta) = 0 \) and \( \epsilon_{cl} (\epsilon_{co}) \) is the dielectric constant for the cladding (host) material and \( \eta \) the radius of the waveguide (we set \( \eta = 10 \mu m \) in our work). Given these conditions, the eigenvalue equation can be solved by the boundary matching conditions. The core
and the cladding regions have the solution $\psi(r) \propto J_l(u r/\eta) e^{i l \phi}$ for $r < \eta$ and $\psi(r) \propto K_l(v r/\eta) e^{i l \phi}$ for $r > \eta$, where $J_l$ ($K_l$) is the (modified) Bessel function of the first (second) kind, $l$ is integer and $\phi$ is the azimuthal angle for $r$. Here $u^2 = [(\omega/c)^2 - \epsilon_{co} - k^2_z + 2k_z \beta] \eta^2$ and $v^2 = [(\omega/c)^2 - \epsilon_{cl} - k^2_z + 2k_z \beta] \eta^2$. The continuity of $\psi(r)$ at the boundary $r = \eta$ gives $u J_l'(u)/J_l(u) = v K_l'(v)/K_l(v)$, the eigenvalue equation for $\beta$. In figure 2a, we show the eigenvalue $\beta$ as a function of $k_z$. The propagation constant $\beta$ obtained here plays a role of the eigenenergy of atoms in solids.

Based on the wave functions with eigenenergy $\beta$ shown in figure 2a, we construct a tight-binding (TB) model for the hexagonal arrays. Among the energy bands, we consider a band which has a rotationally symmetric ($l = 0$) orbital. The creation operators for the orbital at $(m, n)$ site on the A(B) lattice of HPR are written as $a_{m,n}^\dagger (b_{m,n}^\dagger)$ (figure 1b). A generic TB Hamiltonian for the system can be written as $H = -\sum_{m',n'}\sum_{m,n} t_{m'n'}^{m'n} (a_{m',n'}^\dagger b_{m,n} + a_{m',n'}^\dagger a_{m,n}) + (a \leftrightarrow b)$. Here $t_{m'n'}^{m'n}$

Figure 1. (a) Schematic figure for the waveguide array in a hexagonal photonic ribbon (HPR) structure. The z-axis plays the role of time. (b) The honeycomb structure showing the indices of the sites. The two unit vectors are $\mathbf{a}_1 = a_0 (1, 0)$ and $\mathbf{a}_2 = (a_0/2)(1, -\sqrt{3})$, where $a_0 = \sqrt{3} a_w$ and $a_w$ is the centre-to-centre distance between two adjacent waveguides in the array. $\delta_1 = (a_w/2)(0, -2)$, $\delta_2 = (a_w/2)(\sqrt{3}, 1)$ and $\delta_3 = (a_w/2)(-\sqrt{3}, 1)$. (a) and (b) are for 4-HPR and 5-HPR, respectively. (Online version in colour.)
Figure 2. (a) Eigenenergies $\beta$ of one waveguide as a function of $k_z$. We use $\sqrt{\epsilon_{co}} = 1.454$, $\sqrt{\epsilon_{cl}} = 1.45$ and $\omega/c = 9.93$ μm$^{-1}$ that correspond to dielectric constants of fused silica [22–25], and to the frequency of a He/Ne laser [25], respectively. We plot only $\beta < 0.01$ $k_z$ where the paraxial approximation holds. (b) The hopping parameters, $t = [2k_c c^2 / \delta \epsilon \omega^2] t(|d|)$, as a function of distance $d$ between two waveguide centres. $d$ is in a unit of $\eta$. The hopping is assumed to arise between two waveguides in the same mode. Different modes are differentiated by colour. The hopping parameters can be approximated as $t \propto e^{-d/\delta^*}$ where $1/\delta^* \approx -10.67$ (dashed) and $-9.64$ (dotted) (c) Wave functions of two adjacent waveguides. The solid circles denote boundaries of two waveguides. These wave functions correspond to the dashed line in (a,b). (Online version in colour.)

