An elastocapillary model of wood-fibre collapse

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An elastocapillary model for drying-induced collapse is proposed. We consider a circular elastic membrane with a hole at the centre that is deformed by the capillary pressure of simply and doubly connected menisci. The membrane overlays a cylindrical cavity with rigid walls, trapping a prescribed volume of water. This geometry may be suitable for studying structural failures and stiction in microelectromechanical systems during wet etching, where capillary surfaces experience catastrophic transitions. The dry state is determined using the dihedral-angle and volume-turning-point stability criteria. Open and collapsed conformations are predicted from the scaled hole radius, cavity aspect ratio, meniscus contact angle with the membrane and cavity walls, and an elastocapillary number measuring the membrane stretching rigidity relative to the water surface tension. For a given scaled hole radius and cavity aspect ratio, there is a critical elastocapillary number above which the system does not collapse upon drying. The critical elastocapillary number is weakly influenced by the contact angle over a wide range of the scaled hole radius, thus indicating a limitation of surface hydrophobization for controlling the dry-state conformation. The model is applied to the drying of wood fibres above the fibre saturation point, determining the conditions leading to collapse.

1. Introduction

Cellulose fibres are abundant natural resources with desirable characteristics, such as high-strength to weight ratio, corrosion-resistance, biodegradability and biocompatibility [1–3]. Besides their traditional application in papermaking, they have been used as reinforcements for

Isolating fibres from parent wood by pulping is accompanied by dramatic conformational changes, the extent of which depends on the pulping process and the mechanical, physical and structural properties of fibres. The extremities of heating and chemical treatment in pulping also play an important role. Fibres from chemical pulping are delignified and usually collapsed, whereas fibres from thermomechanical pulping are lignin-coated and mostly uncollapsed, imparting contrasting properties to fibre-based composites [5]. For example, papers from chemical pulps have high tensile strength and smooth surfaces, suiting high-quality printing, while those from mechanical pulps have high compressive strength and rough surfaces, suiting fast, economical printing [2].

Wood-fibre deformation upon drying results from complex interactions between water and wood tissues in pulping [2]. The degree to which water is associated with wood can be quantified by differential scanning calorimetry [8,9]. Here, three types of water are identified [8]: (i) free water, having the same transition temperature as bulk water, (ii) freezing-bound (physisorbed) water, having lower transition temperature than bulk water, and (iii) non-freezing-bound (chemisorbed) water, which cannot be detected from the first-order transition. Molecules of non-freezing-bound water are directly bound to cellulose hydroxyl groups, whereas those of freezing-bound water and free water are trapped in the lumen and cell-wall micropores by capillary condensation. Noting that the heat of desorption required to break water-hydroxyl group bonds for non-freezing-bound water is much larger than the excess enthalpy of vaporization because of meniscus formation for freezing-bound and free water [2], drying stresses are expected to increase significantly at low moisture contents where the remaining water is mostly non-freezing bound.

Defining the fibre saturation point (FSP) as the moisture content where all the remaining water is adsorbed (non-freezing bound) [2] identifies two drying-deformation regimes [10]: (i) collapse above the FSP where fibre macro- and microscopic structure is affected and (ii) shrinkage below the FSP where cell-wall submicroscopic structure is affected. Inconsistent reports in the literature as to whether deformations upon drying begin above or below the FSP motivate further studies on wood-fibre collapse and shrinkage at various drying stages [11–13].

Barber & Meylan [14] developed one of the first theoretical models to quantify shrinkage below the FSP. They considered a flat element of the cell wall, comprising only the $S_2$ layer with cellulose microfibrils embedded in an amorphous matrix. This matrix shrinks isotropically with a linear relationship between normal strain and moisture content, while the cell-wall overall shrinkage is anisotropic because the matrix is constrained by crystalline microfibrils. This model was later extended to account for the effect of a cylindrical cell wall [15], a multi-layer cell wall [16], changing lumen shape during drying [12] and moisture-dependent mechanical properties [17].

Key observations from these studies include: (i) drying deformations and changes in the physical and mechanical properties of wood occur below the FSP, (ii) an anisotropic three-dimensional shrinkage is predicted with the radial component lower than the tangential component at small microfibril angles, agreeing with experiments [18,19], and (iii) volumetric shrinkage is proportional to the amount of water removed during drying.

In contrast with the foregoing studies, Hernández & Pontin [13] reviewed experimental evidence in the literature to show that drying deformations can begin at moisture contents well above the FSP. These are attributed to collapsed lumen and micropores resulting from an interplay between water surface tension and cell-wall elastic resistance [10,20]. In fact, non-collapsed fibres can be produced by drying solvent-exchanged pulps where water is substituted with a low-surface tension liquid before drying [2], supporting the latter proposition. Depending on the cell-wall microstructure, mechanical properties and pore size, a spectrum of collapsed, partially collapsed and uncollapsed pores are observed [2,10], implying that the overall volumetric shrinkage in this regime is not linearly related to the volume of removed water. Here, we consider an elastocapillary model of the lumen and cell-wall micropores, rigorously examining the
interactions between elastic and capillary forces during drying to provide a better understanding of drying deformation at moisture contents above the FSP.

Elastocapillary phenomena have been extensively studied over the past two decades in several areas, such as capillary-induced wrinkling [21,22], micro-electromechanical systems [23–25], capillary wrapping and origami [26], and self-assembly, coalescence and bundling of lamellae [27–30]. Mastrangelo & Hsu [23] studied the capillary-driven deformation of beams and plates in micro-machined structures and derived collapse criteria based on the bending stiffness to surface tension ratio. Neglecting the meniscus contribution to the total energy, continuous equilibrium trajectories without stability exchange that connect the initial and collapsed states were identified as leading to collapsed conformations. A two-dimensional model of capillary rise between flexible sheets is given by Kim & Mahadevan [31] based on the linear bending theory of beams. Here, the final conformation is determined by a balance involving gravity, elastic and contact-line forces. This model was further examined by Kwon et al. [32] for systems in which conformations are controlled by the liquid content. Taroni & Vella [33] studied the equilibrium and stability of the same problem. Accounting for the meniscus shape, the Laplace pressure was also incorporated into the equilibrium equation of the beam. Moreover, multiple stable equilibria were found, and final conformations were determined through a dynamic analysis.

