The properties of nonlinear dynamics and chaos are shown to be fundamental to optimal communication signals subject to two practical and realistic design requirements: (i) operation in a noisy environment and (ii) simple hardware implementation. Starting with a simple electronic circuit, a linear filter receiver is presumed, and the matched optimal communication waveform that maximizes the receiver signal-to-noise performance is derived. A return map using samples from this optimal waveform is conjugate to a shift, thereby implying the waveform is chaotic. The optimal communication waveform for a second simple receiver is similarly derived, and it is found to be an exact solution to a physically realizable chaotic oscillator. Thus, a practical consequence of chaos in these waveforms is the potential for simple and efficient signal generation using chaotic oscillators. A conjecture is made that the optimal communication waveform for any stable infinite impulse response filter is similarly chaotic.

1. Introduction

In modern communications technology, the suitability of a method for conveying information is evaluated in terms of quantifiable metrics, such as performance, efficiency, complexity and cost. A communication method is deemed *optimal* if it maximizes one or more of these metrics. Pioneering work by Nyquist, Shannon, Wiener and others in the past century established a rigorous theoretical framework for identifying such optimal methods [1,2]. For example, a fundamental performance metric is the signal-to-noise ratio at the receiver. A famous result from communication theory holds that this ratio is maximized by a receiver that forms the correlation of the incoming received waveform with a reference copy of the transmitted waveform [3]. Such
a correlation receiver can be practically implemented using a linear filter that has an impulse
response that is the time reverse of the transmitted waveform. Such a filter is called a matched
filter because it is specifically matched to the transmitted waveform it uses as a reference [4]. For
an arbitrary waveform, it may not be practical to realize a matched filter using simple analogue
circuits, and more sophisticated digital signal processing is usually required. However, if one
presumes a simple analogue matched filter, it is possible to derive the corresponding waveform
to which it is matched. In this paper, we consider two very simple linear filters and derive the
responding matched waveforms. From this straightforward application of well-established
communication theory, we are surprised to learn that the implied optimal communication
waveforms are chaotic in the sense of modern dynamical systems theory [5]. Importantly, the
presence of chaos enables efficient generation of the waveforms matching these simple filters that
otherwise would be impractical.

Over the past two decades, many have advocated the development of practical data
communications using chaotic waveforms [6–10]. The motivations for such development have
been varied. However, in practice, one would choose chaos for communications only if, among
all possible choices, a chaotic waveform is optimal in the above sense of maximizing some metric.
Here, we consider designing a communication system under two practical and realistic design
requirements: (i) operation in a noisy environment and (ii) simple hardware implementation.
These requirements do not immediately have any obvious relation to chaotic dynamics. The first
requirement suggests the use of a matched filter to obtain a maximal signal-to-noise ratio. The
second requirement can be satisfied by selecting a simple, passive analogue filter as the matched
filter. To be clear, our design makes no a priori assumption of a role for chaos. However, in two
examples, we find that chaos is an implied property of the optimal communication waveforms for
these realistic design constraints. In the conclusion, we extrapolate from these results to conjecture
that optimal waveforms for a large class of stable matched filters are likewise chaotic. Altogether,
these results indicate that the phenomena of nonlinear dynamics and chaos are fundamental and
essential to a physical description of optimal communication signals.

2. Results

To support our claims, we consider two examples of optimal communication waveforms derived
from the design constraints. For each example, we identify an analogue electronic circuit as a filter
with simple hardware realization. We then analytically derive the corresponding communication
waveform that is matched to this filter, thereby yielding the optimal communication waveform
for a noisy environment.

(a) Resistor–inductor–capacitor matched filter circuit

We first consider the simple analogue resistor–inductor–capacitor (RLC) circuit shown in figure 1.
For this filter circuit, the input is the voltage $u(t)$, and the output is the voltage $x(t)$. This passive
linear filter is modelled as

$$\frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + (\omega^2 + \beta^2)(x - u) = 0,$$

where $\beta = T/2RC$, $\omega^2 + \beta^2 = T^2/LC$, and time is in units of a characteristic time $T$.
Arbitrarily choosing the time scale $T = 4\pi RC\sqrt{LC/(4(RC)^2 - LC)}$, we obtain $\omega = 2\pi$ and $\beta = 2\pi\sqrt{LC/(4(RC)^2 - LC)}$.

