Moving breathers and breather-to-soliton conversions for the Hirota equation

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We find that the Hirota equation admits breather-to-soliton conversion at special values of the solution eigenvalues. This occurs for the first-order, as well as higher orders, of breather solutions. An analytic expression for the condition of the transformation is given and several examples of transformations are presented. The values of these special eigenvalues depend on two free parameters that are present in the Hirota equation. We also find that higher order breathers generally have complicated quasi-periodic oscillations along the direction of propagation. Various breather solutions are considered, including the particular case of second-order breathers of the nonlinear Schrödinger equation.

1. Introduction

Recent publications have revealed the intricate relation between the soliton and breather solutions of a certain class of evolution equations [1–3]. Breathers can be converted into solitons with special perturbations [1]. Solitons themselves can evolve into the form of breathers [2]. Even the Peregrine solution can be converted into an infinitely elongated structure at certain conditions [3]. In this paper, we show that there are special cases where breather oscillations can be suppressed, and, in those cases, breathers may be transformed into solitons. Mutual conversions between the two become possible when the evolution equation contains a sufficient number of free parameters which can be used to control their solutions. One of the simplest equations of this type is the so-called Hirota equation (HE) [4,5].
The HE can be written in the following operator form with two free parameters:

\[ i \psi_x + \eta S[\psi(x, t)] - i\alpha H[\psi(x, t)] = 0, \tag{1.1} \]

where \( S \) is the nonlinear Schrödinger operator \( S[\psi(x, t)] = \psi_{tt} + 2|\psi|^2 \), while \( H \) is the Hirota operator, featuring third-order linear dispersion \((\psi_{ttt})\) and nonlinear dispersion \((6|\psi|^2\psi_t)\) terms, \( H[\psi(x, t)] = \psi_{ttt} + 6|\psi|^2\psi_t \).

In equation (1.1), \( x \) is the propagation variable and \( t \) is the transverse variable (time in a moving frame), with the complex function \( \psi(x, t) \) describing the envelope of the waves. Coefficients \( \alpha \) and \( \eta \) are arbitrary real numbers. When both coefficients are non-zero, we have the ‘plain’ integrable HE. If \( \psi(x, t) \) is a real function and \( \eta = 0 \), equation (1.1) is a modified Korteweg–de Vries equation. Below, we assume that the function \( \psi(x, t) \) is always complex. The limit of \( \alpha = 0 \) leaves us with the classic nonlinear Schrödinger equation (NLSE). Conveniently, we can take \( \eta = \frac{1}{2} \) [6], but this is not essential. The choice of \( \eta = \frac{1}{2} \) only becomes important when we want to keep circular symmetry of rogue wave clusters in the \((x, t)\)-plane [7,8]. Remarkably, equation (1.1) remains integrable for any values of the two parameters. Thus, every solution of equation (1.1) will contain free parameters that can be used to control certain features of the solutions. One of these features involves pulsations of the breather solutions. If such pulsations of a localized solution disappear, we can talk about conversion of a breather into a soliton. As we usually fix one of the parameters, namely \( \eta \), we can study such transformations in the simplest situation by changing just one parameter, \( \alpha \). However, keeping both coefficients variable allows us to consider more general cases.

2. Lax pairs

The inverse scattering technique for the NLSE was developed in 1972 by Zakharov & Shabat [9]. This technique is based on the fact that the NLSE can be written in terms of two matrix operators which are known as Lax operators [10]. The latter can be found not only for the NLSE but also for many other integrable equations [11]. The Lax pair for the HE (1.1) has been given in [12–14] with the \( \eta = \frac{1}{2} \) normalization. Below, we provide a more general formulation for the Lax pair with arbitrary \( \eta \) and \( \alpha \). Following [14], we can write the Lax equations in the form

\[ \frac{\partial R}{\partial t} = U \cdot R \quad \text{and} \quad \frac{\partial R}{\partial x} = V \cdot R, \tag{2.1} \]

such that the ‘zero-curvature’ condition \( U_x - V_t + [U, V] = 0 \) will reproduce equation (1.1). Here, \( U \) and \( V \) are \( 2 \times 2 \) matrices with \( U \) given by

\[ U = i \begin{bmatrix} \lambda & \psi(x, t)^* \\ \psi(x, t) & -\lambda \end{bmatrix}, \tag{2.2} \]

