Quantum optics with one or two photons

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We discuss the concept of a single-photon state together with how they are generated, measured and interact with linear and nonlinear systems. In particular, we consider how a single-photon state interacts with an opto-mechanical system: an optical cavity with a moving mirror and how such states can be used as a measurement probe for the mechanical degrees of freedom. We conclude with a discussion of how single-photon states are modified in a gravitational field due to the red-shift.

1. Introduction

The concept of a photon, an elementary excitation of the electromagnetic field, was introduced by Einstein [1] in his 1905 paper on the production and transformation of light. The concept was subsequently elaborated in the development of quantum electrodynamics in terms of a bosonic particle mediating the electromagnetic interaction [2]. Given its long history, it comes as something of a surprise to learn that efforts to generate single-photon states at optical frequencies are rather recent. Mostly, this has been driven by their potential for quantum information-processing tasks. We will discuss quantum optics with single photons with examples ranging from quantum information [3] to opto-mechanics [4].

The application of quantum electrodynamics to non-relativistic optical systems was first undertaken by Glauber [5] in the 1960s, ushering in the field of quantum optics. The development of the He–Ne laser in the early 1960s was the key technological innovation that stimulated Glauber’s careful quantum explanation of the first-order optical coherence exhibited by the output from a laser. As a result of this particular physical context, the early history of quantum optics emphasized quantum states of the electromagnetic field that have a well-defined phase and intensity, or as well defined as the uncertainty principle will permit.
The output of a laser operating well above threshold can be described in terms of Glauber’s coherent states [5] |α⟩ with slow diffusion in phase [6]. It is necessarily the case that, for such fields, the photon number in a time or frequency bin fluctuates. The closest one can get to a single-photon source with a coherent source is to produce a sequence of laser pulses with on average one photon per pulse (α = 1), but the probability that such a pulse contains no photons or more than one photon is greater than 60%.

A single-photon state is a quantum state for which there is one and only one photon per pulse. Such a state has a well-defined intensity and, by the uncertainty principle, random phase. This is not to say that it lacks quantum coherence. In fact, a transform-limited single-photon pulse is a quantum pure state. Single-photon (or indeed N-photon) states offer the potential of significant performance advantages in metrology [7], communication [8], simulation [9] and computation [3]. These applications are driving a considerable effort in finding ways to generate single-photon states, some of which we discuss in §3.

2. Single-photon states

We define a single photon in terms of the operator describing the positive frequency parts of the electromagnetic field. Typically, we use plane wave modes, with plane polarization, to expand the field operator at space–time point (x, t) so that the (dimensionless) positive frequency operator is

\[ E^{(+)}(t,x) = \int_0^\infty d\omega \tilde{a}(\omega) e^{-i(\omega t - k \cdot x)}, \]  

(2.1)

where \( \tilde{a}(\omega) \) and \( \tilde{a}^\dagger(\omega') \) are bosonic annihilation/creation operators satisfying \( [\tilde{a}(\omega), \tilde{a}^\dagger(\omega')] = \delta(\omega - \omega') \).

We will assume that all modes are in the vacuum except for those propagating in the +x-direction with a carrier frequency centred on \( \omega = \Omega \) and a frequency bandwidth \( B \) much less than the carrier frequency. In that case we can change the frequency variable to \( \omega \rightarrow \omega - \Omega \) and extend the lower limit of integration over the displaced variable to \(-\Omega \rightarrow -\infty\). We will refer to this set of approximations collectively as the ‘quantum optics approximation’. We then only need consider those modes that contribute to the positive frequency operator defined as

\[ \tilde{a} \left( t - \frac{x}{c} \right) = \int_{-\infty}^\infty d\omega \tilde{a}(\omega) e^{-i\omega(t-x/c)}. \]  

(2.3)

This operator represents the positive frequency components of the electromagnetic field in the quantum optics approximation. In this form, the operators \( a(t) \) and \( \tilde{a}(\omega) \) appear as Fourier transform pairs. In our units \( a(t) \) has units of s\(^{-1/2}\).

We now define an N-photon (pure) state as a superposition of a single excitation over many frequencies

\[ |N_\xi\rangle = \frac{1}{\sqrt{N!}} \left[ \int_{-\infty}^\infty d\omega \xi(\omega)\tilde{a}^\dagger_\text{in}(\omega) \right]^N |0\rangle. \]  

(2.4)

Here \( \xi(\omega) \) is the probability amplitude that there is a single photon in a frequency band between \( \omega \) and \( \omega + d\omega \). Normalization of the state requires that

\[ \int_{-\infty}^\infty d\omega |\xi(\omega)|^2 = 1. \]  

(2.5)

The average field amplitude of a single-photon state is zero

\[ \langle 1 | \tilde{a}_\text{in}(t) | 1 \rangle = 0. \]  

(2.6)

We can interpret this result as an indication of the random optical phase of a photon number eigenstate. A phase-dependent measurement on the single-photon state, such as homodyne
detection, would give a null signal on average. The quantum coherent nature of a single photon is revealed when we consider the probability per unit time to detect the photon on a photon counter. In the theory of photodetection [5,10,11], the probability of detecting a photon between $t$ and $t + dt$ is given by

$$P_1(t : t + dt) = \gamma dt \int_R dx f(x)(E^-(t;x)E^+(t;x)).$$

(2.7)