is the hopping integral between eigenfunctions of the single photonic waveguide located at the $(m', n')$ and $(m, n)$ sites

$$t_{mn}^{m'n'} \equiv t(|d|) = \frac{\omega^2}{2k_c c^2} \int dr \bar{\psi}(r) \delta \epsilon(\psi(r + d)),$$

where $d = R_{mn}^A - R_{mn}^B$ for $a_{m', n'}^+ b_{mn}$ and $a_{m', n'} a_{mn}$ hoppings, respectively. Our numerical evaluation of $t(|d|)$ (figure 2b) shows that the hopping integral exponentially decays as we increase $|d|$. Therefore, the tight-binding approach is appropriate for the present system. We calculated the tight-binding Hamiltonian for the zigzag ribbon geometry [7] in figure 1b up to the third nearest neighbours with $t \equiv t\langle a_w \rangle$. The overall results in this work, however, do not significantly change even when the calculations are only for the nearest neighbours. This is because the overlaps of the wave functions are sufficiently small, $\int dr \bar{\psi}(r) \psi(r + a_w) \sim 2.79 \times 10^{-7}$. Therefore, from here on, we will provide the results of the analytic expression for the tight binding model with only the nearest neighbour hopping. Meanwhile, all the numerical calculations shown in this work are from the tight-binding model up to the third nearest neighbours.

By introducing Fourier-transformed operators, $a_{k,n}^+ = (1/\sqrt{N_x}) \sum_m e^{ikx} c_{m,n}^+$, where $c = a, b$ $(A, B)$ and $N_x$ is the number of repeated unit cells in the $x$-axis direction, the Hamiltonian can be rewritten as

$$H_k = -t \sum_{k,n} (g_k a_{k,n}^+ b_{k,n} + a_{k,n}^+ b_{k,n+1} + \text{h.c.}),$$

where $g_k = 2 \cos(ka_0/2)$ and $a_0 = \sqrt{3} a_w$. 
one out of the total N where m change into

\[ \delta \phi_\beta - \partial_y \]

\[ \delta \phi_{u_\beta} = 0 \] for the HPR shown in figure 1b, one can solve an eigenvalue equation \( \mathcal{H}_k |\psi_k\rangle = -\beta_k |\psi_k\rangle \). Near the BZ boundary (\( q \equiv k \pi / a_0 \)), and energy of t, the eigenvalue is given by

\[ \beta_q \simeq t + \left[ t a_0 \cos \left( \frac{m}{N} \pi \right) \right] q, \]

where \( m = 1, \ldots, N - 1 \). Hence, at \( q = 0 \) (\( k = \pi / a_0 \)) and \( \beta_q = t \), the eigenstates are (\( N - 1 \))-fold degenerated extended states across the ribbon width. Away from \( q = 0 \), each state has a linear dispersion with a group velocity of \( v_m = t a_0 \cos((m/N)\pi) \) (figure 3b). We note that the remaining one out of the total N states is the edge localized zero energy state (figure 3b).

### 3. The strain effect

Now let us consider the system which has a strain field that depends only on the y coordinate, where k is still the crystal momentum of the system. The Hamiltonian in equation (2.1) changes into

\[ \mathcal{H}'_k = - \sum_{k,n} g_k \left( t + \frac{\partial t}{\partial u} \delta u_{n,N} \right) a_{k,n}^\dagger b_{k,n} - \sum_{k,n} \left( t + \frac{\partial t}{\partial u} \delta u_{n,N+1} \right) a_{k,n}^\dagger b_{k,n+1} + (\text{h.c.}), \]

(3.1)

where \( \delta u_{n,n'} \) is the change in distance between the sites of \( a_{nm} \) and \( b_{n'm} \). Here, the rate of change of the hopping parameter with respect to the inter-waveguide distance is given by \( \partial \ln t/\partial d = -\gamma \) as shown in figure 2b. Assuming \( u(r) = u(y)\hat{y} \), we have \( \delta u_{n,n} = (a_x/2)(\partial u/\partial y) \) and \( \delta u_{n,n+1} = a_x(\partial u/\partial y) \). Solving \( \mathcal{H}'_k |\psi_k\rangle = -\beta_k |\psi_k\rangle \) for the strained HPR numerically, we find that the resulting dispersions around \( k = \pi / a_0 \) show a drastic change from those without strain (figure 3).