To satisfy the constraint for effective operation in a noisy environment, we seek the optimal
communication waveform corresponding to this simple filter, i.e. the waveform to which this
filter is matched. It is a standard result of communication theory that the matched filter for a real
waveform has an impulse response that is the time reverse of the given waveform [3]. Applying
Figure 1. RLC filter proposed as a matched filter for the fixed basis function $\rho(t)$, with input signal $u(t)$ and output $x(t)$.

Figure 2. Basis function $\rho(t)$ corresponding to the simple matched filter circuit with damping $\beta = \ln 2$. (Online version in colour.)

In this result here, we seek the waveform $\rho(t)$ that is the time reverse of the impulse response for the filter (2.1), so that $\rho(t)$ is the basis function for an optimal communication waveform. Equivalently, this basis function satisfies the time-reversed filter equation

$$\frac{d^2 \rho}{dt^2} - 2\beta \frac{d\rho}{dt} + (\omega^2 + \beta^2) \cdot (\rho - \delta(t)) = 0,$$

subject to the final values $\rho(\infty) = d\rho/dt(\infty) = 0$, where the input $\delta(t)$ is a unit impulse function at time $t = 0$. Solving this final value problem yields

$$\rho(t) = \begin{cases} \frac{\omega^2 + \beta^2}{\omega} \cdot \sin(\omega t) \cdot e^{\beta t}, & t < 0 \\ 0, & t \geq 0 \end{cases},$$

which is a rising exponential oscillation that terminates at $t = 0$. For the particular damping $\beta = \ln 2$, the basis function is plotted in figure 2. We note the basis function is not causal, because the waveform has an infinite precursor that is non-zero as $t \to -\infty$.

With the basis function (2.3), we may encode a transmitted communication waveform (to be received by the matched filter in figure 1) using copies of the basis function regularly spaced in time and weighted by bits of a digital message signal. That is, the communication signal is

$$u(t) = \sum_{m=-\infty}^{\infty} s_m \cdot \rho(t - m),$$

where the $s_m$ are the message bits. Each bit is represented by $s_m = \pm 1$ and may be considered random, which is a conventional representation in communication theory [3]. This waveform encodes the message for transmission, and the filter in figure 1 is an optimal receiver for detecting the message at the receiver. A segment of a typical waveform (2.4) for an arbitrary, random bit sequence is shown in figure 3, and the corresponding phase space projection is shown in
Figure 3. Typical waveform $u(t)$ encoding a random bit sequence. (Online version in colour.)

Figure 4. Phase space trajectory of a typical encoded waveform $u(t)$. (Online version in colour.)

Figure 4. The phase space trajectory is similar to a projection of a chaotic trajectory; in fact, for the basis function $\rho(t)$, the waveform $u(t)$ is chaotic in the sense of Li & Yorke [11], which we show in the following.

We first define a return map for the waveform $u(t)$ and examine its properties. For any $\beta > 0$, we define the $n$th return of the waveform as

$$d_n = \frac{e^{\beta/2}}{\omega^2 + \beta^2} \cdot \frac{du}{dt} \left( n - \frac{1}{2} \right),$$

(2.5)

where $n$ is an integer and the multiplicative factor is included for convenience. Using equations (2.3) and (2.4), we find the returns can be evaluated to yield

$$d_n = \sum_{m=0}^{\infty} s_{m+n} \cdot e^{-m\beta},$$

(2.6)

for any $n$. From this representation, it can be shown that successive returns satisfy the recurrence relation

$$d_{n+1} = e^\beta \cdot (d_n - s_n),$$

(2.7)
Figure 5. Waveform return map for $\beta = \ln 2$. (Online version in colour.)

which defines a return map for the encoded waveform. We note that the slope of the map in equation (2.7) is $e^\beta > 1$ for all $\beta > 0$. Furthermore, considering all possible binary sequences yields

$$|d_n| \leq D = \frac{e^\beta}{e^\beta - 1},$$

(2.8)

which bounds the range of waveform return values.

We first consider the special case $\beta = \ln 2$. In this case, we recognize equation (2.6) as a binary expansion covering the closed interval $[-2, 2]$. Furthermore, equation (2.6) implies $\text{sgn}(d_n) = s_n$ for any message signal, where $\text{sgn}$ is the signum function (and neglecting the singular point $d_n = 0$). Thus, the map in equation (2.7) is equivalently written as

$$d_{n+1} = 2 \cdot [d_n - \text{sgn}(d_n)],$$

(2.9)

which provides an explicit, one-dimensional return map for the encoded waveform. The return map is shown in figure 5, for which the map range is bounded by $|d_n| \leq D = 2$ per equation (2.8). This return map is conjugate to a Bernoulli shift, implying that orbits of the map are deterministic and chaotic. Because orbits of the return map correspond to regular returns of the encoded waveform $u(t)$, the waveform is also chaotic (in the sense of Li & Yorke [11]), with positive Lyapunov exponent $\lambda = \ln 2$.

