Uniform stress fields inside multiple inclusions in an elastic infinite plane under plane deformation

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Multiple elastic inclusions with uniform internal stress fields in an infinite elastic matrix are constructed under given uniform remote in-plane loadings. The method is based on the sufficient and necessary condition imposed on the boundary value of a holomorphic function that guarantees the existence of the holomorphic function in a multiply connected region. The unknown shape of each of the multiple inclusions is characterized by a conformal mapping. This work focuses on a major large class of multiple inclusions characterized by a simple condition that covers and is much beyond the known related results reported in previous works. Extensive examples of multiple inclusions with or without geometrical symmetry are shown. Our results showed that the inclusion shapes obtained for the uniformity of internal stress fields are independent of the remote loading only when all of the multiple inclusions have the same shear modulus as that of the matrix. Moreover, specific conditions are derived on remote loading, elastic constants of the inclusions and uniform internal stress fields, which guarantee the existence of multiple symmetric inclusions or multiple rotationally symmetrical inclusions with uniform internal stress fields.

1. Introduction

In the micromechanical analysis of composite materials, the design of inclusions with uniform internal field has received much attention owing to practical significance
of the uniform internal field that does not induce stress peaks within the inclusions. Eshelby [1] pointed out that a two-dimensional elliptical inclusion or a three-dimensional ellipsoidal inclusion within an infinite matrix, subjected to a uniform remote stress field, can achieve a uniform internal stress field, and then conjectured [2] (see also [3]) that they are the only inclusion shapes which enjoy this uniformity property. Based on the complex variables method, Eshelby’s conjecture was proved to be true in two dimensions by Sendeckýj [4] for planar elasticity and by Ru & Schiavone [5] for anti-plane elasticity. Regarding Eshelby’s conjecture in three dimensions, Kang & Milton [6] and Liu [7] proved the weak version for elasticity problems; Liu [7] also showed that the strong version for thermal conductivity problems fails to be true, whereas Ammari et al. [8] made significant progress on the strong version for elasticity problems based on the concepts of generalized elastic moment tensor and generalized polarization tensor (see [9,10]). However, an intermediate layer often exists between the inclusion and matrix in many practical composites for some purposes, for instance to reduce the stress concentration at the inclusion–matrix interface or to improve the attachment between the inclusion and the matrix. Wang et al. [11] modelled the interphase layer as a spring layer with vanishing thickness and found a specific parameter of the spring layer that gives uniform internal stress field inside an elliptical inclusion, whereas Ru et al. [12], Ru [13] and Wang & Schiavone [14] showed that the internal stress field inside a three-phase or even $N$-phase ($N \geq 3$) confocal elliptical inclusion can be uniform and hydrostatic provided that the thickness of the interphase layer is designed appropriately. Unexpectedly, it is found in [15–18] that a non-elliptical (piezoelectric) inclusion with a specific interphase layer also admits a uniform internal elastic (or electro-elastic) field.

More recently, Eshelby’s uniformity property for multiple inclusions has also evoked much interest. For example, Cherepanov [19] first proposed an effective method to establish two ‘equally strong’ holes around which the hoop stress remains constant. It is expected that Cherepanov’s method could also be employed to construct multiple inclusions with Eshelby’s uniformity property (although this issue was not actually discussed in [19]). Based on the potential theory and variational inequality, Liu [7] showed the existence of multiple inclusions in two and three dimensions such that some common uniform eigenstress inside all of the multiple inclusions induces the same uniform internal field, and he also numerically calculated the shapes of these special inclusions (named as E-inclusions) using finite-element method. Liu et al. [20] also showed the existence of periodic multicomponent E-inclusions with individual uniform internal field inside each component of the inclusions characterized by a specific shape matrix. It is worth noting that the method of Liu et al. [20] could be also applied to construct multi-phase inhomogeneous E-inclusions (see [21]), and simple algebraic relations between the shape matrix for each phase and the material property of each phase could be derived by using this method (see [21]). Using the Weierstrass zeta function and the Schwarz–Christoffel formula, Kang et al. [22] studied two-dimensional inclusion pairs with Eshelby’s uniformity property for both anti-plane and plane elasticity, and further showed that the internal strain field of a non-elliptical single simply connected inclusion obtained by adding an extremely narrow bridge between two separated inclusions can be very close to being uniform. Wang [23] introduced a particular conformal mapping to construct inclusion pairs with two different uniform internal fields for anti-plane elasticity and piezoelectricity, and plane elasticity and finite plane elasticity, respectively. It is noted that the multiple E-inclusions shown by Liu [7] and the inclusion pairs considered by Kang et al. [22] are all assumed to have the same elastic constants, whereas the inclusion pairs studied by Wang [23] can have different elastic constants. However, it is difficult to extend Wang’s method [23] to three or more inclusions of different elastic constants. Therefore, we are motivated to reconsider Eshelby’s uniformity property for multiple inclusions (especially for the case of more than two inclusions) of arbitrarily given elastic constants in plane elasticity based on a simpler new method different from those used in the above-mentioned articles [7,22,23].

In this work, an alternative simpler method is developed to construct multiple two-dimensional inclusions with uniform internal stress fields under uniform remote in-plane loadings. Basic formulation of the present method is given in §2. In §3, based on the existence condition for the complex potential defined in the matrix region, the unknown shapes of
the multiple inclusions are determined using Cauchy’s integral formula, Faber series and the Newton–Raphson method. In §4, numerical examples are shown for multiple inclusions with various uniform internal stress fields under various uniform remote loadings, for multiple inclusions of shapes independent of the remote loading, and for multiple symmetrical inclusions, respectively. Finally, the main results of this work are summarized in §5.

2. Basic equations and problem description

(a) Basic equations

Consider an isotropic elastic material referred to a Cartesian coordinate system \((x_1,x_2)\) under plane stress or plane strain, and then the stresses \(\sigma_{11}, \sigma_{12}, \sigma_{22}\) and displacements \((u_1, u_2)\) in the \(x_1 - x_2\) plane can be expressed by two complex potentials \(\varphi(z)\) and \(\psi(z)\) as \([24]\)

\[
\begin{align*}
\sigma_{11} + \sigma_{22} &= 2[\varphi'(z) + \bar{\psi}'(z)], \\
\sigma_{22} - \sigma_{11} + 2i\sigma_{12} &= 2[\bar{\varphi}''(z) + \psi'(z)],
\end{align*}
\]

(2.1)

and

\[
2G(u_1 + iu_2) = \kappa \varphi(z) - z\bar{\varphi}'(z) - \bar{\psi}(z),
\]

(2.2)

where the capital \(I\) is used to denote the imaginary unit, in order to save the symbol \(i\) as a subscript, \(G\) indicates the shear modulus, \(\kappa = (3 - \mu)/(1 + \mu)\) for plane stress and \(\kappa = 3 - 4\mu\) for plane strain with \(\mu\) representing Poisson’s ratio. Additionally, the tractions \((X_1, X_2)\) on a directed curve from point \(A\) to \(B\) in the \(z\)-plane can be written in terms of \(\varphi(z)\) and \(\psi(z)\) as \([24]\)

\[
\int_A^B (IX_1 - X_2) ds = \left[\varphi(z) + z\bar{\varphi}'(z) + \bar{\psi}(z)\right]_A^B,
\]

(2.3)

where \(ds\) is an element of arc length of the curve along its tangent.