In this paper, we study an elastocapillary model of drying-induced collapse for cavities that are characterized by three length scales. The model comprises a circular elastic membrane with a hole at the centre (figure 1). The membrane is anchored above a rigid plate, trapping a prescribed volume of water. We examine membrane deformations caused by a meniscus that forms at the hole as liquid is removed. This furnishes an idealized structure where stability and equilibrium are determined by the elastic–capillary force interactions, providing collapse criteria for lumen and cell-wall micropores at moisture contents above the FSP. Neglecting the contact-line force, we adopt a variational formulation to examine stability and equilibria. Here, the meniscus geometry is exactly treated for the interfacial energies to account for the significant role of the Laplace pressure when collapsed conformations are approached. Using the stability criteria we recently derived based on a spectral analysis of the potential energy [34], the dry-state conformation is determined from the stability of equilibrium branches. The model is also relevant to structural failures and stiction in micro-electromechanical systems during wet etching, where capillary surfaces undergo catastrophic transitions. These transitions, which have not been addressed in previous studies, are modelled as bubble-to-bridge transitions (BBTs) to elucidate their role in the system stability and dry-state conformation.

Previous elastocapillary models in the literature are mostly based on the linear theory of plates, where the bending contribution to the total potential energy is only considered [31–33]. By contrast, the present model only considers the stretching energy, accounting for variable and anisotropic axial forces. The latter is more challenging because the equilibrium equations are nonlinear and the in-plane displacement is not neglected. Moreover, the model does not admit buckling, which typically occurs in geometries with closed structures, such as those of wood fibres. Buckling and the bending rigidity are additional resistances to deformation. Hence, the collapse criteria derived in this paper can be regarded as upper bounds, which are suitable for predicting open conformations from geometrical parameters and mechanical properties.
2. Theory

We consider an elastocapillary model shown in figure 1 comprising a circular elastic membrane with a hole at the centre supported on the sidewall of a cylindrical cavity with rigid walls, trapping a volume \( v_l \) of liquid below the membrane and volume \( v_g \) air between the bounding surface and membrane. Interfacial surface areas are denoted \( \Gamma_{ij} \). The cavity is open to the atmosphere from the top. A meniscus forms at the hole as the liquid (water) is removed, resulting in a difference between the liquid pressure \( p_l \) and atmospheric pressure \( p_g \) that causes the membrane to deform. Here, the membrane radius \( R \), hole radius \( R_0 \) and cylinder height \( H \) are the model length scales that control the interplay between elastic and capillary forces. To determine the equilibria at a given \( v_l \), we consider an imaginary bounding surface (dashed line in figure 1a) that covers the cavity from the top. The system is completely isolated from the surrounding by the bounding surface and cylinder walls. The meniscus is initially a bubble, which can bridge the gap upon contact with the plate at the bottom of the cylinder, forming a free contact line with the plate. The resulting surface \( \Gamma_{sg} \) is assumed to be an ideal solid–gas interface that is free of liquid. Here, \( \theta_c \) and \( \theta_d \) are the thermodynamic contact angle and dihedral angle that the meniscus forms with the plate and membrane, respectively. Assuming that all the dimensions are small compared with the capillary length, the gravity force is neglected. The membrane and meniscus are assumed axisymmetric in equilibrium and perturbed configurations. Drying dynamics are assumed to be slow (quasi-static), so that the system evolves through a sequence of equilibrium states. We also assume that the inner surfaces of the plate and membrane that are in contact with water produce the same thermodynamic contact angle \( \theta_c \).

We derived a variational principle for the stability and equilibrium of the elastocapillary model shown in figure 1 [34]. Neglecting the bending contribution to the elastic strain energy, the membrane in-plane and out-of-plane displacement profiles are (see appendix A)

\[
\tilde{w}(\tilde{r}_p) = \tilde{w}_0(1 - \tilde{r}_p^2) \tag{2.1}
\]

and

\[
\tilde{u}(\tilde{r}_p) = \frac{\tilde{w}_0^2(1 - \tilde{r}_p^2)}{4} \left[ \tilde{r}_p(3 - \nu) + \frac{\kappa^2[(3 - \nu)(1 + \nu) - \kappa^2(1 - \nu^2)]}{\tilde{r}_p[1 - \nu + \kappa^2(1 + \nu)]} \right], \tag{2.2}
\]

where \( \tilde{r}_p \), \( \tilde{u} \) and \( \tilde{w} \) are the radial position in the referential coordinate \( r_p \), in-plane displacement \( u \) and out-of-plane displacement \( w \), all scaled with \( R \); moreover, \( \tilde{w}_0 = (Q_c/N_C)^{1/3} K_{w} \), and

\[
K_w = \left[ \frac{3[1 - \nu + \kappa^2(1 + \nu)]}{\kappa(1 - \kappa^2)(1 - \nu^2)[7 - \nu + \kappa^2(1 + \nu)]} \right]^{1/3}. \tag{2.3}
\]

Here, \( \kappa = R_0/R \), \( Q_c = PR_{\infty}/\gamma_{gl} \) and \( N_C = C/\gamma_{gl} \) are the scaled hole radius, scaled capillary pressure and elastocapillary number where \( P = p_l - p_g \) and \( C, \nu, \gamma_{gl}, R_0 \) are the membrane axial rigidity, Poisson ratio, air–water surface tension and hole radius in the referential configuration. Equations (2.1) and (2.2) are derived using von Kármán’s plate theory, which requires \( dw/dr_p \sim H/R \ll 1 \) [34]. Therefore, we restrict our analysis to cases for which \( H/R \ll 1 \), so that the membrane displacements can be accurately approximated by equations (2.1) and (2.2).

To compute the liquid volume \( v_l \), meniscus meridian curve and minimum gap between the membrane and plate \( h \), we neglect the in-plane displacement, which is a reasonable approximation when \( H/R \ll 1 \) [34]. Neglecting the in-plane displacement implies that the hole expansion resulting from the membrane deflection is negligible \( (R_0 \approx R_{\infty}) \). Furthermore, one must ensure that a given membrane profile corresponding to the capillary pressure of a meniscus does not violate any geometrical constraint. Specifically, the sum of the membrane deflection and the gap between the membrane and plate at the hole edge must equal \( H \), and volume bounded by the meniscus and membrane must equal the liquid volume. These constraints are \( H = h - w(R_0) \).
and \( v_1 + v_g = \pi R^2 h \), which upon substituting equation (2.1) furnish

\[
\kappa \Lambda - \Pi - \left( \frac{Q_c}{N_C} \right)^{1/3} K h (1 - \kappa^2) = 0,
\]

(2.4)

and

\[
\left[ \frac{3}{4} + \frac{3(1 - \kappa^2)}{8k^2} \right] \Lambda + \frac{3(1 - \kappa^2)}{8k^3} \Pi - \hat{v}_a - \hat{v}_l = 0,
\]

(2.5)

where \( \Lambda = h/R_{00}, \) \( \Pi = H/R \) and \( \hat{v}_a \) are the slenderness, aspect ratio and the volume of the air between the meniscus and the plane \( z = h, \) and \( \hat{v}_l = v_l/(4\pi R^3_{00}/3). \)

