Recoverability in quantum information theory

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The fact that the quantum relative entropy is non-increasing with respect to quantum physical evolutions lies at the core of many optimality theorems in quantum information theory and has applications in other areas of physics. In this work, we establish improvements of this entropy inequality in the form of physically meaningful remainder terms. One of the main results can be summarized informally as follows: if the decrease in quantum relative entropy between two quantum states after a quantum physical evolution is relatively small, then it is possible to perform a recovery operation, such that one can perfectly recover one state while approximately recovering the other. This can be interpreted as quantifying how well one can reverse a quantum physical evolution. Our proof method is elementary, relying on the method of complex interpolation, basic linear algebra and the recently introduced Rényi generalization of a relative entropy difference. The theorem has a number of applications in quantum information theory, which have to do with providing physically meaningful improvements to many known entropy inequalities.

1. Introduction

Entropy inequalities are foundational in quantum information theory [1,2], giving limitations not only on which kinds of physical evolutions are possible in principle but also on efficiencies of communication tasks. More generally and for similar reasons, these inequalities find application in many areas of physics such as thermodynamics [3], condensed matter [4] and black hole physics [5,6] to name a few. The most prominent entropy inequalities are the non-increase of quantum relative entropy with respect to the application of a quantum channel [7,8] and the strong subadditivity of quantum entropy [9,10]. In fact, these inequalities are known to be equivalent to each other.
A recent line of research, which has in part been motivated by the posting [11], has been to establish physically meaningful refinements of these entropy inequalities. For example, one might think that if the amount by which the quantum relative entropy decreases is not very much, then it should be possible to perform a recovery channel to reverse the action of the original one. In fact, one of the earliest results in this spirit is due to Petz [12,13], who showed that perfect reversal of a channel acting on two given states is possible if and only if the relative entropy decrease is equal to zero. Furthermore, he gave an explicit construction of the recovery channel which does so (now called the Petz recovery map), such that it depends on the original channel and one of the states used to evaluate the quantum relative entropy. These results were later extended in order to elucidate the structure of quantum states that saturate the strong subadditivity inequality [14] and the structure of states and channels which saturate the non-increase of quantum relative entropy inequality [15,16].

The ‘perfect saturation’ results quoted above are interesting from a fundamental perspective but seem to have little bearing in applications. That is, one might wonder if the results still hold in some form when the entropy inequalities are not fully saturated but are instead nearly saturated. After an initial negative result in this direction [17], a breakthrough result [18] established a long desired refinement of the strong subadditivity inequality. In particular, the new contribution showed that if the strong subadditivity inequality is nearly saturated, then the relevant tripartite state is an approximate quantum Markov chain, in the sense that it is possible to recover one system by acting exclusively on one other system while at the same time preserving the correlations with a third. Later work has further elucidated the form of the recovery channel used in approximate quantum Markov chains [19]. The result from [18] has now found a number of applications in quantum information theory [20–23] and is expected to find more in other areas of physics.

The main contribution of this paper is to establish physically meaningful refinements of the non-increase of quantum relative entropy with respect to quantum channels. One of the main results can be summarized informally as follows: if the decrease in quantum relative entropy between two quantum states after a quantum channel acts is relatively small, then it is possible to perform a recovery operation, such that one can perfectly recover one state while approximately recovering the other. A significant advantage of the proof detailed here is that it is elementary, relying on standard methods from the theory of complex interpolation [24,25], basic linear algebra and the notion of a Rényi generalization of a relative entropy difference [21]. The refinement of strong subadditivity from [18] is now a corollary of theorem 3.3 presented here, but it remains open to determine whether the converse implication is true or whether the more general refinement of strong subadditivity from [19] can be obtained from theorem 3.3. Furthermore, the recovery channel given here obeys desirable ‘functoriality’ properties discussed in [20], which allows for theorem 3.3 to be applied in a wide variety of contexts.

We begin in the next section with some brief background material and a statement of the operator Hadamard three-line theorem. Section 3 details our main result (theorem 3.3) and §4 details the functoriality properties of the recovery channel presented here. Section 5 shows how many refinements of entropy inequalities follow as corollaries of theorem 3.3. We conclude in §6 with a discussion and some open questions.

2. Background

For more background on quantum information theory, we refer to the books [1,2]. Throughout the paper, we deal with density operators and quantum channels. We restrict our developments to finite-dimensional Hilbert spaces, even though it should be possible to extend some of the results here to separable Hilbert spaces. (We leave this for future developments.) Density operators are positive semi-definite operators with trace equal to one—they represent the state of a quantum system. Quantum channels are linear completely positive trace-preserving maps taking density operators in one quantum system to those in another.
An important technical tool in this work is the Schatten $p$-norm of an operator $A$, defined as $\|A\|_p \equiv [\text{Tr}(|A|^p)]^{1/p}$, where $|A| \equiv \sqrt{A^*A}$ and $p \geq 1$. The convention is for $\|A\|_\infty$ to be defined as the largest singular value of $A$ because $\|A\|_p$ converges to this in the limit as $p \to \infty$. In the proof of our main result (theorem 3.3), we repeatedly use the fact that $\|A\|_p$ is unitarily invariant. That is, $\|A\|_p$ is invariant with respect to linear isometries, in the sense that $\|A\|_p = \|ULAV^\dagger\|_p$, where $U$ and $V$ are linear isometries satisfying $U^\dagger U = I$ and $V^\dagger V = I$. From these norms, one can define information measures relating quantum states and channels, with the main one used here known as a Rényi generalization of a relative entropy difference [21], recalled in the next section. A special case of this is the Rényi conditional mutual information defined in [26]. The structure of the paper is to present information measures as we need them, rather than recalling all of them in one place.