(b) Problem description

Shown in figure 1 are \(n\) elastic inclusions (of elastic constants \(G_i\) and \(\kappa_i, i = 1, 2, \ldots, n\)) bounded by the curves \(L_i(i = 1 \ldots n)\) within an infinite elastic matrix (of elastic constants \(G\) and \(\kappa\)) subjected to uniform remote loading \(\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{12}^\infty\). The shapes of inclusions defined by the curves will be determined as part of the solution. Let \(S\) and \(S_i(i = 1 \ldots n)\) denote the regions occupied by the matrix and inclusions, respectively. For the single-inclusion problem, an elliptical shape is the only possible inclusion shape that achieves a uniform internal stress field, and the actual uniform stress field can be arbitrary within a certain admissible range determined by the aspect ratio and orientation of the elliptical inclusion. Here, because the shapes of multiple inclusions that achieve individual uniform internal stress field in each of the inclusions are unknown, the uniform internal stress fields inside the inclusions may be restricted by some special conditions and could be prescribed only within a certain admissible range (see §§3a and 4a), and the problem is reduced to determination of the unknown inclusion shapes. Particularly, we assume that the internal stress field inside each of the inclusions is uniform but may be different from the uniform internal stress fields inside other inclusions.

Therefore, the complex potentials of the matrix and inclusions, \(\varphi(z), \psi(z), \varphi_i(z)\) and \(\psi_i(z)\) \((i = 1 \ldots n)\), have the form of

\[
\begin{align*}
\varphi(z) &= \Gamma z + \varphi_0(z), & \psi(z) &= \Gamma' z + \psi_0(z), \\
\Gamma &= \frac{\sigma_{11}^\infty + \sigma_{22}^\infty}{4}, & \Gamma' &= \frac{\sigma_{22}^\infty - \sigma_{11}^\infty + 2i\sigma_{12}^\infty}{2}
\end{align*}
\]

(2.4)

and

\[
\varphi_i(z) = \Gamma_i z + c_i, \psi_i(z) = \Gamma'_i z + d_i, i = 1 \ldots n,
\]

(2.5)
where the prescribed individual uniform internal stress field, determined by \( \Gamma_i \) and \( \Gamma'_i \), could be restricted within a certain admissible range to ensure the existence of solution, \( c_i \) and \( d_i \) are some complex unknown constants to be determined, whereas \( \varphi_0(z) \) and \( \psi_0(z) \) are holomorphic in the infinite multiply connected region \( S \) and without loss of generality, we stipulate \( \lim_{|z| \to +\infty} \varphi_0(z) = \lim_{|z| \to +\infty} \psi_0(z) = 0. \)

The continuous conditions of the tractions and displacements on the interface \( L_i (i = 1 \ldots n) \) are described, according to equations (2.2) and (2.3), as

\[
\varphi(t) + t \varphi'(t) + \psi(t) = \varphi_i(t) + t \varphi'_i(t) + \psi_i(t), \quad t \in L_i (i = 1 \ldots n)
\]  

and

\[
\frac{\kappa \varphi(t) - t \varphi'(t) - \psi(t)}{2G} = \frac{\kappa \varphi_i(t) - t \varphi'_i(t) - \psi_i(t)}{2G_i}, \quad t \in L_i (i = 1 \ldots n).
\]  

Here, the arbitrary parts of the complex constants \( c_i \) and \( d_i \) defined in equation (2.5) are chosen uniquely, so that the continuity condition of traction (2.3) can be simplified as (2.6). Consequently, the real and imaginary parts of complex constants \( c_i \) and \( d_i \) are now defined uniquely. Substituting equations (2.4) and (2.5) into equations (2.6) and (2.7) leads to

\[
\varphi_0(t) = A_i t + B_i \tilde{t} + C_i, \quad t \in L_i (i = 1 \ldots n),
\]

\[
A_i = \frac{(1 + \kappa_i G/G_i) \Gamma_i + (1 - G/G_i) \Gamma'_i - (1 + \kappa) \Gamma}{1 + \kappa},
\]

\[
B_i = \frac{(1 - G/G_i) \Gamma'_i}{1 + \kappa},
\]

\[
C_i = \frac{(1 + \kappa_i G/G_i) c_i + (1 - G/G_i) d_i}{1 + \kappa},
\]  

and

\[
\psi_0(t) = D_i t + E_i \tilde{t} - B_i \tilde{t} \left( \frac{d \tilde{t}}{dt} \right) + F_i, \quad t \in L_i (i = 1 \ldots n),
\]

\[
D_i = \Gamma'_i - \Gamma' - \tilde{B}_i,
\]

\[
E_i = \Gamma_i + \tilde{\Gamma}_i - 2 \Gamma - A_i - \tilde{A}_i,
\]

\[
F_i = \tilde{c}_i + d_i - \tilde{C}_i,
\]
where, according to the statement after equation (2.5), $A_i, B_i, D_i$ and $E_i$ ($i = 1 \ldots n$) are all known constants determined by the elastic constants of the inclusions and the matrix, individual prescribed uniform internal stress fields and the remote loading, whereas $C_i$ and $F_i$ ($i = 1 \ldots n$) are some unknown complex constants to be determined as part of the solution.

In what follows, we determine the unknown shapes of the inclusions based on the condition for the existence of such two holomorphic functions $\varphi_0(z)$ and $\psi_0(z)$ in the infinite multiply connected region $S$ which meet the boundary conditions (2.8) and (2.9), respectively.

3. General solutions

(a) The existence conditions of the complex potentials $\varphi_0(z)$ and $\psi_0(z)$

In order to ensure the existence of the functions $\varphi_0(z)$ and $\psi_0(z)$ holomorphic in the multiply connected region $S$, the boundary values of $\varphi_0(z)$ and $\psi_0(z)$ on the boundary $L_i (i = 1 \ldots n)$, according to Sokhotski–Plemelj theorem, should satisfy the following necessary and sufficient conditions [24],

$$
\frac{1}{2\pi i} \sum_{j=1}^{n} \int_{L_j} \frac{\varphi_0(t)}{t-z} \, dt = 0, \quad \forall z \in S_i (i = 1 \ldots n)
$$

(3.1)

and

$$
\frac{1}{2\pi i} \sum_{j=1}^{n} \int_{L_j} \frac{\psi_0(t)}{t-z} \, dt = 0, \quad \forall z \in S_i (i = 1 \ldots n).
$$

(3.2)

Substituting equations (2.8) and (2.9) into conditions (3.1) and (3.2) and then using Cauchy’s integral formula, one has

$$
A_i z + \frac{1}{2\pi i} \sum_{j=1}^{n} B_j \int_{L_j} \frac{\bar{t}}{t-z} \, dt = -C_i, \quad \forall z \in S_i (i = 1 \ldots n)
$$