With strain, the states at the BZ boundary lift their massive degeneracy and their dispersions become flat. The split flat energy bands are all doubly degenerate and spread downward from their original degenerate eigenenergy of \( t \). We also find that the energy eigenstates associated with the flat bands are strongly localized (figure 4). Unlike the well-known edge localized zero

![Figure 3](image-url)

**Figure 3.** (a) Full band structure of 21-HPR without strain. Enlarged view (box in (a)) of the band structure of 21-HPR near \( k = \pi / a_0 \) and \( \beta = t \) (b) without and (c) with strain. We apply the deformation as a form of \( u(y) = h_0(y - y_0)^2 \), where \( y_0 \) is the centre of the ribbon in the y-direction and \( h_0 \) is deformation strength. We set \( h_0 = 0.002/a_w \). (Online version in colour.)
energy states [16–18], the present localized states at $k = \pi/a_0$ are strictly localized at the specific line along $y$-direction inside the ribbon, i.e. bulk-regime stripe states and their positions depend on the energy of split flat band (figure 4a,c,e). It is notable that when the momentum deviates from the BZ boundary slightly, the stripe states form a linear combination with left- and right-side localized functions (figure 4b,d,f).

To understand the strain-induced flat bands and their stripe states, we investigate the Hamiltonian near the BZ boundary. By introducing the bond operators between neighbouring $n$th and $(n+1)$th zigzag chains, $c_{q,n}^\dagger = (1/\sqrt{2})(a_{q,n}^\dagger - b_{q,n+1}^\dagger)$ and $d_{q,n}^\dagger = (1/\sqrt{2})(a_{q,n}^\dagger + b_{q,n+1}^\dagger)$, the Hamiltonian in equation (2.1) can be approximated near the BZ boundary as

$$
H_q' \simeq - \sum_n t \sin \left( \frac{q a_0}{2} \right) (c_{q,n}^\dagger + d_{q,n}^\dagger)(c_{q,n-1} - d_{q,n-1})
$$

$$
+ \sum_n \frac{t + \delta t_n}{2} (c_{q,n}^\dagger + d_{q,n}^\dagger)(c_{q,n} - d_{q,n}) + (\text{h.c}),
$$

(3.2)
where $\delta t_n \equiv (\partial t/\partial u)\delta u_{n,n+1}$. Hereafter, we drop the subscript $q$ of the operators for simplicity. Since $q$ is small, we can neglect the mixing between the positive and negative energy states in equation (3.2). Thus, we have the following effective Hamiltonian for the positive energy states:

$$H_{\text{eff}} = -t_q \sum_n (c_n^+ c_{n-1} + c_{n-1}^+ c_n) + \sum_n U_n c_n^+ c_n,$$

(3.3)

where $t_q \equiv t \sin(q a_0/2)$ and $U_n \equiv t + \delta t_n$. Similarly, the effective Hamiltonian for the negative energy states can be obtained using the $d$ operators. Note that the positive (negative) energy states are antisymmetric (symmetric) at the sites of $a_0$'s and $b_0$'s. For a given $q$, the effective Hamiltonian in equation (3.3) is in the same form as the Hamiltonian for one-dimensional atom chains with the hopping $t_q$ between the nearest neighbours and site-dependent local potential energy of $U_n$. The strain field considered here is included in $U_n$ so that it plays a role of an electric field applied along the chain direction rather than its conventional role as the magnetic field in the two-dimensional strained graphene [7]. From equation (3.3), we can immediately note that at $q = 0$ ($t_q = 0$) the state is exactly localized at the $n$-sites and that the states localized at different $n$'s are not mixed together. We also note that the stripe states can appear as a result of the competition between the strain effect and the tunnelling because the localization in the transverse direction of the stripes ($y$-direction) is significant when $t_q < |U_n - U_{n+1}| = |\delta t_n - \delta t_{n+1}|$.