The function $f(x)$ takes into account the spatial extent of a real detector along the direction of propagation. Thus, $f(x) dx$ is the conditional probability that a photon will be counted between $x$ and $x + dx$ over the spatial region of the detector $R$. The constant $\gamma$ characterizes the detector. Thus, $\gamma dt$ is the conditional probability that a photon is counted between $t$ and $t + dt$. We are thus led to define the rate of detection on a point-like detector located at $x = 0$ in the quantum optics approximation as

$$r(t) = \gamma \langle a^\dagger(t)a(t) \rangle \equiv \gamma n(t).$$

(2.8)

In the case of the single-photon state ($N = 1$ in equation (2.4)), this becomes

$$n(t) = |\xi(t)|^2,$$

(2.9)

where

$$\xi(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \tilde{\xi}(\omega).$$

(2.10)

The fact that $n(t)$ appears as the modulus square of a single, complex-valued function in equation (2.9) is a reflection of the underlying purity of the single-photon state. In optical terms, we would say that the pulse is ‘transform limited’, although we need to bear in mind that this is a highly non-classical state with an average field amplitude of zero.

3. Single-photon sources

A single-photon source is a highly non-classical source of light as opposed to the typical semiclassical sources of a laser and thermal radiation. A simple, but not sufficient, metric to characterize non-classical sources is the second-order correlation function. To define this quantity, we need to first define a particular conditional probability distribution: given that a photon is detected at time $t$, what is the conditional probability per unit time that a photon will be detected at a time $t + \tau$, regardless of how many photons are detected between these times. In Glauber’s theory of photodetection [5], this rate is determined by the two-time correlation function for the field

$$G^{(2)}(t : \tau) = \langle a^\dagger(t)a^\dagger(t + \tau)a(t + \tau)a(t) \rangle.$$  

(3.1)

In the case of a stationary source, that is to say one in which all transients due to switch-on have died out, we can define the stationary second-order correlation as

$$G^{(2)}(\tau) = \lim_{t \to \infty} G^{(2)}(t : \tau).$$

(3.2)

In the case of a stationary thermal source, if we detect a photon at time $t$ it is very likely that we will detect another photon a short time later. Thus, $G^{(2)}(\tau)$ has a peak at $\tau = 0$ and decays for increasing $\tau$. This is known as photon bunching. In the case of a photon counting in the output of a laser, $G^{(2)}(\tau)$ is constant.

For photon number states considered in this paper, the photon counting statistics is certainly not stationary. However, most single-photon sources deliver periodic sequences of pulses for which each pulse is a single-photon state. In that case we expect suppression of $G^{(2)}(0)$ and a peak at every period of the pulse sequence. The suppression is not surprising: if one conditions off a single count at $t = 0$ there are no more photons left to detect from that pulse and one must wait at least until the next single-photon pulse comes along. Thus, there are peaks at $\tau = T$, the period of the source.

The suppression of the peak at $\tau = 0$ is a good indication of a single-photon source; however, it does not indicate that the state is a pure quantum state as given in equation (2.4): it could be...
Figure 1. A scheme for a HOM interference experiment. A beam splitter couples two input field modes to two output field modes. The input field modes are prepared in identical single-photon pulses and a variable time delay $\tau$ is introduced. Detectors placed in the path of the output field modes show a suppression of coincidence events when the delay time is zero.

A mixed state with the same intensity distribution as a function of time and not necessarily the modulus square of a complex-valued amplitude function. The key signature of a true pure-state single-photon source is provided by Hong–Ou–Mandel (HOM) interference [12].

HOM interference is a fourth-order interference effect. If two identical single photons arrive on a 50/50 beam-splitter the probability of detecting a single photon at each of the two output ports—a coincidence—is zero. This is because there are two indistinguishable ways this event can occur: both photons are reflected or both photons are transmitted, and the probability amplitudes for each of these paths cancel exactly in the ideal case. If we introduce a time delay between the two photon pulses, the probability of detecting a coincidence drops from a value of 0.5 to near zero as the time delay is reduced to zero.

In figure 1, we show a HOM scheme with two input fields with positive frequency components $a_{\text{in}}(t)$, $b_{\text{in}}(t)$ and two output fields determined by

$$a_{\text{out}}(t) = \frac{1}{\sqrt{2}}(a_{\text{in}}(t) + b_{\text{in}}(t)) \quad (3.3)$$

and

$$b_{\text{out}}(t) = \frac{1}{\sqrt{2}}(a_{\text{in}}(t) - b_{\text{in}}(t)). \quad (3.4)$$

The change in sign of the second equation here is an indication of the time reversal invariance of an ideal beam splitter. Assume that at $t = 0$ the a-mode is prepared in the single-photon state with amplitude function $\xi_a(t)$ while the b-mode is prepared in a single-photon state at $t = \tau$ with amplitude function $\xi_b(t - \tau)$. To be specific we will choose

$$\xi(t) = \begin{cases} \sqrt{\gamma} e^{-\gamma t/2} & t \geq 0 \\ 0 & t < 0. \end{cases} \quad (3.5)$$