(3.3)

and

$$
D_i z + \frac{1}{2\pi i} \sum_{j=1}^{n} E_j \int_{L_j} \frac{\bar{t}}{t-z} \, dt + \frac{1}{2\pi i} \sum_{j=1}^{n} B_j \int_{L_j} \frac{\bar{t}(d\bar{t}/dt)}{t-z} \, dt = -F_i, \quad \forall z \in S_i (i = 1 \ldots n).
$$

(3.4)

For the present problem in plane elasticity, a real challenge is how to satisfy two complex-conform boundary conditions (equations (2.6) and (2.7), or equivalently equations (3.3) and (3.4)) by choosing only one single curve of inclusion shape for each of all multiple inclusions. Actually, one could determine the shape of each inclusion from either equation (3.3) or (3.4). However, the inclusion shapes obtained from equation (3.3) are generally incompatible with those obtained from equation (3.4). In order to guarantee that the inclusion shapes obtained from equation (3.3) are consistent with those obtained from equation (3.4) when $n > 1$, a simple sufficient condition $B_i = 0 (j = 1 \ldots n)$ will be employed in this work, although it remains unclear whether it is also a necessary condition for the uniformity of internal stress fields within multiple inclusions when $n > 1$. Under the condition $B_i = 0 (j = 1 \ldots n$, where $n > 1$), it follows from equation (3.3) that $A_i = C_i = 0 (i = 1 \ldots n$, where $n > 1$), and, thus, we have

$$
I_i = \frac{(1 + \kappa)\Gamma}{2 + (\kappa_i - 1)G/G_i}, \quad I_i' = \begin{cases} 
0, & G_i \neq G \\
\text{unconditional}, & G_i = G
\end{cases} \quad (n > 1).
$$

(3.5)

Thus, an interesting consequence of the condition $B_i = 0 (j = 1 \ldots n$, where $n > 1$) is that the uniform internal stress field inside any inclusion of shear modulus different from that of the
matrix must be hydrostatic and is determined by the remote loading and the elastic constants of the inclusion and matrix but independent of the specific shape of the inclusion, whereas the uniform internal stress field inside an inclusion of the same shear modulus as that of the matrix is not necessarily hydrostatic and can be prescribed within a certain admissible range. It is stated that this result is consistent with basic results obtained in all previous related works for multiple inclusions in plane elasticity such as [7,22,23] which have been also based on some similar restriction conditions equivalent to the present condition $B_j = 0 (j = 1 \ldots n, \text{where } n > 1)$.

Here, it should be stated that for the single-inclusion problem with $n = 1$, the condition $B_j = 0 (j = 1)$ is clearly not a necessary condition for the uniformity of internal stress field. Actually, for single-inclusion problems, the only possible inclusion shape for a uniform internal stress field is elliptical. Because the elliptical shape enjoys some unique properties which all other non-elliptical shapes cannot share, the condition $B_j = 0 (j = 1 \ldots n)$ is not necessary for the uniformity of internal stress field inside a single (elliptical) inclusion with $n = 1$. Actually, the uniform internal stress field inside a single elliptical inclusion depends on the remote loading and is not necessarily hydrostatic. The uniform internal stress field inside a single elliptical inclusion can be hydrostatic only if the remote loading is limited within a certain admissible range, as is discussed in §4a.

For the present problem of multiple inclusions under the condition $B_j = 0 (j = 1 \ldots n, \text{where } n > 1)$, equation (3.4) becomes

$$D_1 z + \frac{1}{2\pi i} \sum_{j=1}^{n} E_j \oint_{L_j} \frac{\bar{t}}{t-z} \, dt = -F_\nu, \quad \forall z \in S_i (i = 1 \ldots n),$$

with

$$D_j = \begin{cases} -\Gamma', & G_i \neq G \\ \Gamma_i' - \Gamma', & G_i = G \end{cases}, \quad E_j = \frac{2\Gamma [\kappa - 1 - (\kappa_i - 1)G/G_i]}{2 + (\kappa_i - 1)G/G_i}.$$  \tag{3.7}

In what follows, we shall determine the shapes of the multiple inclusions based on equation (3.6) with (3.7).

(b) Faber series method

Note that each of the integral expressions $(j = 1 \ldots n)$ on the left side of equation (3.6) can be regarded as a holomorphic function of the argument $z$ in the simply connected regions $S_i (i = 1 \ldots n)$, and thus it can be expanded into a Faber series of the region $S_i$ as [25,26]

$$\frac{1}{2\pi i} \oint_{L_j} \frac{\bar{t}}{t-z} \, dt = \sum_{k=0}^{+\infty} b_{ijk} P_k (z-z_{0i}), \quad z \in S_i (i = 1 \ldots n),$$ \tag{3.8}

where $z_{0i}$ is a specific point in the region $S_i$ (see mapping (3.9)) and $P_k (z-z_{0i})$ is the $k$-th order Faber polynomial defined in the region $S_i$, particularly with $P_0 (z-z_{0i}) = 1$, whereas $b_{ijk}$ are coefficients of the related Faber series. An interesting merit of the present method is that the actual forms of the Faber polynomials are not required. Here, each of the undetermined simply connected regions $S_i (i = 1 \ldots n)$ can be defined by a conformal mapping which maps the exterior of the boundary $L_i$ of the region $S_i$ in the $z$-plane to the exterior of the unit circle (denoted by $\sigma_i = e^{i\theta}$) in the $\xi_i$-plane [24],

$$z - z_{0i} = \omega_i (\xi_i) = R_i \left( \xi_i + \sum_{l=1}^{+\infty} a_{il} \xi_i^{-l} \right), \quad i = 1 \ldots n,$$ \tag{3.9}

where the known complex constant $z_{0i}$ and the known real constant $R_i$ characterize the location and size of the $i$th inclusion, whereas all the unknown complex coefficients $a_{ij}$ determine the actual shapes of the multiple inclusions based on equation (3.6) with (3.7).
shape of the inclusion. In particular, it follows from definition (3.9) that the derivative of \( \omega_j(\xi) \) has no zeros outside the unit circle in the \( \xi_i \)-plane. Then, according to the definition of the Faber series [26] and mapping (3.9), the coefficients \( b_{ijk} \) in equation (3.8) are given by

\[
j = i: \quad b_{ijk} = \int_{|\sigma| = 1} \frac{(z_{0i} + \omega_i(\sigma))\sigma_i^{-k-1}d\sigma_i}{z_{0i} + \omega_i(\sigma) - z_{0i} - \omega_i(\sigma)} = \begin{cases} \tilde{z}_{0i}, & k = 0 \\ R_i a_{i}, & k \geq 1 \end{cases}
\]  