The existence of a continuous and stable solution branch from the fully saturated state \((h = H)\) to the collapsed state \((h = 0)\) is a necessary condition for the elastocapillary system in figure 1 to collapse. Akbari et al. [34] demonstrated that this system loses stability during drying at volume turning points, \(^1\) provided the meniscus is pinned to the hole edge and \( \Pi \lesssim 2^{3/4}(1 - \kappa^2). \) However, assuming that menisci are pinned to the hole edge implies that the contact line is restrained by an external force. This assumption can be relaxed by permitting the contact line to move if it is energetically favourable to do so, which leads to the necessary condition \( \theta_d + \beta_0 > \theta_c \) for menisci to be stably pinned to the hole edge [35], provided the meniscus contact angles with the membrane and plate are the same. Here, \( \beta_0 \) measures the membrane slope at the hole edge (figure 1b), which is negligible because \( \tan \beta_0 \approx \tilde{w}'(\kappa) \sim \Pi \kappa \ll 1. \) This simplifies the foregoing stability criterion to

\[
\theta_d > \theta_c.
\]

(2.6)

Note that, unlike \( \theta_c, \) equilibrium imposes no restriction on \( \theta_d, \) so \( \theta_d \) is calculated as a part of the solution. Moreover, when the meniscus is a bridge, equilibrium solutions of the meniscus meridian curve can intersect the boundaries or be self-intersecting [36]. These menisci are clearly non-physical and must be excluded before examining the stability of equilibrium branches.

The foregoing stability criteria only ensure that the membrane can contact the plate, which corresponds to the ‘contact bound’ defined by Mastrangelo & Hsu [23]. However, these are not sufficient for predicting the dry-state conformation. Accounting for adhesion energy between the membrane and plate is also required to determine whether the membrane and plate remain attached when dry [24]. In the remainder of the paper, assuming that the adhesion energy is always strong enough to maintain the membrane and plate attached upon contact, the dry state is identified as collapsed if there exists a continuous solution branch without volume turning points and non-physical menisci from \( h = H \) to 0 along which equation (2.6) is satisfied.

We solve nonlinear systems of equations using Newton–Raphson-based predictor–corrector techniques. Solution branches are constructed using Keller’s arclength continuation method [37]. Branch continuation begins from a state that either has an analytical solution or can be readily constructed (i.e. cases with low sensitivity to initial guess and problem parameters) and terminates when the solution does not satisfy the geometrical constraints (e.g. negative \( \hat{v}_a \) or \( \Lambda \)).

### 3. Results and discussion

We determine the dry-state conformation in the parameter space \((\kappa, N_C, \theta_c)\) at fixed \( v \) and \( \Pi \). Drying from the fully saturated state, the elastocapillary system of figure 1 must undergo three deformation phases to collapse: (i) The initial phase where the meniscus is a bubble (simply connected), which can bridge the membrane and plate when the bubble apex reaches the plate, (ii) the transition phase where the bubble transforms into a bridge at fixed liquid volume \( v^*_l, \) and (iii) the final phase where the meniscus is a bridge (doubly connected) with the slenderness approaching zero as \( v_1 \) decreases further below \( v^*_l. \)

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\(^1\) Unless stated otherwise, volume turning point refers to a turning point in \( v_l. \)
Figure 2. Special states for simply connected menisci; (a) tangent bubble and (b) rigid membrane ($N_C \to \infty$). (Online version in colour.)

(a) Initial phase

Here, the meniscus is a truncated sphere because gravity is neglected. Accordingly, the scaled bubble volume and capillary pressure are

$$\hat{v}_a = \frac{2 + 3 \cos \hat{\theta}_d - \cos^3 \hat{\theta}_d}{4 \sin^3 \hat{\theta}_d}$$  \hspace{1cm} (3.1)

and

$$Q_c = -2 \sin \hat{\theta}_d.$$  

(3.2)

The system state at any $\hat{v}_l$ does not depend on $\theta_c$. Solving equations (2.4), (2.5), (3.1) and (3.2), the state is specified for a given $\hat{v}_l$, $\kappa$ and $N_C$, where the dependent variables are ($\Lambda$, $\theta_d$, $Q_c$, $\hat{v}_a$, $\hat{v}_l$). The tangent bubble and rigid membrane, shown in figure 2, are special states in this phase that are determined by $\Lambda^* = \cot(\theta_d^*/2)$ and $\Lambda^\lim = \cot(\theta_d^\lim/2) = \Pi/\kappa$, respectively. Here, the superscripts ‘$*$’ and ‘lim’ denote variables in the tangent-bubble and rigid-membrane states. The latter corresponds to $N_C \to \infty$, where the membrane is rigid and does not deform.

Note that the dependent variables in the tangent-bubble state are ($\Lambda^*$, $\theta_d^*$, $Q_c^*$, $\hat{v}_a^*$, $\hat{v}_l^*$), and the system state is specified by fixing $\kappa$ and $N_C$. Choosing $\theta_d^*$ as the solution norm, we first determine equilibria at the tangent-bubble state in the space ($\kappa$, $N_C$, $\theta_d^*$), as shown in figure 3. At fixed $\kappa$, there is a critical $\kappa$ below which the tangent-sphere state has three equilibrium solutions for a given $N_C$. Similarly, at fixed $N_C$, there is a critical $N_C$ above which the tangent-sphere state has three equilibrium solutions for a given $\kappa$. Here, the stability of the equilibrium branch from the fully-saturated to tangent-bubble state must be examined to determine which equilibrium solution is realized for given ($\kappa$, $N_C$). Hereafter, we refer to equilibrium branches parametrized with $v_l$ as ‘drying trajectories’, which can be conveniently represented by $x_l$, $Q_c$, $\theta_d$ and $\Lambda$. Note that

$$x_l = \frac{v_l}{(\pi R^2H)}$$  \hspace{1cm} (3.3)

is a volume fraction, measuring the liquid content relative to the fully saturated state.
Figure 4. (a) Drying trajectory with $\nu = 0.3$, $IT = 0.2$, $\kappa = 0.15$ and $N_C = 5000$. Circle indicates the tangent-bubble state at $x_l \simeq 0.7505$. (b) Conformations along the drying trajectory at $x_l = 0.95$, 0.9, 0.85, 0.8, 0.78, 0.76 and 0.7505 (downward). (Online version in colour.)

Figure 5. (a) Drying trajectory with $\nu = 0.3$, $IT = 0.2$, $\kappa = 0.01$ and $N_C = 8000$. Circles indicate multiple equilibria at the tangent-bubble state. (b) Conformations at the tangent-bubble state corresponding to $x_l \simeq 0.5140$, 0.5932 and 0.6410 (b, downward). (Online version in colour.)

Figure 4 illustrates a drying trajectory corresponding to $(\kappa, N_C)$ for which the tangent-bubble state has one equilibrium solution. The equilibrium branch has no turning point in $v_l$, so no stability exchange occurs from the fully saturated state to the tangent-bubble state. At the tangent-bubble state, $\theta_d^* \simeq 110^\circ$. Note that $\theta_d$ monotonically decreases with decreasing $v_l$. Therefore, when $\theta_c > 110^\circ$, the contact line is detached from the hole edge before the bubble can bridge the membrane and plate according to equation (2.6), implying that the dry-state conformation is open.