while \( V \) is a matrix polynomial in eigenvalue \( \lambda \). For the NLSE, it is a simple quadratic polynomial. The order of the polynomial increases when there is an additional higher order operator. With the Hirota operator, it becomes a cubic polynomial which can be written in general form \( V = \sum_{j=0}^{3} \lambda^j V_j \), where submatrices \( V_j \) are

\[ V_j = i \begin{bmatrix} A_j & B_j^* \\ B_j & -A_j \end{bmatrix}, \tag{2.3} \]

with

\[
\begin{align*}
A_0 &= -\eta |\psi|^2 - i\alpha (\psi_t^* \psi - \psi \psi_t^*), & B_0 &= 2\alpha |\psi|^2 \psi + i\eta \psi_t + \alpha \psi_{ttt}, \\
A_1 &= 2\alpha |\psi|^2, & B_1 &= 2\eta \psi - 2i\alpha \psi_t, & A_2 &= 2\eta, & B_2 &= -4\alpha \psi, \\
A_3 &= -4\alpha & B_3 &= 0.
\end{align*}
\]

It is easy to check that substitution of the matrices \( U \) and \( V \) into the ‘zero-curvature’ condition leads directly to equation (1.1).
3. Breather-to-soliton conversions of the Hirota equation

Taking purely imaginary eigenvalues allows us to derive the first-order Akhmediev breather (AB) solution of the HE. It was first presented in [15]. In [16], more general solutions of the HE were presented. In this work, we generalize the procedure for arbitrary coefficients $\alpha$ and $\eta$ and we use complex eigenvalues, $\lambda = a + i b$. The real part of $\lambda$ leads to oblique propagation of the breather.

The first-order breather solution of the HE with non-zero $\eta$ and $\alpha$ and with complex eigenvalue is given by

$$\psi_1 = \left[ 1 + 2b \frac{G_1 + iH_1}{D_1} \right] e^{i2\eta x}. \quad (3.1)$$

Here, the functions $G_1$, $H_1$ and $D_1$ are combinations of trigonometric and hyperbolic functions,

$$G_1 = \cos(xV_T + t\kappa_r) \cosh(2\chi_i) - \cosh(xV_H + t\kappa_i) \sin(2\chi_T),$$
$$H_1 = \cos(2\chi_r) \sinh(xV_H + t\kappa_i) + \sin(xV_T + t\kappa_r) \sinh(2\chi_I)$$
and
$$D_1 = \cosh(xV_H + t\kappa_i) \cosh(2\chi_I) - \cos(xV_T + t\kappa_r) \sin(2\chi_T),$$

with $\kappa = 2\sqrt{1 + \lambda^2}$ and $\chi = \frac{1}{2} \arccos(\kappa/2)$, while

$$V_T = 2\eta(-b\kappa_i + a\kappa_r) + \alpha(\Omega\kappa_i + \overline{\Omega}\kappa_r) \quad \text{and} \quad V_H = 2\eta(a\kappa_i + b\kappa_r) + \alpha(\overline{\Omega}\kappa_i - \Omega\kappa_r),$$

where $\Omega = 8ab$ and $\overline{\Omega} = 2(1 - 2a^2 + 2b^2)$, with $a$ and $b$ being arbitrary real numbers. An example of this solution is presented in figure 1a. We can clearly see the non-zero angle of propagation of the breather on the $(x, t)$-plane, its periodicity along the propagation direction and also the oblique orientation of each peak on the plane.

One of the new features of this breather is that it can be converted into a non-periodic solution. This occurs when the lines of extrema of the hyperbolic and the trigonometric functions in (3.1) on the $(x, t)$-plane coincide. For this to happen, the following condition:

$$\frac{V_H}{\kappa_i} = \frac{V_T}{\kappa_r} \quad (3.2)$$

must be satisfied. Solving equation (3.2) leads to a simple expression for the real part of the eigenvalue

$$a = \frac{\eta}{4\alpha}. \quad (3.3)$$
Figure 2. Collision of two breathers of the HE with $\eta = \frac{1}{4}$, $\alpha = \frac{1}{16}$ and eigenvalues $\lambda_1 = -0.08 + 0.9i$, $\lambda_2 = 0.08 + 0.8i$. This is an in-phase superposition with maximum amplitude at the origin, $|\psi_2(0,0)| = 4.4$. (Online version in colour.)