4. Atomic chain model for the stripe states

To investigate the structure of the stripes, we use a semiclassical method where we assume a quasi-momentum $p$ along the $y$-direction weakly depending on $n$. Without strain ($\delta t_n = 0$), this effective Hamiltonian gives quasi-momentum eigenstates in the form of $|\psi_p \rangle \propto \sum_n e^{i p n} c_n^+$ which have eigenvalues

$$\beta_p = t + 2t_q \cos p.$$

(4.1)

Now let us consider the case where a strain field $\mathcal{E}$ makes the hopping integral $\delta t_n$ be linearly proportional to the position $n$ as $\delta t_n = (n - n_0)\mathcal{E}$, where the $n_0$th zigzag chain is on the centre of the ribbon. Since the energy of the stripe state can be approximated as $t + (n - n_0)\mathcal{E}$, equation (4.1) thus can be used for describing the position-dependent momentum states as $2t_q \cos p_n \approx (n - n_0)\mathcal{E}$. This equation can be solved in an algebraic manner using $e^{i \xi_n p_n}$ which satisfies $\mathcal{E} e^{i \xi_n p_n} - 2\xi_n e^{i \xi_n p_n} + 1 = 0$, where $\xi_n = (n - n_0)\mathcal{E}/2t_q$. The solution is $e^{i \xi_n p_n} = \xi_n \pm \sqrt{\xi_n^2 - 1}$. For $\xi_n > 1$, i.e. $n - n_0 > 2t \sin(q a_0/2)/\mathcal{E}$, $p_n$ is a pure imaginary number satisfying $e^{i \xi_n p_n} = \xi_n - \sqrt{\xi_n^2 - 1}$. The amplitude of the wave function $|\psi_n| \sim |\xi_n - \sqrt{\xi_n^2 - 1}|^{-n}$ decays asymptotically as $n$ goes away from $n_0$,

$$|\psi_n| \sim \left[ \frac{(n - n_0)\mathcal{E}}{t \sin(q a_0/2)} \right]^{-n} = \left( \frac{|U_n - U_{n_0}|}{t_q} \right)^{-n}.$$

For $|\xi_n| < 1$, i.e. $|n - n_0| < 2t_q/\mathcal{E}$, $p_n$ is a real number and the amplitude of the wave function does not depend on the site index $n$ because $|e^{i \xi_n p_n}| = 1$. These localized states can tunnel to neighbouring sites, which is an effective Bloch oscillation [33] induced by strain. The tunnelling strength becomes significant when $t_q > |U_n - U_{n+1}| = |\delta t_n - \delta t_{n+1}|$. In this case, the dispersion near $k = \pi/a_0$ changes from flat band states to the states with finite group velocity (figure 3c).

5. Conclusion

In conclusion, we investigate the characteristic modes of photonic crystals in a ribbon-shaped honeycomb lattice which is a photonic counterpart of GNRs. The massively degenerate modes are shown to split to form flat bands under applied strains and their eigenstates are found to be stripe states located inside the ribbon. The locations of stripe states and tunnelling are controlled by deformation strength. The applied strain to the stripe states plays the role of a pseudo-electric field so that the dynamics of the stripe states are very similar to those of Bloch oscillation.
6. Computing solution

The eigenstates of the tight-binding Hamiltonian were obtained using a LAPACK library in Fortran77.

Ethics statement. This work did not involve any active collection of human data.

Data accessibility. This work does not have any experimental data.

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