The joint probability of counting one photon in output mode-a and one photon in output mode-b is defined by

$$P_{ab} = \int_0^\infty \int_0^\infty \langle a^\dagger_{\text{out}}(t) b^\dagger_{\text{out}}(t') a_{\text{out}}(t) b_{\text{out}}(t') \rangle \, dt \, dt'. \quad (3.6)$$

In this case, we find that this is a function of $\tau$ and is given by

$$P_{ab}(\tau) = \frac{1}{2}(1 - e^{-\gamma |\tau|}). \quad (3.7)$$

A good example of a single-photon source which exhibits both a suppression of second-order correlation at zero time and good HOM interference visibility was demonstrated by the group of Rempe [13] in Garching, Germany, and also by the group of Kuhn in Oxford, UK [14]. This system is based on a Raman two-photon transition in a three-level system inside an optical cavity.
One transition is driven by a strong time-dependent laser and the other is coupled to a cavity mode. A control pulse transforms the atomic system from the ground to an excited state while simultaneously exciting one photon in a cavity mode. The time taken to excite the cavity photon is determined primarily by the duration of the control pulse and can be short compared with the cavity decay time. The photon is then emitted from the cavity via the usual process of cavity decay. Under these conditions, a good approximation to the single-photon amplitude function is the exponential in equation (3.5). The experiments of Rempe and co-workers [13] and Kuhn and co-workers [14] verified the temporal shape of their photon amplitudes and phases in detection-time-resolved HOM and quantum homodyne experiments.

If we can control the temporal shape of the control pulse on a time scale commensurate with the cavity line width, quite complex single-photon pulse temporal shapes can be achieved [13,14]. This is a consequence of the overall temporal response being determined by the convolution of the cavity decay with the excitation pulse. In fact any scheme capable of pulse shaping a coherent pulse will also work for a single-photon pulse. As an example we consider the filtering of a single photon by a single-sided cavity.

In the input/output theory of quantum optics [16], the external field modes are related to the internal modes through the boundary condition

\[ a_{\text{out}}(t) = \sqrt{\gamma} a(t) - a_{\text{in}}(t). \] (3.8)

From equation (3.8), the propagating input and output field operators \( a_{\text{in/out}}(t) \) have units of \( s^{-1/2} \) while the cavity mode operator \( a(t) \) is dimensionless. The cavity quasi-mode \( a(t) \) is represented by a single harmonic oscillator degree of freedom with frequency \( \omega_c \). For an empty cavity, the quantum stochastic differential equation in the Heisenberg picture is given by

\[ \frac{da(t)}{dt} = -\left( \frac{\gamma}{2} + i\delta \right) a(t) + \sqrt{\gamma} a_{\text{in}}(t), \] (3.9)

where \( \delta = \omega_a - \omega_c \), where \( \omega_c \) is the carrier frequency of the single-photon pulse. Solving this equation and substituting into the input–output relations, we can write the transformation induced by the cavity as

\[ a_{\text{out}}(\omega) = \frac{\gamma/2 + i(\omega - \delta)}{\gamma/2 - i(\omega - \delta)} a_{\text{in}}(\omega). \] (3.10)

In writing this equation we have neglected the term that results from the initial condition \( a(0) \) in the quantum stochastic differential equations. As the cavity field starts in the vacuum, and we will only need to evaluate normally ordered moments, this is justified. In other words, we are only considering the field emitted from the cavity that is due to absorption of the incoming field state. The single-photon state defined in equation (3.5) is

\[ |\psi\rangle_{\text{in}} = \int_{-\infty}^{\infty} \xi(\omega) a_{\text{in}}^\dagger(\omega) |0\rangle. \] (3.11)

After interacting with the cavity this state is transformed to

\[ |\psi\rangle_{\text{out}} = \int_{-\infty}^{\infty} \xi(\omega) a_{\text{out}}^\dagger(\omega) |0\rangle. \] (3.12)

It is then easy to see that the probability per unit time of detecting a single photon in the output field is

\[ n(t) = \left| \int_{-\infty}^{\infty} d\omega \left( \frac{\gamma/2 + i(\omega - \delta)}{\gamma/2 - i(\omega - \delta)} \right) \xi(\omega) e^{-i\omega t} \right|^2. \] (3.13)

Shifting the variable of integration and writing the first factor in the integrand as a phasor, this represents a transformation of the probability amplitude \( \xi(\omega) \),

\[ \xi(\omega) \rightarrow \xi(\omega + \delta) e^{i\phi(\omega)}, \] (3.14)

where

\[ \tan(\phi(\omega)) = \frac{\nu \omega}{\gamma^2/4 - \omega^2}. \] (3.15)
In the time domain, the output pulse is a convolution of the input pulse and the cavity response. Thus, the temporal pulse is phase shifted by $\delta t$ and broadened. We will use this result below when we consider an opto-mechanical system.

By ignoring transients in the preceding calculation, we are effectively ignoring the possibility of reflection of photons reflected directly off the cavity from the source to the detector. If the detector cannot distinguish these photons from those that enter the cavity to be re-emitted, an interference dip can occur in the overall detection rate. See [17] for further details.