Figure 5 shows a drying trajectory where the tangent-bubble state has three equilibrium solutions. The equilibria at $B$ and $C$ are past the turning point in $v_l$ on an unstable branch [34], and, thus, cannot be practically realized. The equilibrium at $A$, where $\theta_d^* \simeq 121^\circ$, is on a stable branch; thus, drying from the fully saturated state, the tangent-bubble state at $A$ can be practically realized. Similarly to figure 4, when $\theta_c > 121^\circ$, the contact line is detached from the hole edge before the bubble can bridge the membrane and plate.

To determine the region in $(\kappa, N_C)$ in which the tangent-bubble state is reached before stability is lost at a volume turning point, we determine tangent-bubble states that coincide with volume
turning points (thin-dashed lines in figure 6). These states are determined by \((\partial \hat{v}_l / \partial \theta_d)_{\kappa,N_C} = 0\), where \((\Lambda^*, \theta_d^*, Q_{C}, \hat{v}_d^*, \hat{v}_l^*, N_C)\) are specified for a given \(\kappa\). Moreover, at fixed \(\kappa\), there is a critical \(N_C\), determined by \((\partial \hat{v}_l / \partial \theta_d)_{\kappa,N_C} = 0\) and \((\partial^2 \hat{v}_l / \partial \theta_d^2)_{\kappa,N_C} = 0\), at which the two volume turning points on drying trajectories coalesce into an inflection point; above this elastocapillary number, drying trajectories have no volume turning points and no stability exchange occurs along equilibrium branches. These criteria define another state (thick-dashed lines in figure 6) that is specified for a given \(\kappa\), similarly to the tangent-bubble state at volume turning point. The shaded region in figure 6a, constructed based on the foregoing states, furnishes a range of \((\kappa, N_C)\) for which the tangent-bubble state is reached before the volume turning point. However, one must also ensure that equation (2.6) is satisfied along the drying trajectory from the fully saturated to tangent-bubble state. The shaded regions in figure 6b–f are subregions of the shaded region in figure 6a in which the dihedral-angle and turning-point stability criteria are simultaneously satisfied, therefore, providing a range of \((\kappa, N_C)\) in which the tangent-bubble state can be realized.

The tangent-bubble state coincides with volume turning points only for \(\theta_d^* < \pi/2\). To prove this, we choose \(\theta_d\) as an independent variable, so that \(\hat{v}_l = \hat{v}_l(\kappa, N_C, \theta_d)\) and \(\Lambda = \Lambda(\kappa, N_C, \theta_d)\) for the equilibria from the fully saturated to tangent-bubble state. Differentiating equation (2.5) with respect to \(\theta_d\) furnishes

\[
\left(\frac{\partial \hat{v}_l}{\partial \theta_d}\right)_{\kappa,N_C} = \left[\frac{3}{4} + \frac{3(1 - \kappa^2)}{8\kappa^2}\right] \left(\frac{\partial \Lambda}{\partial \theta_d}\right)_{\kappa,N_C} - \frac{d \hat{v}_a}{d \theta_d}, \quad (3.4)
\]

leading to

\[
\left(\frac{\partial \hat{v}_a}{\partial \Lambda}\right)_{\kappa,N_C} = \left[\frac{3}{4} + \frac{3(1 - \kappa^2)}{8\kappa^2}\right] > 0, \quad (3.5)
\]
at volume turning points. Because \( \lambda \) decreases (increases) with increasing \( \hat{v}_l \) when the bubble is smaller (larger) than a hemisphere, it follows that
\[
\left( \frac{\partial \hat{v}_a}{\partial \lambda} \right)_c < 0 \quad \left( \frac{\partial \hat{v}_a}{\partial \lambda} \right)_c > 0
\]
when \( \theta_d > \pi/2 \) (\( \theta_d < \pi/2 \)). Noting that equation (3.5) holds at all volume turning points, specifically those coinciding with the tangent-bubble state, it follows that \( \theta^*_{d} < \pi/2 \) along the thin-dashed lines in figure 6.

(b) Transition phase

When the bubble reaches the plate in the tangent-bubble state, it transforms into a doubly connected meniscus by bridging the membrane and plate. The dynamics of such transformations are comparable to those of stability loss in liquid-bridge breakup [38,39], so the time scale of the BBT is very small compared with that of drying. Therefore, this transition is volume preserving (\( v_l = \text{const.} \)), and the state after the transition is determined by Akbari et al. [36]

\[
\sqrt{1 + a^2 + 2a \cos \tau_0} - |Q_c| = 0, \quad (3.6)
\]

\[
\int_{\tau_0}^{\tau_1} \frac{1 + a \cos t}{\sqrt{1 + a^2 + 2a \cos t}} \, dt + \Lambda Q_c = 0, \quad (3.7)
\]

\[
\text{sign}(Q_c) \frac{1 + a \cos \tau_0}{a \sin \tau_0} - \tan \theta_d = 0, \quad (3.8)
\]

\[
\text{sign}(Q_c) \frac{1 + a \cos \tau_1}{a \sin \tau_1} + \tan \theta_c = 0 \quad (3.9)
\]

and

\[
\int_{\tau_0}^{\tau_1} \sqrt{1 + a^2 + 2a \cos t}(1 + a \cos t) \, dt + \frac{4}{3} Q_c^3 \hat{v}_a = 0. \quad (3.10)
\]

Equation (2.5) with \( \hat{v}_l = \hat{v}^{**}_l(\kappa, N_C) \), and equation (2.4), furnishing the dependent variables \((Q^{**}_c, \tau^{**}_0, \tau^{**}_1, a^{**}, \theta^{**}d, A^{**}, \theta^{**}c)\). Here, the superscript ‘**’ denotes variables after the BBT, and the system state is specified for a given \( \kappa, N_C \) and \( \theta_c \). Note that \( \tau_0 \) and \( \tau_1 \) are, respectively, the mean-curvature-scaled arclengths at the hole edge and the bridge contact line with the plate, and \( a \) is the bridge shape parameter [36,40–42].