(3.10)

and

\[
j \neq i: \quad b_{ijk} = -\frac{1}{4\pi^2} \int_{|\sigma| = 1} \int_{|\sigma| = 1} \frac{(z_{0j} + \omega_j(\sigma))\omega_j(\sigma)\sigma_i^{-k-1}d\sigma_j}{z_{0j} + \omega_j(\sigma) - z_{0j} - \omega_j(\sigma)} d\sigma_i d\sigma_j = -\frac{1}{4\pi^2} \int_{|\sigma| = 1} \int_{|\sigma| = 1} \frac{\omega_j(\sigma)\omega_j(\sigma)\sigma_i^{-k-1}d\sigma_j}{z_{0j} + \omega_j(\sigma) - z_{0j} - \omega_j(\sigma)} d\sigma_i d\sigma_j, \quad k \geq 0.
\]  

(3.11)

Then, substituting equation (3.8) into equation (3.6) and using the formula \( P_{01}(z - z_{0i}) = 1 \) and \( P_{11}(z - z_{0i}) = (z - z_{0i})/R_i \), we obtain

\[
D_i R_i P_{11}(z - z_{0i}) + \sum_{k=1}^{+\infty} \left( \sum_{j=1}^{n} E_j b_{ijk} \right) P_{ik}(z - z_{0i}) = -\sum_{j=1}^{n} E_j b_{ij0} - D_i z_{0i} - F_i, \quad \forall z \in S_i(i = 1 \ldots n).
\]  

(3.12)

In order to satisfy equation (3.12) for any given \( z \) in the region \( S_i(i = 1 \ldots n) \), clearly the sufficient and necessary conditions are

\[
D_i R_i + \sum_{j=1}^{n} E_j b_{ij1} = 0, \quad \sum_{j=1}^{n} E_j b_{ijk} = 0(k \geq 2), \quad i = 1 \ldots n
\]  

(3.13)

and

\[
\sum_{j=1}^{n} E_j b_{ij0} + D_i z_{0i} + F_i = 0, \quad i = 1 \ldots n.
\]  

(3.14)

Here, in equation (3.13), the loading parameters \( D_i \) and \( E_i(i = 1 \ldots n) \), determined by the remote loading and the prescribed uniform internal stress fields, and the geometry parameters \( z_{0i} \) and \( R_i(i = 1 \ldots n) \) are all known, and the unknowns are the coefficients \( a_{il}(i = 1 \ldots n, l = 1 \ldots +\infty) \) introduced in mapping (3.9) which determine the actual shapes of the multiple inclusions. In what follows, the infinite series form of the conformal mapping (3.9) of the region \( S_i \) will be truncated into an \( N_i \)-order polynomial of \( N_i \) unknown coefficients \( a_{il}(i = 1 \ldots n, l = 1 \ldots +N_i) \), and thus the infinite number of nonlinear equations (3.13) (for \( i = 1 \ldots n \), and \( k \geq 2 \)) are truncated into a finite number of nonlinear equations (3.13) with \( k = 2 \ldots N_i(i = 1 \ldots n) \), respectively. Numerical methods will be employed to obtain these \( \sum_{l=1}^{N_i} N_i \) coefficients by solving the \( \sum_{l=1}^{N_i} N_i \) equations (3.13). Once the shapes of the multiple inclusions are obtained by solving the \( \sum_{l=1}^{N_i} N_i \) equations (3.13), the unknown constants \( F_i(i = 1 \ldots n) \) can be determined from equation (3.14) and then the complex constants \( c_i \) and \( d_i(i = 1 \ldots n) \) introduced in equation (2.5) is solved uniquely by the known \( C_i = 0 \) and \( F_i(i = 1 \ldots n) \) according to the relations between \( c_i, d_i, C_i \) and \( F_i(i = 1 \ldots n) \) shown in equations (2.8) and (2.9).
(c) Newton–Raphson iteration

By defining two vectors $\alpha$ and $F(\alpha)$ on the real and imaginary parts of the truncated coefficients $a_{il}(i = 1 \ldots n, l = 1 \ldots N_i)$,

$$
\alpha = \begin{bmatrix}
\Re(a_{11}) \\
\Im(a_{11}) \\
\vdots \\
\Re(a_{nN_n}) \\
\Im(a_{nN_n})
\end{bmatrix},
F(\alpha) = 
\begin{bmatrix}
\Re(D_1R_1) \\
\Im(D_1R_1) \\
0 \\
0 \\
\vdots \\
\Re(D_nR_n) \\
\Im(D_nR_n) \\
0 \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
\sum_{j=1}^{N_i} \Re(E_j b_{ij1}) \\
\sum_{j=1}^{N_i} \Im(E_j b_{ij1}) \\
0 \\
0 \\
\vdots \\
\sum_{j=1}^{N_i} \Re(E_j b_{ijN_n}) \\
\sum_{j=1}^{N_i} \Im(E_j b_{ijN_n}) \\
0 \\
0
\end{bmatrix}
\right)
$$ (3.15)

the truncated real form of equation (3.13) can be rewritten as

$$
F(\alpha) = 0,
$$ (3.16)

and the related Jacobian matrix $[\partial F(\alpha)/\partial \alpha]$ can be easily obtained based on the corresponding mapping (3.9) and expressions (3.10) and (3.11). The iterative process is then given by

$$
(\alpha)^{(p+1)} = (\alpha)^{(p)} - \left[\frac{\partial F(\alpha)}{\partial \alpha}\right]_{\alpha=(\alpha)^{(p)}}^{-1} F(\alpha)^{(p)}, \quad p = 0, 1, \ldots,
$$ (3.17)

where the superscript ‘$-1$’ indicates the inverse of the Jacobian matrix and $\alpha^{(p)}$ represents the value of the vector $\alpha$ after the $p$th iteration.

To guarantee convergence of the iterative process (3.17), here the initial value $\alpha^{(0)}$ with the known geometry parameters $z_{0i}$ and $R_i (i = 1 \ldots n)$ will be given by, say, $n$ disjoint ellipses (or circles) in the $z$-plane. If the iterative process (3.17) does not converge for any reasonable initial values $\alpha^{(0)}$, it implies that, most likely, the prescribed uniform internal stress fields cannot actually be achieved under the given elastic constants, geometry conditions and remote loadings. In addition, even a convergent solution will be considered inadmissible if either the corresponding boundaries $L_i (i = 1 \ldots n)$ intersect in the $z$-plane or the derivative of any of the corresponding mapping (3.9) has zero(s) outside the unit circle in the $\xi_i$-plane.

It is noted that Liu [7] and Liu et al. [20] have shown the existence and uniqueness of multicomponent inclusions with uniform internal fields (E-inclusions) for a given piecewise quadratic obstacle in the variational inequality. For the present general problem of multiple inclusions, however, we have not achieved a simple sufficient and necessary condition imposed on the elastic constants, geometry conditions, prescribed uniform internal stress fields and remote loadings, which guarantees the existence and uniqueness of the required inclusion shapes. A systematical theoretical investigation of the existence of solution is out of the scope of the present work. However, for given elastic constants, geometry conditions, prescribed uniform internal stress fields and remote loadings, our numerical results indicated that the solution is unique, because the iteration process always converges to the same inclusion shapes for different reasonable initial values.