4. Driving nonlinear systems with a single photon

In the previous section, we considered the response of an empty single-sided cavity to a single-photon input. A rather more interesting case is to consider the response of a nonlinear cavity to a single-photon input. The nonlinearity could result by including an atom in the cavity that interacts with the intra-cavity field, by including a harmonically bound mirror [17], or a medium with a significant nonlinear polarizability [18]. In these cases, an elegant master equation method has been developed by Combes and co-workers [19] which we now summarize.

If the initial state of the system is $|\eta\rangle$ and the initial state of the external input field is the one photon state $|1_\xi\rangle$, then the joint initial state is $|\eta1_\xi\rangle = |\eta\rangle \otimes |1_\xi\rangle$. The total state of the external field and cavity field at time $t > 0$ is $|\psi(t)\rangle = U(t)|\eta\rangle \otimes |1_\xi\rangle$. As the interaction proceeds, the external field and the cavity field become entangled states with one photon in the external field and no photons in the cavity superposed with no photons in the external field and one photon in the cavity. Initially, the cavity field is in the vacuum state. It then becomes excited before eventually decaying back to the vacuum as the photon is emitted into the external field.

Let $\hat{X}$ be a system operator (e.g. $a^\dagger a$). First define the Heisenberg operator

$$j_t(\hat{X}) = U^\dagger(t)(\hat{X} \otimes 1_f)U(t),$$

(4.1)

where $1_f$ is the identity operator acting on the external field. Note that, because $U(t)$ describes an interaction between the system and the field external to the cavity, $j_t(\hat{X})$ is, generally, a joint system–field operator.

The objective is to find the moments of such a time-evolved operator. For initial state $|\eta1_\epsilon\rangle = |\eta\rangle \otimes |1_\epsilon\rangle$, we can define matrix elements of $j_t(\hat{X})$ in the Heisenberg picture as

$$\sigma^{mn}_t(\hat{X}) = \langle \eta, m_\xi | j_t(\hat{X}) | \eta, n_\xi \rangle.$$

(4.2)

For example, if $\hat{X} = a^\dagger a$ then $\sigma^{11}_t(11) \equiv \langle \hat{n}(t) \rangle$ is simply the mean photon number inside the cavity at time $t$. Equivalently, we can work in the Schrödinger picture in which states evolve. To this end, we define the time-dependent system operators $\rho_{m,n}(t)$ to give the same moments

$$\text{Tr}[\rho_{mn}(t)\hat{X}] = \sigma^{mn}_t(\hat{X}).$$

(4.3)

Taking the time derivative of this expression and using the cyclic property of trace, we find the Fock state master equation for an optical cavity driven by number state input fields

$$\frac{d\rho_{mn}}{dt} = -\frac{i}{\hbar} [H, \rho_{mn}] + \kappa D[a]\rho_{mn} + \sqrt{\kappa m_\xi}(t)\rho_{m-1n, a^\dagger} + \sqrt{\kappa n_\xi^*}(t)\rho_{mn-1}$$

$$\equiv L\rho_{mn} + \sqrt{\kappa m_\xi}(t)\rho_{m-1n, a^\dagger} + \sqrt{\kappa n_\xi^*}(t)\rho_{mn-1},$$

(4.4)

where $D[a]\rho = a\rho a^\dagger - a^\dagger a\rho/2 - \rho a^\dagger a/2$. Just as in equation (4.2), the indices here, $m, n$, take the values of the possible single-photon excitations in the external field, in this case simply $1, 0$. 

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$$\equiv L\rho_{mn} + \sqrt{\kappa m_\xi}(t)\rho_{m-1n, a^\dagger} + \sqrt{\kappa n_\xi^*}(t)\rho_{mn-1},$$

(4.4)

where $D[a]\rho = a\rho a^\dagger - a^\dagger a\rho/2 - \rho a^\dagger a/2$. Just as in equation (4.2), the indices here, $m, n$, take the values of the possible single-photon excitations in the external field, in this case simply 1, 0.
This set of coupled equations is solved subject to the initial condition \( \rho_{m0} = \rho(0) \), the initial density operator for the cavity, while the off-diagonal operators are initially zero. For example, to find the average photon number in the cavity given a single-photon input \( \langle a^\dagger a \rangle = \text{Tr}[a^\dagger a \rho_{11}] \), we need to solve the equations of motion for \( \rho_{11}, \rho_{10}, \rho_{01} \) and \( \rho_{00} \). Note that when \( n = m = 0 \) we recover the usual master equation for a single cavity mode as the equation for \( \rho_{00} \). For further details see [20].

We can now apply this formalism to a nonlinear cavity: a cavity with a harmonically bound moving mirror, an opto-mechanical system. As the length of the cavity is not fixed, the frequency details see [20].

\[ H_c = \hbar \omega_c a^\dagger a + \hbar \omega_c' a^\dagger a x = \hbar \omega_c a^\dagger a \]

From the perspective of the mechanical element, this represents a linear potential with a slope proportional to the total photon number in the cavity. This is the radiation pressure force. From the perspective of the cavity field, the moving mirror represents a frequency modulation. In passing to the quantum description, we write the displacement operator as

\[ \hat{x} = x_{r.m.s.} (b + b^\dagger), \]

where \( x_{r.m.s.} = (\hbar/2m\omega_m)^{1/2} \) is the ground state displacement uncertainty for a mechanical resonator with frequency \( \omega_m \) and \( b, b^\dagger \) are the annihilation and creation operators for the mechanical resonator. The Hamiltonian for this system is then given by [21]

\[ H = \hbar \omega_c a^\dagger a + \hbar \omega_m b^\dagger b + \hbar g_0 a^\dagger a (b + b^\dagger), \]

where \( g_0 \) is the opto-mechanical interaction rate in frequency units. The radiation pressure force due to a single photon is \( \hbar g_0/x_{r.m.s.} \). The mechanical system is taken to be initially in the ground state \( |0\rangle_b \) and the initial cavity state is vacuum \( |0\rangle_a \).