The catenoidal bridge and rigid membrane are special limiting states in this phase, as shown in figure 7. After the transition, the meniscus mean curvature is zero \( (Q_c \to 0) \), and the membrane is undeflected in the catenoid limit. The dependent variables in this state \((\theta^\text{cat}_d, \hat{v}^\text{cat}_a, N_c)\) are determined by

\[
\frac{1}{\sqrt{s_0^2 + 1}} \ln \left( \frac{\hat{s}_1 + \sqrt{s_1^2 + 1}}{\hat{s}_0 + \sqrt{s_0^2 + 1}} \right) - \frac{\Pi}{\kappa} = 0 \quad (3.11)
\]

and

\[
\hat{s}_1 \sqrt{s_1^2 + 1} - \hat{s}_0 \sqrt{s_0^2 + 1} + (\Pi/\kappa) \sqrt{s_0^2 + 1} \frac{8 \hat{v}_a}{(s_0^2 + 1)^{3/2}} = 0, (3.12)
\]
Figure 8. System state after the BBT with \( \nu = 0.3 \) and \( \Pi T = 0.2 \); the bridge after the transition is (a) a catenoid and (b) at the dihedral-angle stability limit \( (\theta_d^{**} = \theta_c) \) according to equation (2.6). Dashed line in the right figure indicates states where the bridge after the transition is a catenoid with \( \theta_c = 120^\circ \). Labels denote contact angle in degrees. (Online version in colour.)

and equation (2.5) with \( \hat{\theta} = \hat{\theta}^\ast (\kappa, N_C) \). Moreover, \( \hat{\theta}_0 = \cot \theta^\ast_d \), \( \hat{\theta}_1 = -\cot \theta_c \) \cite{43}, \( Q^\ast_d = 0 \) and \( \Lambda^\ast = \Pi/\kappa \) with the superscript ‘cat’ denoting variables in the catenoid limit. This state is specified by fixing \( \kappa \) and \( \theta_c \). Similarly to the initial phase, the rigid-membrane limit \( (N_C \to \infty) \) is determined by \( \Lambda^\ast \equiv \cot(\theta^\ast_d/2) = \Pi/\kappa \), where the membrane is not deformed by the capillary pressure.

Figure 8a shows the system state after the BBT at fixed \( \theta_c \) in the catenoid limit. For a given \( \theta_c \), the catenoidal-catenoid state separates \((\kappa, N_C)\) on the left for which \( Q_C < 0 \) from those on the right for which \( Q_C > 0 \) after the transition. This contrasts with simply connected menisci, where \( Q_C \) is always negative. The membrane is deflected downward (upward) when \( Q_C \) is negative (positive), so \((\kappa, N_C)\) on the right of the catenoid curves are desirable if collapse is to be prevented. Moreover, no equilibrium solution exists for the catenoid state in figure 8a when \( \theta_c \leq 30^\circ \). This is because at any given \( \theta_c \) there is a slenderness \( \Lambda_{\text{max}} \) above which no catenoid exists \cite{43}. Since \( \Lambda_{\text{max}} \approx 0.2 \) at \( \theta_c = 30^\circ \), and noting that \( \Lambda_{\text{max}} \) monotonically decreases with decreasing \( \theta_c \) \cite{43}, \( \kappa_{\text{min}} = \Pi/\Lambda_{\text{max}} \geq 1 \) at \( \Pi = 0.2 \) when \( \theta_c \leq 30^\circ \), which is not physically possible. Therefore, hydrophilically modifying the membrane and plate enlarges the undesirable region of \((\kappa, N_C)\), which can span the entire parameter space if \( \theta_c \) becomes smaller than a critical value corresponding to a given \( \Pi \).

Figure 8b shows the dihedral-angle stability limit at fixed \( \theta_c \) with \( \theta_d^{**} = \theta_c \). Here, \( N_C \) is a dependent variable, and the state is specified for a given \( \kappa \) and \( \theta_c \). For a fixed \( \theta_c \), there is a \( \kappa \) at which \( N_C \to \infty \) at the stability limit, corresponding to the rigid-membrane limit (figure 7b). When \( \theta_c < 90^\circ \), \((\kappa, N_C)\) in the complement of the region \( acba \) correspond to states in which the bridge after the transition satisfies equation (2.6). However, this behaviour changes when \( \theta_c > 90^\circ \). For example, at \( \theta_c = 120^\circ \), the curve corresponding to the dihedral-angle stability limit intersects the catenoid curve for the same contact angle at \( E \). Here, \((\kappa, N_C)\) in the region \( dEfd \) correspond to states in which the bridge after the transition satisfies equation (2.6) and \( Q_C < 0 \). Interesting to note is the opposite behaviour of the dihedral-angle stability region for hydrophobic and hydrophilic surfaces. For hydrophobic surfaces, the stability region spans an area of the parameter space in which \( \kappa \) and \( N_C \) are large. As will be discussed in the next section, when the meniscus is a bridge, the elastocapillary system tends to lose stability at a volume turning point before collapse at large \( N_C \). By contrast, for hydrophilic surfaces, decreasing \( \theta_c \) enlarges the stability region, which spans a wide range of large and small \( \kappa \) and \( N_C \). Therefore, hydrophilically modifying the membrane and plate broadens the range of \( \kappa \) and \( N_C \) in which the dihedral-angle and volume-turning-point stability criteria are simultaneously satisfied during drying, thereby favouring collapsed conformations.
**Final phase**

Similarly to the initial phase, drying trajectories in this phase, where the meniscus is a bridge, describe how the system evolves with decreasing $v_i$. One must determine whether stability is lost from the state following the foregoing BBT to the collapsed state along drying trajectories. Thus, equilibrium branches, parametrized with $v_i$, must be constructed to identify volume turning points on drying trajectories. This is accomplished by solving equations (2.4), (2.5) and (3.6)–(3.10), providing the dependent variables $(Q_c, r_0, r_1, a, \theta_d, A, \dot{\theta}_a)$ for a given $v_i, \kappa$, $N_C$ and $\theta_c$. We first consider the collapsed state, which is identified by $A = 0$, furnishing

$$Q_c^\text{col} = -\frac{\Pi N_C}{[K_{it}(1 - \kappa^2)]^3},$$

$$\cos \tau_0^\text{col} = -\frac{1 + \tan \theta_c \sqrt{(a^\text{col})^2(1 + \tan^2 \theta_c) - 1}}{1 + \tan^2 \theta_c},$$

and

$$\dot{v}_1^\text{col} = \frac{3(1 - \kappa^2)\Pi}{8\kappa^3} \quad \text{and} \quad \chi_1^\text{col} = \frac{1}{2}(1 - \kappa^2),$$

where $a^\text{col}$ is a solution of

$$1 + (a^\text{col})^2 + \frac{-2 + 2 \tan \theta_c \sqrt{(a^\text{col})^2(1 + \tan^2 \theta_c) - 1}}{1 + \tan^2 \theta_c} = -(Q_c^\text{col})^2 = 0.$$  

(3.16)

Note that, in the limit $A \to 0$, the meniscus is vanishingly small and the meniscus meridian curve is linear. Consequently, $r_1^\text{col} = r_0^\text{col}$, $\theta_d^\text{col} = \pi - \theta_c$ and $\dot{v}_1^\text{col} = 0$, where the superscript ‘col’ denotes variables in the collapsed state. Furthermore, $\theta_d^\text{col} \leq \theta_c$ when $\theta_c \geq \pi/2$, implying that, according to equation (2.6), the collapsed state is always unstable for hydrophobic surfaces. Therefore, in the remainder of this section, we focus on hydrophilic surfaces with $\theta_c < \pi/2$.