4. Numerical examples

For the present multiple inclusion problem under the condition $B_j = 0 \ (j = 1 \ldots n$, where $n > 1$), whether the shear moduli of the multiple inclusions are identical to that of the matrix plays a key role in classifying the solutions, for convenience, we shall classify all examples into two categories. In the first category, all of the multiple inclusions have shear moduli different from that of the matrix, whereas the multiple inclusions of the second category have all exactly the same shear
modulus as that of the matrix. Particularly, under the present condition $B_j = 0 (j = 1 \ldots n$, where $n > 1$), as indicated by the condition (3.5), the uniform internal stress fields inside inclusions of the first category are always hydrostatic.

Our extensive numerical examples (including all examples described below) confirmed that moderately large numbers $N_i (7 \leq N_i \leq 12) (i = 1 \ldots n)$ are sufficient to achieve reasonably accurate convergent solution with relative errors less than 1%.

(a) Admissible range of the remote loading for first-category multiple inclusions with uniform internal hydrostatic stress fields and admissible range of prescribed uniform internal stress fields inside second-category multiple inclusions

All examples discussed in this paper are limited to cases under the condition $B_j = 0 (j = 1 \ldots n$, where $n > 1$), which can impose a restriction on the remote loading or the achieved uniform internal stress field. For example, for a single elliptical inclusion of the first category, the uniform internal stress field can meet the condition $B_j = 0 (j = 1)$ and be hydrostatic only if the remote loading is limited within an admissible range. Actually, for a single inclusion ($n = 1$), substituting equation (3.10) into equation (3.13), we obtain

$$a_{11} = \frac{-D_1}{E_1}, \quad a_{1l} = 0, \quad l = 2, 3, \ldots$$

(4.1)

where the condition $|a_{11}| < 1$ must be met for a correct conformal mapping (3.9). Here, because $E_1 = 0$ corresponds to some trivial cases in which the shape of the inclusion can be arbitrary, in what follows $E_1 \neq 0$ is always assumed. Thus, equation (4.1) indicates that the single inclusion must be elliptical and, on the other hand, equation (4.1) gives the admissible range of the remote loading for a single inclusion of the first category, or equation (4.1) gives the admissible range of the uniform internal stress field inside a single inclusion of the second category, as given by

$$\frac{D_1}{E_1} = g_1, \quad |g_1| < 1,$$

(4.2)

which, according to equation (3.7), can be expressed in a detailed form as

$$\Gamma' = -\frac{2 [\kappa - 1 - (\kappa_1 - 1)G/G_1]}{2 + (\kappa_1 - 1)G/G_1} g_1, \quad |g_1| < 1 (G_1 \neq G),$$

(4.3)

and

$$\Gamma'_1 = \Gamma' + \frac{2(\kappa - \kappa_1)g_1 \Gamma'}{\kappa_1 + 1}, \quad |g_1| < 1 (G_1 = G).$$

(4.4)

Equation (4.3) indicates that for a single elliptical inclusion of the first category that achieves a uniform internal hydrostatic stress field given by equation (3.5), the remote loading given by $\Gamma$ and $\Gamma'$ cannot be arbitrary and must be restricted within a limited range defined by the above arbitrary complex constant $g_1$ of absolute value less than unity which depends on the aspect ratio and orientation of the elliptical inclusion. On the other hand, for a single elliptical inclusion of the second category under an arbitrary remote loading, equations (3.5) and (4.4) show that the uniform internal stress field inside the elliptical inclusion is restricted within a certain admissible range defined by the above arbitrary complex constant $g_1$ of absolute value less than unity which depends on the aspect ratio and orientation of the elliptical inclusion.

For multiple inclusions discussed in all of the following examples for which $E_i \neq 0 (i = 1 \ldots n)$ are assumed, the complex parameters $g_i$ ($i = 1 \ldots n$) can be defined by $D_i$ and $E_i (i = 1 \ldots n)$ in a similar way as formula (4.2). Particularly, because all the complex parameters $g_i$ of the inclusions of the first category are associated with each other by a common remote loading (see equation (4.3)), only one of them can be prescribed to define the common remote loading and the others are then determined by this prescribed complex parameter or equivalently the defined remote loading. Thus, if there exist some inclusions of the first category, we shall first use one of the complex parameters $g_i$ (see equation (4.3)) to define a specific remote loading (the others
of the complex parameters $g_i$ will be determined by the specific remote loading), and then use the complex parameters $g_i$ (see equation (4.4)) of the other inclusions of the second category to define the prescribed the uniform internal stress fields inside the inclusions of the second category. If all of the inclusions belong to the second category, we shall use the complex parameters $g_i$ ($i = 1 \ldots n$) (see equation (4.4)) to define the prescribed uniform internal stress fields inside all of the inclusions for arbitrarily given remote loading. However, unlike the single-inclusion problem in which both admissible remote loading for an inclusion of the first category and admissible uniform internal stress field inside an inclusion of the second category must meet the condition $|g_1| < 1$, our results will show that admissible remote loading with $|g_i| > 1$ for some of the first-category inclusions and admissible uniform internal stress fields with $|g_i| > 1$ inside some of the second-category inclusions can exist.

As the first two examples, figure 2 shows various shapes of multiple inclusions of the first category under various uniform remote loadings, and figure 3 shows various shapes of multiple inclusions of the second category which achieve various prescribed uniform internal stress fields under given remote loadings.

**Figure 2.** (a–d) Multiple inclusions of the first category under various uniform remote loadings admissible for a single elliptical inclusion of the first category. (Online version in colour.)
Figure 3. (a–d) Multiple inclusions of the second category with various prescribed uniform internal stress fields admissible for a single elliptical inclusion of the second category.

It is noted that figure 2a given here (with \(N_1 = N_2 = 7\)) is in good agreement with fig. 6.1 of [22] and fig. 2 of [23], and figure 3a given here (with \(N_1 = N_2 = 8\)) recovers fig. 1 of [7] and fig. 3 of [23]. Here, it should be stressed that although the inclusion shapes shown in fig. 1 of [7] and figs 2 and 3 of [23] are obtained in anti-plane shear, the methods of [7] and [23] suggest that they could also achieve uniform internal stress fields in plane stress/strain for specific elastic constants and remote in-plane loadings. Our results shown in figure 2 indicate that any admissible remote loading with \(|g_1| < 1\) for a single elliptical inclusion of the first category is admissible for multiple inclusions of the first category, or, more precisely, any remote loading will be admissible for multiple inclusions of the first category if all (not some) of the corresponding complex parameters \(g_i\) of the inclusions satisfy \(|g_i| < 1\) simultaneously. The results shown in figure 3 indicate that any admissible uniform internal stress field with \(|g_1| < 1\) for a single elliptical inclusion of the second category is achievable for multiple inclusions of the second category. It is also shown in figure 2d that the uniform internal hydrostatic stress fields inside two first-category inclusions with different shear moduli can be exactly identical, and in figure 3b that
Figure 4. (a–d) Multiple inclusions of the mixed first and second categories under various uniform remote loadings admissible for a single first-category elliptical inclusion and with various prescribed uniform internal stress fields inside all involved second-category inclusions admissible for a single second-category elliptical inclusion.

the uniform internal stress fields inside multiple inclusions of the second category with the same Poisson’s ratio can be different from each other. Particularly, figure 3b,d indicates that the shear stresses inside any two inclusions of multiple inclusions of the second category need not be the same, which extends the scope of the validity of a conclusion made in [23] (see after equation (77) of [23]).