It is easier to solve this problem in the interaction picture for which the Hamiltonian is

\[ H_I(t) = \hbar g_0 a^\dagger a (b e^{-i\omega_m t} + b^\dagger e^{i\omega_m t}). \]

One easily sees that the initial state is a fixed point for the lowest order equation in the system for \( \rho_{m0,1} \), thus

\[ \rho_{00}(t) = \rho_{00}(0) = |0\rangle_a \langle 0 | \otimes |0\rangle_b \langle 0 |. \]

Substituting this into the equation for \( \rho_{10}(t) \), we find that it satisfies the same equation as \( \rho_{00}(t) \) but with an inhomogeneous source term of the form \( -\tilde{\xi}(t)|1\rangle_a \langle 0 | \otimes |0\rangle_b \langle 0 | \). The equation becomes

\[ \frac{d\rho_{10}}{dt} = \mathcal{L}[\rho_{10}] - \sqrt{\kappa} \tilde{\xi}(t) \hat{S}_{10}, \]

where \( \hat{S}_{10} = |1\rangle_a \langle 0 | \otimes |0\rangle_b \langle 0 | \) and

\[ \mathcal{L}[A] = -i\xi_0 a^\dagger a (b e^{-i\omega_m t} + b^\dagger e^{i\omega_m t}) A + \kappa (a A a^\dagger - \frac{1}{2} a^\dagger a A - \frac{1}{2} A a^\dagger a). \]

The top-level equation is

\[ \frac{d\rho_{11}}{dt} = \mathcal{L}[\rho_{11}] + \sqrt{\kappa} \tilde{\xi}(t) \rho_{11} + \sqrt{\kappa} \tilde{\xi}(t) \rho_{01} a^\dagger. \]

The solution to the off-diagonal equation is

\[ \rho_{10}(t) = -\sqrt{\kappa} \int_0^t dt' \tilde{\xi}(t') e^{-\kappa (t-t')/2} \hat{R}^\dagger(t) \hat{R}(t') \hat{S}_{10}, \]

where \( \hat{R} = b - b^\dagger \) (r.m.s. motion).
where \( R(t) \) is given by
\[
R(t) = 1 + ig_0 \int_0^t dt_1 (b(t_1) + b^\dagger(t_1)) + (ig_0)^2 \int_0^t dt_2 \int_0^{t_2} dt_1 (b(t_1) + b^\dagger(t_1))(b(t_2) + b^\dagger(t_2)) + (ig)^3 \ldots
\]
(4.14)
which defines the anti-time-ordering operator and \( b(t) = b e^{-i\omega_0 t} \). Clearly \( \hat{R}^{-1}(t) = \hat{R}^\dagger(t) \).