This condition, in effect, enables conversion of the breather into a soliton. The slope, i.e. the tangent of the propagation angle (relative to the $x$-axis) of the soliton on the $(x,t)$-plane, is given by

$$\frac{V_T}{\kappa_T} = \frac{V_H}{\kappa_H} = \frac{\eta^2}{4\alpha} + 2\alpha(1 + 2b^2).$$

The imaginary part of the eigenvalue, $b$, is arbitrary while the coefficient $\alpha$ cannot be zero. This means that the conversion can happen only for the HE. NLSE breathers cannot be converted into solitons. An example of a converted solution is shown in figure 1b. The solution is indeed non-periodic, like an ordinary soliton. However, it is located on a non-zero background and has oscillating tails. The latter appear due to the trigonometric functions involved in the solution (3.1). Oscillations are minimal when $a \to 0$, while for large $a$ they increase in amplitude. The directions of the oscillations and their decay are orthogonal to the line of the soliton maximum. The expression for the second-order breather solution is given in appendix A. One general example calculated using equation (A 4) is shown in figure 2. The solution is a nonlinear superposition of two first-order breathers. Their parameters are given by the two eigenvalues $\lambda_1 = a_1 + ib_1$ and $\lambda_2 = a_2 + ib_2$. Each of the breathers involved in this solution can individually be converted into a soliton. The condition of conversion is the same as before, i.e.

$$\frac{V_{Hj}}{\kappa_{Hj}} - \frac{V_{Tj}}{\kappa_{Tj}} = 0, \quad (3.4)$$

where $j = 1, 2$ is an integer labelling the breathers. From equation (3.4), we have the same expression as before for each real part,

$$a_{1,2} = \frac{\eta}{4\alpha}.$$ 

This expression is valid for every breather in a multi-component solution for an arbitrary number of breathers. The angular orientations of individual breathers in the superposition depend on the real parts of the eigenvalues, $a_j$.

Figure 3a,b shows two- and three-component breather solutions with each converted to a soliton. For the eigenvalues, we use the same value for the real parts, $a = a_1 = a_2 = a_3 = 1/8\alpha$, but have different $b_j$. In these examples, we have chosen the relatively high value $\alpha = \frac{1}{6}$. This reduces the values of $a_j$, resulting in fewer tail oscillations. The converted solitons have maximum amplitudes in the middle and quickly decaying oscillations on each tail.
Figure 3. (a) A second-order breather converted to a two-soliton solution with the eigenvalues $\lambda_1 = 0.75 + 0.9i$ and $\lambda_2 = 0.75 + 0.85i$. (b) A third-order breather converted to a three-soliton solution with the eigenvalues $\lambda_1 = 0.75 + 0.9i$, $\lambda_2 = 0.75 + 0.95i$ and $\lambda_3 = 0.75 + 0.8i$. Equation parameters are $\eta = \frac{1}{2}$ and $\alpha = \frac{1}{6}$. (Online version in colour.)

4. Higher order breather superpositions

Another interesting possibility when dealing with multi-component solutions is aligning the directions of propagation for different breathers. The condition of alignment is

$$\frac{V_{H1}}{\kappa_1} = \frac{V_{H2}}{\kappa_2}. \quad (4.1)$$

Then the lines of maxima for the two breathers on the $(x,t)$-plane become parallel. In this case, each breather remains periodic. We do not introduce spatial shifts for the breathers, and let them overlap. The solution of equation (4.1) is complicated and has to be found numerically. We can fix one of the eigenvalues, say $\lambda_2$, and find the other one, $\lambda_1$, using equation (4.1). For each $\lambda_2$, we obtain a curve on the complex plane for $\lambda_1$. The results are presented in figure 4.