At a fixed $\kappa$, there is a critical $N_C$ below which equation (2.6) is satisfied along the entire drying trajectory, from the state after the BBT to the collapsed state. This critical $N_C$ corresponds to a state, which we refer to as the dihedral-angle turning point limit, where $\theta_d = \theta_c$ at a dihedral-angle turning point where the dependent parameters $(Q_c^\text{dtp}, r_0^\text{dtp}, r_1^\text{dtp}, a^\text{dtp}, \theta_d^\text{dtp}, \dot{\theta}_a, \dot{v}_1^\text{dtp}, N_C)$ are determined by $(\partial \theta_d/\partial v_i)_{\kappa,N_C,\theta_c} = 0$, equations (2.4), (2.5) and (3.6)–(3.10). Here, the superscript ‘dtp’ denotes variables in this state, which is specified for a given $\kappa$ and $\theta_c$. When the meniscus is a bridge, this state (thin-solid lines in figure 9) provides a boundary in the parameter space, separating $(\kappa, N_C)$ for which equation (2.6) is satisfied along the drying trajectory from those for which equation (2.6) is not satisfied on a segment of the drying trajectory. Overlying this stability region with those constructed in figures 6 and 8 provides a region in the parameter space in which the volume-turning-point and dihedral-angle stability criteria are satisfied along drying trajectories from the fully saturated to collapsed state, except the volume-turning-point stability criterion for doubly connected menisci. Moreover, one must ensure that all menisci are physically realizable along drying trajectories. The resulting region is, therefore, an upper bound on the region of collapsed conformations in the parameter space.

Figure 9 illustrates the foregoing region for $\theta_c = 30^\circ$ and $60^\circ$. The curve corresponding to the dihedral-angle turning point limit itself has a turning point in $\kappa$ at $\kappa \approx 0.0006$ and 0.0062 for $\theta_c = 30^\circ$ and $60^\circ$, respectively. When the meniscus is a bridge and below these values, equation (2.6) is satisfied along the entire drying trajectory for all $N_C$, and the dry-state conformation is determined only by the volume-turning-point stability criterion. This appears as a sudden jump in the boundary of the shaded areas in figure 9, from the dihedral-angle stability limit at the tangent-bubble state (dash-dotted line) to the dihedral-angle turning point limit (thin-solid line). Moreover, it is clear from figure 9 that, except for $\kappa \ll 1$, the stability region is not significantly influenced by the contact angle, and $N_C(v = 0.3, \Pi = 0.2) \approx 10^4$ furnishes an upper bound on the critical elastocapillary number, separating open and collapsed conformations the elastocapillary system of figure 1 assumes upon drying.
Figure 9. Phase diagrams providing upper bounds on the region of collapsed conformation in the parameter space from the locus of volume turning points coinciding with the tangent-bubble state (dotted), volume inflection points when the meniscus is a bubble (thick-solid), the dihedral-angle stability limit ($\theta_d = \theta_c$) according to equation (2.6) at the tangent-bubble state (dash-dotted), after the BBT (dashed) and at dihedral-angle turning points when the meniscus is a bridge (thin-solid) with $\nu = 0.3$, $\Pi = 0.2$ for (a) $\theta_c = 30^\circ$ and (b) $\theta_c = 60^\circ$. Shaded area indicates a region of the parameter space where the stability criteria are satisfied from the fully saturated to collapsed state, except the volume-turning-point stability criterion for doubly connected menisci. (Online version in colour.)

Figure 10. Drying trajectories in volume versus pressure diagrams for doubly connected menisci with $\nu = 0.3$, $\Pi = 0.2$, $\kappa = 0.1$ and $\theta_c = 60^\circ$. The former is suitable for examining the volume-turning-point stability criterion as stability can be related to the slope of drying trajectories [34], whereas the latter is useful for ascertaining the dihedral-angle stability criterion along the drying trajectories. (Online version in colour.)

We illustrate how to examine the volume-turning-point stability criterion from the state after the BBT to the collapsed state by an example. Drying trajectories are shown in volume versus pressure and dihedral angle versus slenderness diagrams in figures 10 and 11 when $\nu = 0.3$, $\Pi = 0.2$, $\kappa = 0.1$ and $\theta_c = 60^\circ$. The former is suitable for examining the volume-turning-point stability criterion as stability can be related to the slope of drying trajectories [34], whereas the latter is useful for ascertaining the dihedral-angle stability criterion along the drying trajectories.
When $\kappa = 0.1$, the dihedral-angle stability limit after the BBT has the solutions $\log N_C \simeq 3.7538$ and $6.0523$ (figure 8b), and the dihedral-angle turning point occurs at $\log N_C \simeq 3.4576$ (thin-solid line in figure 9b). As shown in figure 10, the state after the BBT and collapsed state lie on disconnected branches at $\log N_C = 3.75$ and 3.6, and the system loses stability at a volume turning point before collapse. Decreasing the elasto-capillary number to $\log N_C = 3.55$, the two disconnected segments merge, forming a continuous solution branch from the state after the BBT to the collapsed state. However, equation (2.6) is not satisfied on the middle part of the branch (figure 11c), and the contact line is detached from the hole edge before collapse. When the elasto-capillary number drops below $\log N_C \simeq 3.4576$, for example, at $\log N_C = 3$ (figures 10e and 11e), both stability criteria are satisfied simultaneously, and the system collapses upon drying. Therefore, in this example, the stability, and, thus, dry-state conformation of the system is determined by the dihedral-angle turning point limit.

Figures 10f and 11f show the drying trajectory when $\log N_C = 6.1$. Although the corresponding point in the parameter space lies in the dihedral-angle stability region after the BBT (figure 8b), the state after the BBT is unstable because the corresponding point lies on an unstable branch, according to lemmas 2 and 3 of Akbari et al. [34]. The dihedral-angle stability criterion is not satisfied along the entire branch, and the system does not evolve towards the collapsed state. In this case, stability is lost during the BBT.