We now substitute this into the equation of motion for \( \rho_{11} \) to obtain
\[
\frac{d\rho_{11}}{dt} = \mathcal{L}[\rho_{11}] + \kappa [\langle 1 \rangle_a |1\rangle_a - |0\rangle_a \langle 0\rangle_a] \left[ \int_0^t dt' \xi^\ast(t') \xi(t') e^{-\kappa(t-t')/2} \hat{R}^\dagger(t') \hat{R}(t') |0\rangle_b \langle 0| + \text{h.c.} \right].
\]
(4.16)
Then we see that the mean photon number in the cavity is given by the solution to the equation
\[
\frac{d\langle a^\dagger a \rangle}{dt} = -\kappa \langle a^\dagger a \rangle + \kappa \left( \int_0^t dt' \xi^\ast(t') \xi(t') |\psi_b(t)\rangle \langle \psi_b(t')| e^{-\kappa(t-t')/2} + \text{c.c.} \right),
\]
(4.17)
where
\[
|\psi_b(t)\rangle = \hat{R}(t)|0\rangle_b.
\]
(4.18)
The equation of motion for the cavity amplitude does not have any contribution from the single-photon driving term and is given by
\[
\frac{d\langle a \rangle}{dt} = -ig_0 \langle a (b e^{-i\omega_0 t} + b^\dagger e^{i\omega_0 t}) \rangle.
\]
(4.19)
This indicates an effective phase modulation chirp of the single-photon state inside the cavity due to the oscillatory motion of the mirror. Indeed this is what is indicated in equation (4.17). Inspection of the integrand in the second term shows that the input field two-time correlation function, \( \langle a^\dagger(t)a(t) \rangle \), is modified by the mechanical response,
\[
\langle a^\dagger(t)a(t) \rangle = \xi^\ast(t)\xi(t) \rightarrow \xi^\ast(t)\xi(t) \langle \psi_b(t) | \psi_b(t') \rangle.
\]
(4.20)
We can also see that the second term in equation (4.16) cannot contribute to any mechanical moments as the operator \( |1\rangle_a \langle 1| - |0\rangle_a \langle 0| \) is traceless. Thus, the equation of motion for the mechanical amplitude is
\[
\frac{d\langle b \rangle}{dt} = -ig_0 \langle a^\dagger a \rangle e^{-i\omega_0 t}.
\]
(4.21)
In the case of an empty cavity with no opto-mechanical nonlinearity, we set \( g_0 = 0 \), so that \( \hat{R}(t) = 1 \) and the mechanical degree of freedom can be traced out. We find that for the optical degree of freedom alone
\[
\rho_{10} = -\sqrt{\kappa} \int_0^t dt' \xi(t') e^{-\kappa(t-t')/2} |1\rangle_a \langle 0|.
\]
(4.22)
Substitution into the top-level equation gives
\[
\frac{d\rho_{11}}{dt} = \kappa \mathcal{D}[a] \rho_{11} + \kappa \left[ \int_0^t dt' \xi^\ast(t') \xi(t') e^{-\kappa(t-t')/2} \right] (|1\rangle_a \langle 1| - |0\rangle_a \langle 0|).
\]
(4.23)
Multiplying by \( a^\dagger a \) and taking the trace we then find
\[
\frac{d\langle a^\dagger a \rangle}{dt} = -\kappa \langle a^\dagger a \rangle + \kappa \left( \xi^\ast(t) \int_0^t dt' \xi(t') e^{-\kappa(t-t')/2} + \text{c.c.} \right).
\]
(4.24)
Using the form of \( \xi(t) \) given in equation (3.5), this equation can be solved to give
\[
\langle a^\dagger a \rangle = \frac{4\gamma\kappa}{(\gamma - \kappa)^2} e^{-\kappa t} (1 - e^{-(\gamma - \kappa) t/2})^2.
\]
(4.25)
This is plotted in figure 2 (dotted curve). We see that the occupation starts at zero and rises as the incoming photon excites the cavity before eventually decaying as the photon is emitted.

In the case of an opto-mechanical cavity $g_0 \neq 0$, the effect of the mechanical phase modulation on the single photon is captured by the function $\langle \psi(t) | \psi(t') \rangle$. The state to lowest order is close to an oscillator coherent state with the amplitude $\beta(t)$. This is the semiclassical amplitude of an oscillator, with an initial zero position and momentum, subject to a forcing term of the form in equation (4.8) with the cavity in a one-photon eigenstate. The transformation of the input single-photon amplitudes then appears as average phase modulation, average over the initial vacuum state of the mechanical resonator. In figure 2, we show the intra-cavity photon number as a function of time for increasing values of $g_0/\omega_m$. We see that when this ratio is large the excitation number of the cavity field is low and even goes to zero for certain times. At these times, the photon is reflected from the cavity. These oscillations are evidence of the strong self-phase modulation of the field in the cavity as the mechanical resonator begins to move. This is evident in figure 3, where we plot the average excitation energy in the mechanical resonator. The oscillations are at the mechanical frequency and in phase with the mean photon number in the cavity. For low values of the ratio $g_0/\omega_m$, the mechanical system is simply driven to a non-zero amplitude coherent state.
Usingsinglephotonsasameasurementprobe

We now consider a single-photon pulse incident on a single-sided opto-mechanical cavity. From the optical perspective, the opto-mechanical interaction is a position-dependent phase shift of the optical field: to what extent can this be used as a measurement channel for the mechanical displacement? To put this another way: the opto-mechanical interaction correlates the optical output field with the mechanical system. How can this correlation be used to infer something about the mechanical system? We will solve this using the stochastic quantum Langevin equations.

The quantum Langevin equations for the cavity field amplitude operator \( a \) and the mirror displacement amplitude \( b \) are

\[
\frac{da}{dt} = -i\left(\Delta + g_0 X\right)a - \frac{\kappa}{2}a + \sqrt{\kappa}a_{\text{in}}
\]

and

\[
\frac{db}{dt} = -i\omega_m b - ig_0 a^\dagger a,
\]

with \( X = b + b^\dagger \), and we have neglected the mechanical damping.

In a typical situation, \( \gamma \gg g_0 \). Let us assume that the time duration of the single-photon pulse \( (\gamma^{-1}) \) is much shorter than the mechanical period \( 2\pi\omega_m^{-1} \) so that we can regard the mirror as stationary over the time during which the cavity fills and empties. We can then assume \( X \) is a constant and integrate the field Langevin equation directly. In the frequency domain, the output optical field operator is then given by equation (3.10) with \( \Delta \rightarrow \Delta + g_0 X \). In the corresponding Schrödinger picture, this implies that the output field is entangled with the mechanical degree of freedom. In other words, the output field is correlated with the position of the mirror at the time of its interaction with the input single-photon pulse.

To extract information on the mechanical displacement, we will consider a HOM interferometer in which the output field from the cavity is mixed on a 50/50 beam splitter with a single-photon input modified to match the response of the opto-mechanical cavity with no mechanical coupling. Ordinary homodyne detection will not work here as the single-photon output state has zero-field amplitude on average.