Here, the solid curve is the locus of points $\lambda_1$ found for fixed $\lambda_2 = -0.08 + 0.9i$. Moving along the curve allows us to change the common angle of propagation of the two superposed breathers. Another similar curve is calculated for $\lambda_2 = 0.08 + 1.5i$. It is shown by the dashed curve. Having the freedom of moving along the curves offers infinite possibilities for generating various breather structures. Here, we give only a few examples indicated by the asterisks, triangle and squares in figure 4. Solving equation (4.1) for fixed $\lambda_2 = -0.08 + 0.9i$ and fixed imaginary part of $\lambda_1 = a_1 + 0.8i$ provides us with three solutions for the real part of $\lambda_1$, i.e. $a_1 = 1.69952$, $-0.36428$ and $-0.228223$. Two of them are shown in figure 4 by the solid squares. These are $a_1 = 1.69952$ and $a_1 = -0.36428$. We omit the illustration for the case $a_1 = -0.228223$, as it is similar to the other cases. The first two breather profiles are presented in figure 5a,b. There is a beating pattern on top of regular periodicity. This is related to the presence of two frequencies of modulation of the two breathers. The beating periods are generally incommensurate. This is to be expected, as we did not match the basic periods. As the phases of the two breathers have not been altered, the two breathers have constructive interference at the origin. Thus, the absolute maximum of the whole pattern in each case is reached at the centre, i.e. $|\psi(0,0)|$ is the maximum amplitude. This can be seen by comparing its brightness with one of the other peaks.

Two other examples correspond to the two asterisks in figure 4 with $b_1 = 1.5$. The real parts of the first eigenvalue in this case are $a_1 = -0.874443$ and $1.92315$, while $\lambda_2 = -0.08 + 0.9i$. These asterisks are located in the upper parts of the solid curves in figure 4. The corresponding wave profiles are shown in figure 5c,d. These are similar to the previous two cases, apart from the
Figure 4. Locus of the eigenvalues $\lambda_1 = a_1 + ib_1$ found as solutions of equation (4.1) for three fixed eigenvalues, $\lambda_2$. Solid curves correspond to $\lambda_2 = -0.08 + 0.9i$, while the dashed curves correspond to $\lambda_2 = 0.08 + 1.5i$. Equation parameters are $\alpha = \frac{1}{6}$ and $\eta = \frac{1}{2}$. The asterisks, triangle and squares on the curves are chosen for the illustrations below. (Online version in colour.)

Figure 5. Breather superpositions for (a) $\lambda_1 = 1.69952 + 0.8i$ and $\lambda_2 = -0.08 + 0.9i$, and (b) $\lambda_1 = -0.36428 + 0.8i$ and $\lambda_2 = -0.08 + 0.9i$. The points corresponding to these cases are indicated by the two squares on the solid curves in figure 4. Breather superpositions for (c) $\lambda_1 = -0.874443 + 1.5i$ and $\lambda_2 = -0.08 + 0.9i$, and (d) $\lambda_1 = 1.92315 + 1.5i$ and $\lambda_2 = -0.08 + 0.9i$. These choices correspond to the stars on the solid curves in figure 4. (Online version in colour.)
longer beating periods in comparison with the basic periods of individual breathers. One more example corresponds to the eigenvalues $\lambda_1 = a_1 + 1.5i$ and $\lambda_2 = 0.08 + 1.5i$. Of the two solutions for $a_1$ ($a_1 = 0.08$ and $a_1 = 0.850848$), we choose only one, namely $a_1 = 0.850848$. This choice corresponds to the triangle in figure 4. The breather profile for this case is shown in figure 6. The breather has a slope that is opposite to the previous cases. In addition, it has a ‘chain-shaped’ structure.

The choice $a_1 = 0.08$ represents a degenerate case, as $\lambda_1 = \lambda_2$. It needs a special consideration [17] which will be handled elsewhere.

5. Moving breathers of the nonlinear Schrödinger equation

The real part of the eigenvalue that follows from conditions (3.2) and (3.3) does not exist when $\alpha = 0$. This shows that the type of the breather-to-soliton conversion discussed above cannot happen for solutions of the NLSE. However, there is a transition from breather to soliton when the imaginary part of the eigenvalue changes from values below 1 to values above 1 when the real part of the eigenvalue is zero [18]. The non-zero real part of the eigenvalue, $a_1$, introduces velocity both to the breather and to the soliton. The mathematical analysis of moving breathers has been presented earlier in [19]. Here, we provide a few more clarifications.