We conclude this section by demonstrating how the elasto-capillary system of figure 1 evolves along drying trajectories that lead to open and collapsed conformations upon drying. Figure 12 illustrate the drying trajectory at $\kappa = 0.1$, $\log N_C = 3.6$ and $\theta_c = 60^\circ$. These parameters correspond to a point inside the shaded region in figure 6c and outside the shaded region in figure 9b. Starting from the fully saturated state at $A$ and removing water (decreasing $x_l$), the system follows the dashed line until the bubble reaches the plate at $B$, corresponding to the tangent-bubble state, where $\Lambda^* \simeq 0.8097$, $\theta_d^* \simeq 102^\circ$ and $x_l^* \simeq 0.6970$. As predicted in figure 6c, the tangent-bubble state is on a stable branch, between the volume turning point and fully saturated state, where equation (2.6) is satisfied. Then, the system undergoes a volume-preserving BBT ($x_l = \text{const.}$) to the state $C$ where $\Lambda^{**} \simeq 0.8165$ and $\theta_d^{**} \simeq 73^\circ$, which also satisfies equation (2.6) (figure 8b). According to lemmas 2 and 3 of Akbari et al. [34], this state is on a stable equilibrium.
Figure 12. System conformations along a drying trajectory when the meniscus is a bubble (dashed) and bridge (solid) with $\nu = 0.3$, $\Pi = 0.2$, $\kappa = 0.1$, $\theta_c = 60^\circ$ and $\log N_C = 3.6$. (Online version in colour.)

Figure 13. System conformations along a drying trajectory when the meniscus is a bubble (dashed) and bridge (solid) with $\nu = 0.3$, $\Pi = 0.2$, $\kappa = 0.1$, $\theta_c = 60^\circ$ and $\log N_C = 3$. (Online version in colour.)

branch (figure 10b). Decreasing $x_l$ below $x_l^*$, the system follows the solid line and reaches the dihedral-angle stability limit at $D$ where $\Lambda \simeq 0.7601$, $\theta_d = 60^\circ$ and $x_l \simeq 0.6825$. Beyond this point, the contact line does not remain pinned to the hole edge with decreasing $x_l$, as predicted in figure 9b. Note that the drying trajectory from $C$ to the collapsed state at $G$ is not continuous, and the system loses stability at the volume turning point $E$ before collapse, even if the contact line is maintained pinned to the hole edge by an external force. The dry-state conformation is consequently open.

Figure 13 shows the drying trajectory at $\kappa = 0.1$, $\log N_C = 3$ and $\theta_c = 60^\circ$. These parameters correspond to a point inside the shaded regions in figures 6c and 9b. Here, the system behaves similarly to the previous case (figure 12) until the point C. According to lemmas 2 and 3 of Akbari et al. [34], the corresponding state is stable because C lies on a stable equilibrium branch (figure 10e). However, the drying trajectory has a turning point in $\theta_d$ at $F$, and no turning point in $v_1$ from $C$ to the collapsed state at $G$. As predicted in figure 9b, $\theta_d > \theta_c$ along the entire trajectory from the fully saturated to collapsed state, and stability is not lost with respect to the dihedral-angle stability criterion. Moreover, the drying trajectory continuously connects $C$ to $G$, implying
Figure 14. Schematic representation of wood fibres, highlighting the characteristic length scales.

that both stability criteria are satisfied along the entire trajectory from A to G, and the dry-state conformation is collapsed. Note also that in figures 12 and 13, all menisci are physically realizable from the state after the BBT at C to the collapsed state at G.

4. Wood-fibre lumen collapse

Here, we apply the elastocapillary model to lumen collapse of wood fibres, providing a quantitative estimate of the critical $N_C$, determining the dry-state conformation. The model has the length scales $R_0$, $R$ and $H$, which we adopt to characterize the meniscus curvature, deformation length and maximum deflection. Wood-fibre geometry, mechanical properties and ultrastructure have been extensively studied for various species in the literature in the past few decades [2]. Here, we choose Norway spruce as an example with which to estimate the foregoing length scales (figure 14) and approximate the critical $N_C$.

Cristian Neagu et al. [44] reported the average lumen width $W = H + 2b \approx 20 \mu m$ for early and late wood, and Yan & Li [45] measured an elastic modulus $E \approx 1.4 \text{ GPa}$ and wall thickness $b \approx 1.4 \mu m$ for Kraft spruce. Sirviö & Kärenlampi [46] reported the normalized size of bordered pits $R_0/H \approx 0.075$ with an average longitudinal spacing $\delta \approx 10 \mu m$. Taking $R \approx \delta/2$, we find $\kappa \approx 0.22$, $\Pi \approx 3.4$ and $N_C \approx 30 500$. However, from figure 9, $N_C \approx 3700, 2300$ are the upper bounds on elastocapillary numbers for which the dry-state conformation is collapsed when $\Pi = 0.2$ and $\theta_c = 30^\circ, 60^\circ$. Noting that the critical $N_C$ decreases with increasing $\Pi$, it follows that the capillary pressure of menisci, having length scales of the same order as the pit-hole diameter and lumen width, is smaller by at least an order of magnitude to overcome the cell-wall stretching resistance and induce collapse.

5. Concluding remarks

We have developed an elastocapillary model of drying-induced collapse to predict the dry-state conformation from geometrical and mechanical properties. This model simulates drying-induced structural deformations arising in cavities with three characteristic length scales, providing a low-dimensional model to describe the collapse of lumen or cell-wall micropores at moisture contents above the FSP. The dry-state conformation was determined based on the dihedral-angle and volume-turning-point stability criteria. Fixing the cavity aspect ratio, an upper bound on the critical elastocapillary number, corresponding to a scaled hole radius and contact angle, was provided, above which the system does not collapse upon drying. The critical elastocapillary number has a weak dependence on the contact angle over a wide range of the scaled hole radius, indicating a limitation of surface hydrophobization for controlling the dry-state conformation. Accounting for the BBT on the system stability and dry-state conformation, this model is also relevant to structural failures and stiction in micro-electromechanical systems during wet etching.

Applying the model to lumen collapse based on the structural and mechanical properties of Norway spruce fibres revealed that the capillary pressure of menisci, spanning pit-hole openings and/or the lumen width, is not strong enough to overcome the cell-wall stretching resistance, so is unlikely to cause collapse. However, the capillary-induced collapse of cell-wall micropores is yet to be understood. Granted, this is more challenging because cell-wall microstructural properties, such as pore geometry and pore-size distribution have not been established in the literature. Nevertheless, the nonlinear trend of the overall volumetric shrinkage versus moisture content
observed above the FSP [13] as compared with the linear trend below the FSP can serve as a suggestive piece of evidence. This nonlinear trend signifies deformations in the collapse regime and may be attributed to the capillary-induced collapse of micropores in a transition phase of drying when the moisture content approaches the FSP from above. Understanding the transition between the collapse and shrinkage regimes during drying demands reproducible and reliable characterizations of cell-wall microstructural properties to provide an accurate prediction of the dry-state conformation.