Shown in figure 4 are multiple inclusions of the mixed first and second categories, with various prescribed uniform internal stress fields inside the inclusions of the second category, under various remote loadings.

The results shown in figure 4 indicate that any admissible remote loading with $|g_i| < 1$ for a single elliptical inclusion of the first category is admissible for multiple inclusions of the mixed first and second categories, or more precisely, any remote loading will be admissible for multiple inclusions of the mixed first and second categories if all (not some) of the corresponding complex parameters $g_i$ of the involved first-category inclusions satisfy $|g_i| < 1$ simultaneously. On the other
hand, any admissible uniform internal stress field with \(|g_1| < 1\) for a single elliptical inclusion of the second category is achievable for all involved second-category inclusions of multiple inclusions of the mixed first and categories.

It is of great interest to see whether the remote loading for first-category inclusions of multiple inclusions could be beyond the admissible range with \(|g_1| < 1\) for a single elliptical inclusion of the first category, and to see whether the uniform internal stress fields for second-category multiple inclusions could be outside the admissible range with \(|g_1| < 1\) for a single elliptical inclusion of the second category. Figures 5–7 show several examples of multiple inclusions of the first category, multiple inclusions of the second category and multiple inclusions of mixed first and second categories, respectively, in which the remote loading with \(|g_i| > 1\) for some of the first-category multiple inclusions is a little beyond the admissible range with \(|g_1| < 1\) for a single elliptical inclusion of the first category, and some of the second-category multiple inclusions achieve uniform internal stress fields with \(|g_i| > 1\) slightly beyond the admissible range with \(|g_1| < 1\) for a single elliptical inclusion of the second category.

Our extensive numerical examples showed that it seems almost impossible to construct multiple inclusions of the first category, so that the remote loading corresponds to \(|g_i| > 1\) for all of the multiple inclusions, and almost impossible to construct multiple inclusions of the second category, so that all of the multiple inclusions simultaneously achieve the uniform internal stress fields with \(|g_i| > 1\), and also it seems almost impossible to construct multiple inclusions of the mixed first and second categories, so that the remote loading corresponds to \(|g_i| > 1\) for all of the involved first-category inclusions and simultaneously all of the involved second-category inclusions achieve the uniform internal stress fields with \(|g_i| > 1\). The extensive numerical examples also indicated that when one prescribed parameter \(|g_i|\) is relatively large (such as \(|g_i| > 2\)), it is difficult to construct an inclusion of the first category under a remote loading much beyond the admissible range with \(|g_1| < 1\) for a single elliptical inclusion of the first category, and it is also difficult to construct an inclusion of the second category which achieves a uniform internal stress field much beyond the admissible range with \(|g_1| < 1\) for a single elliptical inclusion of the second category. Thus, this work suggested that, for only some of multiple inclusions of the first category, the remote loading can be moderately beyond the admissible range of the remote loading for a single elliptical inclusion of the first category, and only some of multiple inclusions of the second category can simultaneously achieve uniform internal stress fields which are moderately beyond the admissible range of uniform internal stress field for a single elliptical inclusion of the second category.

From various numerical examples described above, we can see that a remarkable feature of the present method is that it can be used to construct multiple inclusions of arbitrarily given elastic constants, and the uniform internal stress fields are simply defined by a few complex parameters \(g_i(i = 1 \ldots n)\) which outline the rough shapes of the inclusions.

(b) Multiple inclusions of shapes independent of remote loading

Unlike the single inclusion problem in which any arbitrary elliptical inclusion always enjoys uniform internal stress field under any arbitrary remote loading, the shapes of inclusions shown in §4a may correspond to specific remote loadings, and their internal stress fields may be no longer uniform under other different remote loadings. Here, we focus on such multiple inclusions whose achieved internal stress fields are always uniform under arbitrary remote loadings. Note that the shapes of the inclusions determined by equation (3.13) can be rewritten as

\[
\frac{D_i}{E_i} R_i + \sum_{j=1}^{n} \frac{E_j}{E_i} b_{ij1} = 0, \quad \sum_{j=1}^{n} \frac{E_j}{E_i} b_{ijk} = 0 (k \geq 2), \quad i = 1 \ldots n. \tag{4.5}
\]

If the prescribed uniform internal stress fields and the elastic constants of inclusions and matrix can be given in such way that all ratios \(D_i/E_i (i = 1 \ldots n)\) and \(E_j/E_i (i, j = 1 \ldots n)\) in equation (4.5) are
Figure 5. (a, b) Multiple inclusions of the first category under the remote loading moderately beyond the admissible range of the remote loading for a single elliptical inclusion of the first category.

independent of the remote loading, then the corresponding inclusion shapes will be independent
of the remote loading.

For inclusion of the first category whose uniform internal hydrostatic stress field is solely
determined by the remote loading (see equation (3.5)), the corresponding ratio $D_i/E_i$ (see
equation (3.7)) cannot be independent of the remote loading, and thus it is impossible to construct
such multiple inclusions that their internal stress fields are always uniform under arbitrary remote
loadings when one or more of the multiple inclusions belong(s) to the first category. In other
words, all of the inclusions shown in figures 2, 4, 5 and 7 will no longer achieve uniform internal
stress fields when at least one of the ratios between the components ($\sigma^{\infty}_{11}$, $\sigma^{\infty}_{22}$ and
$\sigma^{\infty}_{12}$) of the remote loading is changed from those given in the figures.

However, there, indeed, exist multiple inclusions of the second category each of which always
enjoys uniform internal stress field under any arbitrary remote loadings. According to
equations (3.5) and (3.7), the internal stress fields ($\Gamma_i, \Gamma'_i$) inside such multiple inclusions of
the second category should be prescribed as ($g_i = D_i/E_i, i = 1 \ldots n$)

$$\Gamma_i = \frac{(1 + \kappa)\Gamma}{\kappa_i + 1}, \quad \Gamma_i' = \Gamma' + \frac{2(\kappa - \kappa_i)g_i\Gamma}{\kappa_i + 1}, \quad i = 1 \ldots n,$$

which is equivalent to equation (75) of [23]. Here, equation (4.6) gives the conditions on the
uniform internal stress fields which guarantee the existence of such inclusions of the second
category whose shapes are independent of the remote loading for any given elastic constants
$\kappa$ and $\kappa_i (i = 1 \ldots n)$. Actually, the condition (4.6) is already used to construct such inclusions of
the second category independent of the remote loading in §4a. For example, all of the inclusions
shown in figures 3 and 6 will still have uniform internal stress fields even when the ratios between
the components ($\sigma^{\infty}_{11}$, $\sigma^{\infty}_{22}$ and $\sigma^{\infty}_{12}$) of the remote loading are changed arbitrarily. In §4c, the
condition (4.6) will be also used to construct symmetric and rotationally symmetrical inclusions
of the second category whose shapes are independent of the remote loading.