A standard AB of the NLSE with $\lambda = 0.6i$ is shown in figure 7a. It starts with the plane wave solution at $x = -\infty$. The solution reaches its highest modulation depth at $x = 0$. This stage of evolution is shown in figure 7a. This periodic structure exists over an infinite range, $-\infty < t < \infty$. Adding a small real part to the eigenvalue, $a = 0.08$, transforms the infinite periodic structure into a localized one which is still located on a constant background. It is shown in figure 7b. This solution was called a ‘quasi-AB’ in [20]. The presence of the real part of the eigenvalue also leads to a movement of the modulated part of the solution with constant velocity. A further increase of the real part of the eigenvalue increases the velocity of this motion, leading to stronger localization and a smaller oscillation period. This can be seen in figure 7c,d. A two-dimensional plot of the breather evolution is shown in figure 8a. This plot shows that the periodic structure within the breather moves with a velocity that is different from the velocity of the breather itself. A collision of two such breathers is shown in figure 8b. As was shown in [20], this solution may be highly interesting from a practical point of view. In particular, it describes modulation instability on a variable background.
Figure 7. (a) AB of the NLSE at maximum amplitude stage $x = 0$ (solid curve). Equation parameters are: $\alpha = 0$, $\eta = \frac{1}{2}$. The eigenvalue $\lambda = 0.6i$. The envelope (dashed line) extends to infinity. (b) When a small real part is added to the eigenvalue, giving $\lambda = 0.08 + 0.6i$, the breather becomes localized. (c) Increasing the real part of the eigenvalue, giving $\lambda = 10 + 0.6i$, leads to a higher degree of localization. (d) Same as in (c) but $x = 0.31$. The breather acquires velocity and moves to the left. (Online version in colour.)

6. Superposition of moving breathers of the nonlinear Schrödinger equation

In this section, we consider an interaction of two moving breathers that is more specific than the straight collision shown in figure 8a. Namely, we study the superposition of two breathers propagating at the same angle on the $(x,t)$-plane. The condition for such a superposition is the following:

$$\frac{V_{H1}}{\kappa_1} = \frac{V_{H2}}{\kappa_2}. \quad (6.1)$$

Here, we assume that $\eta = \frac{1}{2}$ and $\alpha = 0$. Acting the same way as before, we fix the eigenvalue $\lambda_2$ and find the eigenvalues $\lambda_1$ that satisfy the condition of equation (6.1). Now, equation (6.1) can be solved analytically for $b_1$. We find that $b_1 = \pm \sqrt{p_1/p_2}$, where

$$p_1 = 4a_1^2a_2^2 + a_2^2(1 + 2a_2^2 - 2b_2^2) - 2a_1^2a_2(2 + 6a_2^2 - 2b_2^2 + m_1m_2)$$
$$- a_1a_2[3a_1^2 + (-1 + b_2^2)(-2 + b_2^2 - m_1m_2) + a_2^2(7 - 4b_2^2 + m_1m_2)]$$
$$+ a_1^2[11a_1^2(-1 + b_2^2)(-1 + b_2^2 - m_1m_2) + a_2^2(9 - 8b_2^2 + 3m_1m_2)]$$

and

$$p_2 = a_2[a_2 + 4a_1^2a_2 + 2a_2^3 - 2a_2b_2^2 + 2a_1(-1 - 3a_2^2 + b_2^2)],$$

with $m_1^2 = a_2^2 + (1 - b_2)^2$ and $m_2^2 = a_2^2 + (1 + b_2)^2$. The solutions for three fixed values of $\lambda_2$ are shown on the complex plane of $\lambda_1$ presented in figure 9a. These solutions appear as complex conjugate pairs. The sign reversal also produces a solution. In order to obtain co-propagating superposition, we must choose the solution in such a way that the signs of real parts, $a_1$ and $a_2$, of the two eigenvalues coincide. For opposite signs of $a_1$ and $a_2$, the moving breathers will cross each other and form a breather collision rather than breather superposition.
Figure 8. (a) Moving breather of the NLSE described by equation (3.1) with \( \lambda = 10 + 0.6i \). (b) Collision of two moving breathers of the NLSE with eigenvalues \( \lambda_1 = -10 + 0.6i \) and \( \lambda_2 = 10 + 0.6i \). Equation parameters \( \eta = \frac{1}{2} \) and \( \alpha = 0 \). Black areas correspond to the amplitude level \( |\psi| = 1 \). (Online version in colour.)