This result explicitly shows that the optimal communication waveform derived for practical and realistic design requirements is provably chaotic. We believe this is a remarkable and surprising result. The derivation did not assume chaos a priori; rather, a chaotic waveform is chosen over other, non-chaotic waveforms because it best satisfies the design requirements.

The waveform (2.4) with basis function (2.3) is an example of a chaotic waveform constructed by linear superposition [12–16]. The choice of a simple analogue matched filter results in a basis function for the communication waveform that has the necessary properties for chaos [12]. In particular, the exponential decay of the matched filter’s impulse response defines an acausal basis function with an infinite precursor and exponentially rising oscillation. The exponentially rising oscillation provides an exponential divergence of nearby trajectories and a positive Lyapunov exponent for the waveform. The infinite precursor implies that the current value of the waveform depends on all future bits $s_m$ that define the waveform. Specifically, in equation (2.4), the value
Figure 6. Waveform return map for $\beta > \ln 2$. (Online version in colour.)

of $u(t)$ at time $t$ includes contributions from basis functions positioned at times $t + m$ for all positive integers $m$. This fact explains determinism in the waveform, because the entire future evolution of the waveform is contained in and, hence, determined by the present state [12]. The property of determinism is the fundamental difference between chaotic signals and merely random signals.

The presence of chaos in optimal communication waveforms extends to the general case with $\beta \neq \ln 2$. To explain, we first consider the case of strong damping, with $\beta > \ln 2$. Although equation (2.6) is no longer a simple binary expansion, we again find $\text{sgn}(d_n) = s_n$, and the map in equation (2.7) similarly yields an explicit one-dimensional return map,

$$d_{n+1} = e^\beta \cdot [d_n - \text{sgn}(d_n)],$$

(2.10)

which is illustrated in figure 6. Because the slope of this map is everywhere greater than one, bounded iterates exhibit exponential divergence, and the corresponding communication waveforms are characterized by the positive Lyapunov exponent $\lambda = \beta$. Hence, the waveform return map is deterministic and also exhibits a chaotic character. However, as an iterated map, chaotic orbits of equation (2.10) are unstable, because typical trajectories are unbounded and characterized by a chaotic transient [5]. The range $|d_n| < E$, where

$$E = (1 - 2e^{-\beta}) \cdot D,$$

(2.11)

is identified as an escape region in figure 6. For any $|d_n| < E$ equation (2.10) implies $|d_{n+1}| > D$, and any orbit that visits the escape region immediately maps outside the allowable waveform range and subsequent iterates grow unbounded. Because the communication waveform returns are bounded by construction, the escape region and its preimages under the map (2.10) are not visited by the waveform returns. Instead, the waveform returns are restricted to a Cantor set constructed by repeated removal of the middle fraction $1 - 2e^{-\beta}$ of the interval.

A chaotic interpretation is also present in the case of weak damping, with $\beta < \ln 2$. In this case, the information bit $s_n$ does not necessarily equal the sign of $d_n$, so that the map in equation (2.7) does not reduce to an explicit one-dimensional return map as in the previous cases. Instead, the following return $d_{n+1}$ depends on both $d_n$ and $s_n$ for a range of return values, as indicated by the two line segments in figure 7. In the plot, the range of multi-value map returns is defined
by $|d_n| < -E$, where $E < 0$ is defined in equation (2.11). Because the slope of each line segment is again greater than one, nearby trajectories exponentially diverge, and the divergence rate is characterized by the positive exponent $\lambda = \beta$. Thus, the waveform returns again exhibit a chaotic characteristic. However, in this case the map is not a chaotic oscillator, because the map is not deterministic. That is, for a given return value $d_n$, the future information bits $s_m$, $m \geq n$, are ambiguous, and the future state of the system is not uniquely determined. We note that determinism can be restored by enforcing a suitable grammar restriction on the encoded message signal [15].