(c) Symmetric multiple inclusions

In the $z$-plane, consider two closed curves $L_1$ and $L_2$ which are symmetric to each other about a
line passing a certain point and at an angle of $\alpha$ to the positive $x$-axis. For any two symmetric
Figure 6. (a, b) Multiple inclusions of the second category with uniform internal stress fields moderately beyond the admissible range of the uniform internal stress field for a single elliptical inclusion of the second category.

Figure 7. (a, b) Multiple inclusions of the mixed first and second categories with the remote loading moderately beyond the admissible range of the remote loading for a single first-category elliptical inclusion or with uniform internal stress fields inside the second-category inclusions moderately beyond the admissible range of uniform internal stress field for a single second-category elliptical inclusion.

Points $z_1$ and $z_2$ about the line of symmetry, one can verify that

$$\frac{1}{2\pi i} \int_{L_1} \frac{t}{t - z_1} dt = \text{constant} + \frac{e^{2\alpha t}}{2\pi i} \int_{L_2} \frac{t}{t - z_2} dt, \quad \forall z_1 \in S_1,$$

and

$$\frac{1}{2\pi i} \int_{L_1} \frac{t}{t - z_1} dt = \frac{e^{2\alpha t}}{2\pi i} \int_{L_2} \frac{t}{t - z_2} dt, \quad \forall z_1 \notin S_1,$$

where $S_1$ denotes the finite region bounded by the curve $L_1$. 

Using equations (4.7) and (4.8), we can derive the condition on elastic constants, remote loading and prescribed uniform internal stress fields which guarantees the existence of even-numbered symmetric inclusions about a certain line. Here, we give an example of two symmetric inclusions about a certain line. For two symmetric inclusions about a line which passes a certain point and is at an angle of $\alpha$ to the positive $x$-axis, conjugating the two sides of equation (3.6) ($n = 2$) and using equations (4.7)–(4.8), one obtains

$$
\begin{align*}
\tilde{D}_1 e^{-2i\alpha} z + \frac{\tilde{E}_2 e^{2i\alpha}}{2\pi i} \int_{L_1} \frac{\bar{t}}{t-z} \, dt + \frac{\tilde{E}_1 e^{2i\alpha}}{2\pi i} \int_{L_2} \frac{\bar{t}}{t-z} \, dt &= -H_1, \quad \forall z \in S_2, \\
\tilde{D}_2 e^{-2i\alpha} z + \frac{\tilde{E}_2 e^{2i\alpha}}{2\pi i} \int_{L_1} \frac{\bar{t}}{t-z} \, dt + \frac{\tilde{E}_1 e^{2i\alpha}}{2\pi i} \int_{L_2} \frac{\bar{t}}{t-z} \, dt &= -H_2, \quad \forall z \in S_1,
\end{align*}
\tag{4.9}
$$

where $H_1$ and $H_2$ are two new unknown constants. Considering that equation (4.9) has to be equivalent to equation (3.6) ($n = 2$), we require ($g_i = D_i/E_i, i = 1, 2$)

$$
K = \frac{E_2}{E_1} = \frac{\tilde{E}_1}{\tilde{E}_2}, \quad \frac{D_1}{E_1} = e^{-4i\alpha} \frac{D_2}{E_2}. \tag{4.10}
$$

Here, according to equations (3.7) and (4.10), $K = \pm 1$ must be met and leads to the following relation between the elastic constants of the two inclusions,

$$
\begin{align*}
\begin{cases}
\frac{\kappa_1 - 1}{G_1} = \frac{\kappa_2 - 1}{G_2}, & (K = 1) \\
\kappa + 1 & 2 + (\kappa_1 - 1)G/G_1 + \frac{\kappa + 1}{2 + (\kappa_2 - 1)G/G_2} = 2, & (K = -1)
\end{cases}
\end{align*}
\tag{4.11}
$$

Furthermore, the complex parameters $g_i = D_i/E_i$ ($i = 1, 2$) for two symmetric inclusions of the first category should satisfy

$$
\begin{align*}
g_1 &= g_2, \quad \text{arg}(g_1) = -2\alpha \text{ or } \pi - 2\alpha, (K = 1) \\
g_1 &= -g_2, \quad \text{arg}(g_1) = \pm\frac{\pi}{2} - 2\alpha, (K = -1)
\end{align*}
\tag{4.12}
$$

whereas the complex parameters $g_i = D_i/E_i$ ($i = 1, 2$) for two symmetric inclusions involving at least one second-category inclusion (say, the second inclusion always belongs to the second category) only need to meet

$$
g_1 = e^{-4i\alpha} \bar{g}_2, \quad (K = \pm 1). \tag{4.13}
$$

Based on conditions (4.11)–(4.13), figures 8, 9 and 10 give a series of two symmetric inclusions of the first category, of the second category, and of the mixed first and second categories, respectively.

Note that the inclusion shapes shown in figures 8 and 10 are based on the specific remote loadings, whereas those shown in figure 9 are independent of the remote loading. As shown in figures 2a and 3a, it is expected that two symmetric inclusions may have the same elastic constants. However, as shown in figures 8–10, the elastic constants of two symmetric inclusions can be totally different, and even their Poisson ratios can also be different for two symmetric inclusions of the second category. If the elastic constants of the two symmetric inclusions correspond to $K = -1$ in equation (4.11) (figures 8c,d, 9b and 10b), the two inclusions tend to be ‘sharp cornered’ when they are close to each other.

**d) Rotationally symmetrical multiple inclusions**

Now, let us consider rotationally symmetrical multiple inclusions. Consider two closed curves ($L_1$, $L_2$) and two points ($z_1, z_2$) in the $z$-plane, where the curve $L_2$ and the point $z_2$ are generated, respectively, from the curve $L_1$ and the point $z_1$ by an anticlockwise rotation of an angle $\alpha$ about
a point \( z_0 \) in the \( z \)-plane. Then, one can show
\[
\frac{1}{2\pi I} \int_{L_2} \frac{i}{t - z_2} \, dt = \text{constant} + \frac{e^{-i\alpha}}{2\pi I} \int_{L_1} \frac{i}{t - z_1} \, dt, \quad \forall z_2 \in S_2,
\]  
(4.14)

and
\[
\frac{1}{2\pi I} \int_{L_2} \frac{i}{t - z_2} \, dt = \frac{e^{-i\alpha}}{2\pi I} \int_{L_1} \frac{i}{t - z_1} \, dt, \quad \forall z_2 \notin S_2,
\]  
(4.15)

where \( S_2 \) denotes the finite region bounded by the curve \( L_2 \).

From equations (4.14) and (4.15), one can derive the condition on elastic constants, remote loading and prescribed uniform internal stress fields, which guarantees the existence of rotationally symmetrical inclusions around a certain point. For \( n \) rotationally symmetrical inclusions around a point \( z_0 \) in the \( z \)-plane, the \((i + 1)\)-th inclusion is generated from the \(i\)th inclusion by an anticlockwise rotation of the angle \( \alpha = 2\pi/n \) around the point \( z_0 \). By employing

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**Figure 8.** (a–d) Two symmetric inclusions of the first category.
Figure 9. (a, b) Two symmetric inclusions of the second category independent of the remote loading.