Figure 9. (a) Locus of points on the complex plane, \( \lambda_1 = a_1 + ib_1 \), that lead to co-propagating moving breathers for fixed \( \lambda_2 = 0.08 + 0.9i \) (solid curves), \( \lambda_2 = -0.08 + 1.5i \) (dotted curves) and \( \lambda_2 = -0.08 + 0.9i \) (dashed curves). The solid triangle and square are chosen to illustrate the moving breather superpositions. (b,c) Two examples of moving breather superpositions of the NLSE. The eigenvalues are (b) \( \lambda_1 = 1.09458 + 0.9i \) and \( \lambda_2 = 0.08 + 0.9i \) (square in (a)) and (c) \( \lambda_1 = -0.0528537 + 1.2i \) and \( \lambda_2 = -0.08 + 1.5i \) (triangle in (a)). (Online version in colour.)

There is an unlimited number of combinations that allow us to construct moving breather superpositions. Here, we present only two examples that correspond to the triangle and square in figure 9a. Figure 9b corresponds to \( \lambda_1 = 1.09458 + 0.9i \) and \( \lambda_2 = 0.08 + 0.9i \) (solid square). Figure 9c corresponds to \( \lambda_1 = -0.0528537 + 1.2i \) and \( \lambda_2 = -0.08 + 1.5i \) (solid triangle).

A common feature of these two examples is the beating pattern of the superposition. As the periods of the two breathers are generally incommensurate, none of the patterns ever exactly repeats itself. One of the differences between the two cases is that the two superpositions are propagating in opposite directions, as they should because of the opposite signs of the real parts of the eigenvalues in these two examples. Another difference is the width of the superpositions. This is also caused by the differences in eigenvalues.

7. Conclusion

We have considered various cases of transformations of breather solutions into moving solitons and breathers. This relates to the HE with two arbitrary real parameters. The presence of free parameters allows us to treat the general case, as well as particular cases. The latter include the NLSE. We have shown that there is a specific relationship between the coefficients of the equation and the real part of the eigenvalue that causes transformations of breather solutions to
solitons. We have also found the equation which connects the eigenvalues of separate breathers that allows us to transform them into moving breather superpositions.

Data accessibility. All the mathematical results are in analytic form and are reproducible.

Authors’ contributions. A.C. derived all the solutions and generated all the figures. All the authors contributed equally to the preparation of the manuscript.

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Appendix A. Explicit expressions for the second-order breather solution with complex eigenvalues

The Darboux transformation technique was first applied to breather solutions in [21]. Here, we will follow our previous calculations, given explicitly in appendices A and B of [19], except that we keep arbitrary coefficients of the HE in the results. Also, some of the notations will be changed in the present calculation. We separate the real and imaginary parts of variables and the function arguments where necessary. For example, \( \kappa_j = 2 \sqrt{1 + \lambda_j^2} = \kappa_{jr} + i \kappa_{ji} \) with \( \lambda_j = a_j + ib_j \) and \( \chi_j = \arccos(\kappa_j/2) = \chi_{jr} + i \chi_{ji} \). We can write the \( r \) and \( s \) functions in terms of trigonometric functions of \( G = A_r + i A_i \) and \( H = B_r + i B_i \):

\[
\begin{align*}
r_{1,1} &= 2ie^{-inx} \sin(G) \\
s_{1,1} &= 2e^{inx} \cos(H),
\end{align*}
\]

(A 1)

where

\[
\begin{align*}
A_r &= \chi_{1r} + \frac{1}{2}(\kappa_{1r} t + d_{1r} x) - \frac{\pi}{4}, \\
A_i &= \chi_{1i} + \frac{1}{2}(\kappa_{1i} t + d_{1i} x), \\
B_r &= -\chi_{1r} + \frac{1}{2}(\kappa_{1r} t + d_{1r} x) - \frac{\pi}{4} \quad \text{and} \quad B_i = -\chi_{1i} + \frac{1}{2}(\kappa_{1i} t + d_{1i} x),
\end{align*}
\]

while \( d_1 = d_{1r} + id_{1i} \) is the complex coefficient in front of the argument \( x \) with