We considered an idealized model of drying-induced collapse in wood fibres, capturing the basic elements of the underlying physics. The bending contribution to the total potential energy and membrane profile were neglected, limiting the accuracy for thick-walled and slender cavities. Accounting for the bending energy requires a more complicated test function for the membrane deflection in the bending- and stretching-dominated regimes. However, implementing complicated test functions is computationally expensive and algorithmically challenging. Moreover, our model does not account for geometrical complexities of cavities in wood fibres. Specifically, the lumen and cell-wall micropores have a closed structure, exhibiting a buckling resistance to compressive loads. However, the membrane in the present model can continuously deform from the fully saturated to collapsed state without buckling. The critical elastocapillary number is consequently smaller than predicted in this work. Nevertheless, our results furnish an upper bound on the critical elastocapillary number, and can reasonably estimate the range of parameters for which the dry-state conformation is open. More realistic geometries will surely improve quantitative predictions, but at higher computational cost. We hope that this study will motivate future investigations into the underlying mechanisms of wood–water interactions, to improve understanding of drying deformation in the shrinkage and collapse regimes over the full moisture-content range.

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Appendix A

The in-plane and out-of-plane equations of equilibrium for the membrane in the elastocapillary model of figure 1 are [34]

\[ r_p N_{rr} + N_{rr} - N_{tt} - P(r_p + u)w' = 0 \]  \hspace{1cm} (A 1)

and

\[ (N_{rr}w'r_p)' + P(r_p + u)(1 + u') = 0 \]  \hspace{1cm} (A 2)

with boundary conditions

\[ N_{rr} = 0 \hspace{0.5cm} \text{at} \hspace{0.5cm} r_p = R_0 \]  \hspace{1cm} (A 3)

and

\[ u = 0, \hspace{0.5cm} w = 0 \hspace{0.5cm} \text{at} \hspace{0.5cm} r_p = R, \]  \hspace{1cm} (A 4)

where

\[ N_{tt} = C \left( \ddot{u}' + \frac{\ddot{w}^2}{2} + \frac{\nu}{r_p} \dddot{u} \right) \]  \hspace{1cm} (A 5)
and
\[ N_{tt} = C \left( v\ddot{u}' + \frac{v\ddot{w}'^2}{2} + \frac{\dddot{u}}{r_p} \right). \] (A 6)

As we focus on cases where \( \Pi \ll 1 \), we have \( w', u \ll 1 \), and the capillary pressure can be regarded as acting in the \( z \)-direction, furnishing
\[ r_pN_{rr}' + N_{rr} - N_{tt} = 0 \] (A 7)
and
\[ (N_{rr}w'r_p)' + Pr_p = 0. \] (A 8)

The above equations are nonlinear and do not have tractable closed-form solutions. Note that the nonlinearity can be removed if \( N_{rr} \) is known in equation (A 8). Thus, as a first approximation, \( w \) is estimated from the out-of-plane equilibrium by substituting the radial distribution of the axial force from a simpler problem: undeflected circular plates under boundary tension, such that the boundary conditions equations (A 3) and (A 4) are satisfied. The radial and tangential components of the axial force in this problem are [47]
\[ N_{rr} = \frac{A}{r_p} + B \] (A 9)
and
\[ N_{tt} = -\frac{A}{r_p} + B \] (A 10)
with \( A = -TR^2R_{00}'/(R^2 - R_{00}^2) \) and \( B = TR^2/(R^2 - R_{00}^2) \), where \( T \) is the radial tension at \( r_p = R \).

Solving equation (A 8) using the radial component of the axial force in equation (A 9) furnishes
\[ \ddot{w}(\ddot{r}_p) = \ddot{w}_0(1 - \dddot{r}_p^2). \] (A 11)

The radial distribution of the in-plane displacement \( \dddot{u}(\dddot{r}_p) \), given by equation (2.2), is obtained by solving equation (A 7) using equation (A 11).

We use equation (A 11) as a test function to derive a variational approximation for the membrane equilibrium. The total potential energy \( \Omega_T \) comprises the membrane stretching energy
\[ \Omega_S = \frac{2\pi C}{2} \int_{R_{00}}^R \left( u'^2 + u'w'^2 + \frac{2uum'}{r_p} + \frac{nuw'^2}{r_p} + \frac{u^2}{r_p} + \frac{w'^4}{4} \right) r_p \, dr_p \] (A 12)
and the work of the capillary pressure
\[ \Omega_P = -2\pi \int_{R_{00}}^R Pwr_p \, dr_p. \] (A 13)

Introducing the dimensionless energies \( \Omega^*_i = \Omega_i/(2\pi CR^2) \) and the scaling forms
\[ \ddot{w}(\ddot{r}_p) = \ddot{w}_0\ddot{w}(\ddot{r}_p), \] (A 14)
\[ \ddot{u}(\ddot{r}_p) = \ddot{w}_0^2\ddot{u}(\ddot{r}_p), \] (A 15)
\[ \Omega_S^* = \ddot{w}_0^2\ddot{\Omega}_S \] (A 16)
and
\[ \Omega_P^* = \ddot{w}_0^2\ddot{\Omega}_P \] (A 17)
the total potential energy can be written
\[ \Omega_T^* = \ddot{w}_0^2\ddot{\Omega}_S + \ddot{w}_0^2\ddot{\Omega}_P, \] (A 18)
where

\[
\tilde{\Omega}_S = \frac{1}{2} \int_{\bar{r}_p}^{1} \left( \bar{u}'^2 + \bar{w}' \right) + \frac{1}{2} \tilde{\bar{u}} \tilde{\bar{w}}' + \frac{\bar{u}'^2}{\bar{r}_p} + \frac{\bar{w}'^4}{4} \right) \bar{r}_p \, d\bar{r}_p
\]  

(A 19)

and

\[
\tilde{\Omega}_P = - \frac{Q_c}{\kappa N_C} \int_{\bar{r}_p}^{1} \tilde{\bar{w}} \bar{r}_p \, d\bar{r}_p.
\]  

(A 20)

Substituting \(\bar{u}\) and \(\bar{w}\) from equations (2.1) and (2.2) in equations (A 19) and (A 20) leads to

\[
\tilde{\Omega}_S = \frac{(1 - \kappa^2)^3 (1 - \nu + \kappa^2 (1 + \nu))}{48[1 - \nu + \kappa^2 (1 + \nu)]}
\]  

(A 21)

and

\[
\tilde{\Omega}_P = - \frac{Q_c (1 - \kappa^2)^2}{4N_C \kappa}.
\]  

(A 22)

The membrane equilibrium is identified by \(\delta \Omega_T^\dagger = (d\Omega_T^\dagger / d\bar{w}_0) \delta \bar{w}_0 = 0\), furnishing

\[
\bar{w}_0 = \left( -\frac{\tilde{\Omega}_P}{4\tilde{\Omega}_S} \right)^{1/3} = \left( \frac{Q_c}{N_C} \right)^{1/3} K_w.
\]  

(A 23)

Since the test function in equation (A 11) is derived based on the axial-force distribution of undeflected circular plates, the variational approximation must approach the exact solution at vanishingly small deflections. For the same reason, this approximation is expected to be less accurate at large deflections.

References