Figure 10. (a, b) Two symmetric inclusions of the mixed first and second categories.

equations (4.14) and (4.15), equation (3.6) becomes

\[
\begin{align*}
D_1 e^{i\alpha z} + \frac{E_1 e^{-i\alpha}}{2\pi i} \oint_{L_n} \frac{t}{t - z} dt + \sum_{j=2}^{n} E_j e^{-i\alpha} \oint_{L_{n-1}} -i = H_n, \quad \forall z \in S_n, \\
D_{i+1} e^{i\alpha z} + \frac{E_1 e^{-i\alpha}}{2\pi i} \oint_{L_n} \frac{t}{t - z} dt + \sum_{j=2}^{n} E_j e^{-i\alpha} \oint_{L_{n-1}} -i = H_i, \quad \forall z \in S_i(i = 1 \ldots n - 1),
\end{align*}
\]

(4.16)
where $H_i (i = 1 \ldots n)$ are some new constants to be determined. It follows from the equivalency between equation (4.16) and equation (3.6) that

$$K = \frac{E_1}{E_n} = \frac{E_{i+1}}{E_i}, \quad i = 1 \ldots n - 1$$

and

$$D_{i+1} = \frac{D_i}{E_i} e^{-2i\alpha}, \quad i = 1 \ldots n - 1.$$ (4.17)

Here, according to equations (3.7) and (4.17), one has $K = 1$ for an odd $n$ and $K = \pm 1$ for an even $n$, which further results in the following relation between the elastic constants of the $n$ inclusions,

$$\begin{cases}
\kappa_1 - 1 = \ldots = \kappa_n - 1, (K = 1) \\
\kappa + 1
\end{cases}
\begin{cases}
\frac{G_1}{\kappa} + \frac{G_n}{\kappa + 1} = 2,
2 + (\kappa_1 - 1)G_i G_i + 2 + (\kappa_2 - 1)G_i G_{i+1} = 2, \\
(K = -1, \quad i = 1 \ldots n - 1)
\end{cases}$$

Then, based on equations (3.7) and (4.17), the complex parameters $g_i = D_i/E_i (i = 1 \ldots n)$ for rotationally symmetrical inclusions of the first category should meet

$$K = 1: \begin{cases}
g_1 = g_2, (n = 2) \\
g_1 = \ldots = g_n = 0, (n \geq 3)
\end{cases}$$

$$K = -1: \begin{cases}
g_1 = -g_2 = g_3 = -g_4, (n = 4) \\
g_1 = \ldots = g_n = 0, (n = 2, 6, 8, 10, \ldots),
\end{cases}$$

and the complex parameters $g_i = D_i/E_i (i = 1 \ldots n)$ for rotationally symmetrical inclusions of the second category only need to satisfy

$$g_{i+1} = g_i e^{-2i\alpha} (K = \pm 1, \quad i = 1 \ldots n - 1).$$ (4.20)

However, for rotationally symmetrical inclusions of the mixed first and second categories, it is difficult to derive a unified condition like (4.19) or (4.20) from the condition (4.17).

Figures 11 and 12 show several examples of rotationally symmetrical inclusions of the first category or the second category, respectively, based on the conditions (4.18)–(4.20), whereas figure 13 gives a few examples of rotationally symmetrical inclusions of the mixed first and second categories derived directly from the condition (4.17). In particular, the inclusion shapes shown in figures 11 and 13 correspond to the specific remote loading, whereas those shown in figure 12 are independent of the remote loading.

It is noted that the rotationally symmetrical inclusions of a shape shown in figure 11 are well consistent with those shown in fig. 2 of [7]. Similar to two symmetric inclusions, rotationally symmetrical inclusions shown in figures 11–13 can also have quite different elastic constants. Particularly, for rotationally symmetrical inclusions whose elastic constants correspond to $K = -1$ in equation (4.18) (figures 11c,d, 12b and 13b), each of the inclusions tends to have some ‘sharp corners’ when they are close to adjacent inclusions.

5. Conclusion

A simpler new method is developed to construct multiple elastic inclusions with uniform internal stress fields in an infinite elastic plane under uniform remote in-plane loadings. All inclusions discussed in this paper are classified into two categories: the inclusions of the first category have shear moduli different from that of the matrix and consequently have uniform hydrostatic internal stress fields, whereas the inclusions of the second category have all the same shear modulus as that of the matrix. The unknown shape of each of the inclusions is characterized by a conformal mapping whose unknown coefficients can be determined by a system of nonlinear equations with the aid of Cauchy’s integral formula and Faber series. A remarkable merit of the present method when compared with all previous methods is that the present method can be used to construct...
complicated examples of more than two inclusions, such as multiple inclusions of the mixed first and second categories. Extensive numerical examples are given to verify the validity and accuracy of the present method. Specific examples are shown for multiple inclusions of shapes independent of the remote loading, as well as for multiple symmetrical or rotationally symmetrical inclusions, which are very much beyond the known examples already reported in the existing literature. Among others, some conclusions can be drawn as follows:

— the uniform internal hydrostatic stress field inside each of the first-category inclusions is determined solely by the elastic constants of the inclusion and matrix and the remote loading, which can not only cover, but also be moderately beyond the admissible range of the remote loading for a single elliptical inclusion of the first category;

— the uniform internal stress field inside each of the second-category inclusions cannot be determined completely from the given remote loading and the elastic constants of the inclusion and matrix, but can vary within an admissible range which covers and slightly exceeds the admissible range of the uniform internal stress field for a single elliptical inclusion of the second category under the same remote loading;
to achieve multiple inclusions with uniform internal stress fields whose shapes are independent of the remote loading, all of the inclusions have to be of second category. On the other hand, if certain multiple inclusions of the second category have uniform internal stress fields under a specific remote loading, they can also achieve uniform internal stress fields under any other remote loadings; and

— symmetrical inclusions with uniform internal stress fields can be constructed not only for pure first or second category, but also for mixed first and second categories, and the elastic constants of the inclusions are not necessarily identical as long as they are compatible with each other to meet some specific relations.

Data accessibility. This work focuses on developing a new method, and the related parameters for all the numerical examples are shown in the figures. Readers can perform the method to reproduce the present numerical results by any program language such as C, Fortran or Matlab.
Funding statement. This work was supported by the National Natural Science Foundation of China (11232007 and 11472130), a Project supported by the Priority Academic Programme Development of Jiangsu Higher Education Institutions (PAPD) and the Natural Science and Engineering Research Council of Canada (NSERC-RGPIN204992).

Author contributions. M.D. participated in devising the method, designed the solving procedure, carried out the numerical examples and drafted the manuscript; C.-F.G. participated in devising the method, analysed the numerical results and helped drafting the manuscript; C.Q.R. conceived of the study, participated in devising the method and revised the manuscript critically. All authors gave final approval for submission and publication.

Conflict of interests. All authors have declared that no competing interests exist.

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