\[
d_{1r} = 2\eta(-b_1 \kappa_{1i} + a_1 \kappa_{1r}) + \alpha(\Omega_1 \kappa_{1i} + \bar{\Omega}_1 \kappa_{1r})
\]

and

\[
d_{1i} = 2\eta(a_1 \kappa_{1i} + b_1 \kappa_{1r}) + \alpha(\bar{\Omega}_1 \kappa_{1i} - \Omega_1 \kappa_{1r}),
\]

where \( \Omega_1 = 8a_1 b_1 \) and \( \bar{\Omega}_1 = 2 - 4a_1^2 + 4b_1^2 \). Then, the first-order result is

\[
\psi_1 = e^{2inx} \left[ 1 + \frac{8}{D_1} ib_1 \cosh(B_i - iB_r) \sinh(A_i + iA_r) \right],
\]

where \( D_1 = \cos(2B_r) + \cosh(2A_i) + \cosh(2B_i) - \cos(2A_r) \). Two other linear functions \( r_{1,2} \) and \( s_{1,2} \) can be written similarly, namely \( C = C_r + iC_i \) and \( D = D_r + iD_i \):

\[
\begin{align*}
r_{1,2} &= 2ie^{-inx} \sin(C) \\
s_{1,2} &= 2e^{inx} \cos(D),
\end{align*}
\]

(A 2)

where

\[
\begin{align*}
C_r &= \chi_{2r} + \frac{1}{2}(\kappa_{2r} t + d_{2r} x) - \frac{\pi}{4}, \\
C_i &= \chi_{2i} + \frac{1}{2}(\kappa_{2i} t + d_{2i} x), \\
D_r &= -\chi_{2r} + \frac{1}{2}(\kappa_{2r} t + d_{2r} x) - \frac{\pi}{4} \quad \text{and} \quad D_i = -\chi_{2i} + \frac{1}{2}(\kappa_{2i} t + d_{2i} x),
\end{align*}
\]

while \( d_2 = d_{2r} + id_{2i} \) with

\[
d_{2r} = 2\eta(-b_2 \kappa_{2i} + a_2 \kappa_{2r}) + \alpha(\Omega_2 \kappa_{2i} + \bar{\Omega}_2 \kappa_{2r})
\]
and

\[ d_{2j} = 2 \eta (a_2 \kappa_{2j} + b_2 \kappa_{2j}) + \alpha (\Omega_2 \kappa_{2j} - \Omega_2 \kappa_{2j}), \]

where \( \Omega_2 = 8a_2 b_2 \) and \( \Omega_2 = 2 - 4a_2^2 + 4b_2^2 \). In these notations, the second-order linear functions \( r_{2,1} \) and \( s_{2,1} \) are (see eqn. (7) of [21]):

\[
\begin{align*}
    r_{2,1}^* &= -\frac{2}{D_1} e^{i\eta (4i b_1 \cosh(b_1 - i b_2) \cosh(D_1 + i D_2) \\
    &\quad \times \sinh(A_i + i A_r) + \sinh(C_i + i C_r) \\
    &\quad \times \{ \cosh(2 A_i) [-a_1 + a_2 + i(b_1 + b_2)] + \cosh(2 A_r) [a_1 - a_2 - i(b_1 - b_2)] \\
    &\quad + [\cos(2B_i) + \cosh(2 B_i)] [-a_1 + a_2 - i(b_1 + b_2)] \} \\

    s_{2,1} &= -\frac{2}{D_1} e^{i\eta (4i b_1 \cosh(b_1 - i b_2) \sinh(A_i + i A_r) \\
    &\quad \times \sinh(C_i - i C_r) + \cosh(D_1 - i D_2) \\
    &\quad \times \{ \cosh(2 A_i) [a_1 - a_2 - i(b_1 + b_2)] + \cosh(2 A_r) [-a_1 + a_2 + i(b_1 + b_2)] \\
    &\quad + [\cos(2B_i) + \cosh(2 B_i)] [a_1 - a_2 + i(b_1 - b_2)] \} \\
\end{align*}
\]

(A 3)

The second-order solution of the HE follows from here directly:

\[
\psi_2 = \psi_1 + \frac{2(\lambda_2^* - \lambda_2) r_{2,1}^* s_{2,1}}{|r_{2,1}|^2 + |s_{2,1}|^2}.
\]

(A 4)

References