In HOM interferometry, the coincidence rate for photodetections at each of the output ports goes to zero [22] for two indistinguishable photons incident on the beam splitter. The coincidence rate is given by

\[
C = \frac{1}{2} - \frac{1}{2} \int d\omega \int d\omega' \xi_1(\omega)\xi_1^*(\omega')\xi_2(\omega')\xi_2^*(\omega).
\]

In the case considered here we find that (with \( \Delta = 0 \))

\[
C = \frac{1}{2} - \frac{1}{2} \text{Tr} \left[ d\omega |\tilde{\xi}(\omega)|^2 \left( \frac{\kappa}{2} + i(\omega - g_0 X) \right) \left( \frac{\kappa}{2} - i(\omega - g_0 X) \right) \right]^2,
\]

where the trace is over the mechanical degrees of freedom. To lowest order in \( g_0 \), this is given by

\[
C = g_0^2 \mu^2 <X^2>.
\]

where

\[
\mu = \kappa^2 \int d\omega |\tilde{\xi}(\omega)|^2 \left( \frac{\kappa^2}{4} + \omega^2 \right)^{-1}
\]

accounts for the pulse distortion by the empty cavity response. The HOM visibility thus encodes information on the mean square of the mechanical displacement. This could be used as a temperature measurement of the mechanical resonator. One would need to first calibrate the experiment by holding the mechanical element fixed to observe high HOM visibly. This could be done with an external laser field.
6. Single photon in a gravitational field

As a final example of how a single-photon state can undergo a phase modulation we take the case of a single photon climbing out of a gravitational field. This is due to unitary evolution as the classical gravitational field simply acts as a classical controller. Indeed it has often been noted that the propagation of light in a gravitational field is analogous to the propagation of light in a medium with a nonlinear refractive index \[ 23 \]. There are a number of proposals for space-based quantum communication protocols using single photons. In this technology, it will eventually become necessary to account for the effect of gravitational red-shifts, just as we do now for the Global Navigation Satellite System. The propagation of a single photon in a gravitational field is often used as a text-book example to derive the gravitational red-shift on light. Simple derivations however do not take into account the temporal structure of the photon and thus cannot fully determine the effect of the red-shift on the rate of single-photon detection.

A first attempt to understand quantum optics in curved space–time might begin by simply using the covariant version of Maxwell’s equations on curved space–time as a basis for conventional canonical field quantization. This is an old subject \[ 24,25 \] and includes the well-known predictions of Unruh radiation \[ 26 \] for an accelerated detector and Hawking radiation \[ 27 \] for the horizon of a black hole. In these examples, the key feature is the non-uniqueness of a global vacuum state for a field in curved space–time. The best we can do is to define local vacuum states and thus local single-photon states. The full covariant wave equation for a vector field is not simple. Acedo & Tung \[ 28 \] have derived the covariant classical Maxwell equations in this space–time. We will replace this with a simple scalar theory in one spatial dimension. While this may not be a very good approximation in most cases, it will illustrate the key features of the gravitational red-shift.

Consider the case of constant gravitational acceleration in one spatial dimension. This is not an acceptable physical solution to Einstein’s equations but can be used (with caution) in some finite region of space–time near the Earth. We will use the coordinate system of Desloge \[ 29,30 \]. One observer, the reference observer, O, is assigned the spatial coordinate, \( x = 0 \). Each of the other observers is assigned a spatial coordinate, \( x \), equal to the rod distance of that observer from the reference observer. Time values, \( t \), are assigned to the reference observer using that observer’s standard clock. The other observers in the frame are assigned a coordinate clock, the rate of which is determined by the rate of arrival of signals sent at unit time intervals by the reference observer. This coordinate clock for the observer at \( x \) is then used to assign times, \( t \), to the points on that observer’s world line. The coordinate clock of each observer is synchronized with the reference clock so that a signal pulse that originates at \( O \) at time \( t_1 \) and terminates at \( O \) at time \( t_2 \) being reflected back from the observer at \( x \) at a time \( (t_1 + t_2) / 2 \) as measured by the coordinate clock at \( x \).

The metric for this space–time is \[ 29,30 \]
\[
d s^2 = -d x^2 + e^{2 g x} d t^2, \tag{6.1}
\]
where \( g \) is the proper acceleration of the reference observer at \( x = 0 \). We have used units in which \( c = 1 \). Normal units will be replaced in the final result. The metric matrix is thus
\[
\Sigma_{\mu \nu} = \begin{pmatrix} e^{2 g x} & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.2}
\]

Here we will consider a massless scalar field for simplicity. A classical scalar field, \( \phi(t) \), in this space–time satisfies the wave equation
\[
\frac{\partial^2 \phi}{\partial t^2} - e^{2 g x} \frac{\partial^2 \phi}{\partial x^2} - g e^{2 g x} \frac{\partial \phi}{\partial x} = 0. \tag{6.3}
\]
This equation is separable, so we may expand the field in positive and negative frequency components as
\[
\phi(x, t) = \int_0^\infty d \omega b_{\omega}^* u_{\omega}(x) e^{-i \omega t} + b_{\omega} u_{\omega}(x) e^{i \omega t}, \tag{6.4}
\]
where the spatial mode functions are chosen as
\[ u_\omega(x) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{i}{\hbar} (e^{-gx} - 1)\right], \]  (6.5)
and we have chosen a phase factor so that this mode function approaches that for a plane wave at the reference observer \((x \to 0)\). It also approaches a plane wave as the acceleration approaches zero.