(b) Integrate-and-dump resistor–inductor–capacitor matched filter

A second simple analogue matched filter suggests that the presence of chaos in a communication waveform offers practical benefits for efficient signal generation. For this example, we consider a filter that has been previously identified as a matched filter for an exactly solvable chaotic oscillator [17]. This filter also includes the RLC circuit of figure 1, but it additionally contains an integrate-and-dump circuit. This composite filter is modelled as

$$\frac{dv}{dt} = u(t+1) - u(t)$$  \hspace{1cm} (2.12)

and

$$\frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + (\omega^2 + \beta^2)(x - y) = 0,$$  \hspace{1cm} (2.13)

where $u(t)$ is the filter input, $y(t)$ is an intermediate state and $x(t)$ is the output.

The previous results may be extended to reveal the optimal basis function corresponding to the matched filter including the integrate-and-dump operation. The basis function for the composite matched filter is then

$$P(t) = \int_0^1 \rho(t - \tau) \, d\tau,$$  \hspace{1cm} (2.14)
where \( \rho(t) \) is the basis function in equation (2.3) and the integral results from the integrate-and-dump operation in equation (2.12). Direct evaluation of the definite integral (2.14) yields

\[
P(t) = \begin{cases} 
(1 - e^{-\beta}) \cdot \left( \cos \omega t - \frac{\beta}{\omega} \sin \omega t \right) \cdot e^{\beta t}, & t \leq 0 \\
1 - \left( \cos \omega t - \frac{\beta}{\omega} \sin \omega t \right) \cdot e^{\beta(t-1)}, & 0 < t \leq 1 \\
0, & 1 < t.
\end{cases}
\] (2.15)

This acausal pulse has been previously identified as a basis function for the linear synthesis of chaos [14,17].

This particular basis function (2.15) is practically significant, because the corresponding communication waveform,

\[
v(t) = \sum_{m=-\infty}^{\infty} s_m \cdot P(t - m),
\] (2.16)

is an exact analytic solution to a physically realizable chaotic oscillator [17–23]. Remarkably, the autonomous oscillation of this system generates a waveform constructed of acausal basis functions. Historically, relatively little attention has been given to acausal basis functions, because the conventional approach to signal generation is to use a pulse generator to drive a causal filter [1,2,4]. In such a scheme, generation of an acausal basis function would require the exciting pulses to originate at \( t \to -\infty \). Here, we see that a chaotic oscillator naturally and efficiently accomplishes this task. Moreover, it has long been known that the symbolic content of a chaotic waveform can be controlled using small perturbations [6]. As such, the chaotic oscillator offers a simple and practical optimal waveform source.

3. Discussion

In this paper, we have shown that, under certain realistic and practical design constraints, the optimal communication waveform is chaotic. Specifically, we required matched filter performance using simple analogue receivers and derived waveforms that are chaotic. The first example used a simple RLC circuit, which is a passive analogue filter that is well known to electrical engineering students. The simplicity of the circuit enables exact analytic calculations, and the resulting optimal communication waveform is provably chaotic. The second example extends the first filter circuit with an integrate-and-dump operation, and the resulting optimal waveform is found to be a solution to an exactly solvable chaotic oscillator that has been reported in the literature. As such, the second example suggests that a chaotic oscillator may be used to practically generate an optimal communication signal, possibly using control to encode information for transmission [6].

In this analysis, the RLC circuits chosen for the matched filter are some of the simplest examples of infinite impulse response (IIR) filters. Time reversal of a circuit’s impulse response leads to an exponentially rising, acausal basis function that persists for all prior time. An infinite precursor is an essential element for determinism, and an exponentially rising character is necessary for the synthesis of chaotic waveforms by linear superposition [12]. These attributes would be ascribed to the basis function corresponding to any IIR filter that exhibits exponential decay. Thus, we conjecture that any stable IIR filter can similarly yield optimal communication waveforms that are chaotic. Historically, waveforms constructed from acausal basis functions have been deemed impractical, but recently discovered chaotic oscillators naturally and efficiently generate them. Thus, the identification of chaos in optimal communication waveforms opens up new avenues for communication system design. More fundamentally, if our conjecture is true, then nonlinear dynamics and chaos are necessary ingredients in the physical description of practical communication signals.

**Ethics.** This work did not involve the collection of human or animal data.
Data accessibility. This work does not have any experimental data.

Authors’ contributions. Both authors contributed equally to the work and preparation of the manuscript. Both authors gave final approval for publication.

Competing interests. The authors have no competing interests.

Funding. We received no funding for this study.

Acknowledgements. The authors acknowledge Seth Cohen, Daniel Hahs and Shawn Pethel for helpful discussions regarding the interpretation and presentation of the research results.

References