The field is quantized by replacing the classical complex-valued functions \(b_\omega\) with operators so that
\[ \hat{\Phi}(x, t) = \int_0^\infty \! d\omega b_\omega^* u_\omega^*(x) e^{-i\omega t} + b_\omega^* u_\omega(x) e^{i\omega t}, \]  (6.6)
where \([b_\omega, b_\omega^*] = \delta(\omega - \omega')\) and \([b_\omega, b_\omega^*] = 0\). We now define the vacuum state \(\text{with respect to these modes as } b_\omega|0\rangle_b = 0.\) As we can make arbitrary changes of coordinates the mode functions are not unique and neither is the definition of the vacuum state.

A source at \((x_s = 0, t_s = 0)\) excites a single photon so that the field state on the space-like hyper surface \(t = 0\) is \(|1\rangle = \int d\omega \xi(\omega) b_\omega^*|0\rangle_b\). The detection rate for a detector on the world line \(x_d(\tau)\), where \(\tau\) is the proper time as measured in the detector’s rest frame, is given by \([24]\)
\[ R_d = (1|\hat{\phi}^{-}(x_d(\tau))\hat{\phi}^{+}(x_d(\tau))|1\rangle, \]  (6.7)
where \(\hat{\phi}^{\pm}\) are the positive and negative frequency components of the field in equation (6.4).

As an example we consider an exponential temporal pulse, equation (2.4), with \(N = 1\), and with decay rate \(\gamma\) and carrier frequency \(\omega_c\)
\[ \xi(\omega) = \frac{\sqrt{\gamma}}{\gamma / 2 + i(\omega - \omega_c)}. \]  (6.8)
Then
\[ R_d = e^{-g x_d/c^2} |\bar{\xi}(f(x_d) - \tau)|^2, \]  (6.9)
where the pulse is still Lorentzian but with decay rate \(\tilde{\gamma} = \gamma e^{-g x_d/c^2}\) and carrier frequency \(\tilde{\omega}_c = \omega_c e^{-g x_d/c^2}\) and \(f(x_d) = c^2(1 - e^{-g x_d/c^2})/g = h_b\), where \(h_b\) is the radar distance defined as \((t_2 - t_1)/2c\), where a light pulse sent from the source at time \(t_1\) is reflected by the detector back to the source, where it is received at time \(t_2\) from the detector as measured at the source. In the case that \(g x_d/c^2 \ll 1\), this rate is as if the detector sees a source pulse with red-shifted bandwidth and carrier frequency.

Note that, under the gravitational red-shift, the pulses remain transform limited and the state remains a pure state. That is to say, there is no noise added by the gravitational field. It simply acts like a classical control of a unitary quantum channel. If the gravitational decoherence proposals of Penrose \([31]\) and Diosi \([32]\) are correct this will no longer be the case. Phase noise will necessarily broaden single-photon pulses. Single photons may provide a good way to test these ideas, perhaps using HOM interference visibility as an experimental signature. In the approach of Kafri et al. \([33]\), these decoherence models are shown to be equivalent to gravity acting as a classical measurement that necessarily adds noise.

7. Conclusion

It is surprising that, while the idea of a photon heralded the birth of quantum theory, it has only been in the last decade or so that we have developed the technology capable of generating single-photon states of the electromagnetic field. A single-photon pulse is a pure quantum state with zero average optical phase but a definite photon number when integrated over all time. This is in contrast to a pulse from a laser which produces coherent states. These can certainly have on average one photon in the pulse but the actual number of counts detected, from trial to trial, is subject to fluctuations (in fact Poisson distributed).

In this review, we have defined a pure single-photon pulse and considered how it can be coherently controlled by passive and nonlinear optics. We gave one example showing how a
single-photon state can be used as a measurement probe for the mechanical resonator in an opto-mechanical system. Finally, we considered how the gravitational field transforms the single photon so as to give a red-shift for all relevant frequency parameters that characterize the state.

Recently, the new field of superconducting quantum circuits has enabled quantum optics at microwave frequencies. Many of the features of visible light quantum optics can be realized more easily in the microwave domain largely because of the very large dipole moments that can be obtained using superconducting Josephson junctions in a co-planar microwave cavity. On the other hand, the absence of good single-photon detectors in the microwave regime has forced the development of quantum-limited amplifiers followed by microwave homo/heterodyne detection. We do not have space here to review these developments. The interested reader is referred to [34] and the experimental work on itinerant single microwave photons in the Chalmers group and their collaborators [35,36].

The motivation for creating true single-photon sources comes from new kinds of coherent quantum communication protocols and quantum information processing. We are at a very elementary stage of the technology of single photonics. This is expected to change in coming years as we learn how to efficiently engineer the kinds of optical nonlinearities required for information coding and processing in single-photon states. We anticipate a new quantum photonics that parallels the development of coherent optical photonics that is the basis of our current optical communication technology. No doubt there will be other applications of single photonics, for example sensors [37] and fundamental tests of physics [38]. A single-photon state may be next to nothing but single photonics has a bright future.

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