Throughout we adopt the usual convention and define $f(A)$ for a function $f$ and a positive semi-definite operator $A$ as follows: $f(A) \equiv \sum_i f(\lambda_i)|i\rangle\langle i|$, where $A = \sum_i \lambda_i|i\rangle\langle i|$ is a spectral decomposition of $A$ such that $\lambda_i \neq 0$ for all $i$. We denote the support of $A$ by $\text{supp}(A)$, and we let $\Pi_A$ denote the projection onto the support of $A$.

Another important technical tool for proving our main result is the operator version of the Hadamard three-line theorem given in [27], in particular, the very slight modification stated in [28]. We note that the theorem below is a variant of the Riesz–Thorin operator interpolation theorem (e.g. [24,25]).

**Theorem 2.1.** Let

$$ S \equiv \{z \in \mathbb{C} : 0 \leq \text{Re}[z] \leq 1 \}, $$

and let $L(H)$ be the space of bounded linear operators acting on a Hilbert space $H$. Let $G : S \to L(H)$ be a bounded map that is holomorphic on the interior of $S$ and continuous on the boundary.\(^1\) Let $\theta \in (0, 1)$ and define $p_0$ by

$$ \frac{1}{p_0} = 1 - \theta \frac{p_1}{p_0} + \frac{\theta}{p_1}, $$

where $p_0, p_1 \in [1, \infty]$. For $k = 0, 1$ define $M_k = \sup_{t \in \mathbb{R}} \|G(k + it)\|_{p_k}$. Then

$$ \|G(\theta)\|_{p_0} \leq M_0^{1-\theta} M_1^\theta. $$

**3. Bounds for a difference of quantum relative entropies**

This section presents our main result (theorem 3.3), which is a refinement of the monotonicity of quantum relative entropy. For the lower bounds given in this paper, we take states $\rho$ and $\sigma$ and the channel $\mathcal{N}$ to be as given in the following definition.

**Definition 3.1.** Let $\rho$ be a density operator and let $\sigma$ be a positive semi-definite operator, each acting on a finite-dimensional Hilbert space $H$ and such that $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$. Let $\mathcal{N} : L(H) \to L(H_B)$ be a quantum channel with finite-dimensional output Hilbert space $H_B$.

A Rényi generalization of a relative entropy difference is defined as [21]

$$ \tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) \equiv \frac{2\alpha}{\alpha - 1} \log \|([\mathcal{N}(\rho)]^{(1-\alpha)/2\alpha}[\mathcal{N}(\sigma)]^{(\alpha-1)/2\alpha} \otimes I_E)U^\dagger_{S \to BE}\mathcal{N}(\sigma)^{(1-\alpha)/2\alpha} \rho^{1/2}\otimes I_2\|_{2\alpha}, $$

where here and throughout this paper log denotes the natural logarithm, $\alpha \in (0, 1) \cup (1, \infty)$, and $U_{S \to BE}$ is an isometric extension of the channel $\mathcal{N}$. That is, $U_{S \to BE}$ is a linear isometry satisfying $\text{Tr}_E[U_{S \to BE}(\psi)U_{S \to BE}^\dagger] = \mathcal{N}(\psi)$ and $U_{S \to BE}^*U_{S \to BE} = I_S$. All isometric extensions of a channel are related by an isometry acting on the environment system $E$, so that the definition in (3.1) is invariant under any such choice. Recall also that the adjoint $N^\dagger$ of a channel is given in terms of an isometric extension $U$ as $N^\dagger(\cdot) = U^\dagger((\cdot) \otimes I_E)U$. This can be used to verify that the definition given in (3.1) is the same as the definition given in [21].

\(^1\)A map $G : S \to L(H)$ is holomorphic (continuous, bounded) if the corresponding functions to matrix entries are holomorphic (continuous, bounded).
The following limit is known for positive definite operators [21, §6] and we provide a proof in appendix A that it holds for \( \rho, \sigma \) and \( \mathcal{N} \) as given in definition 3.1:

\[
\lim_{\alpha \to 1} \tilde{A}_\alpha(\rho, \sigma, \mathcal{N}) = D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \tag{3.2}
\]

It is one reason why we say that \( \tilde{A}_\alpha(\rho, \sigma, \mathcal{N}) \) is a R\'enyi generalization of a relative entropy difference, in addition to the fact that \( \tilde{A}_\alpha(\rho, \sigma, \mathcal{N}) \geq 0 \) for all \( \alpha \in [\frac{1}{2}, 1) \cup (1, \infty] \) [29]. The quantum relative entropy \( D(\omega\|\tau) \) is defined for a density operator \( \omega \) and a positive semi-definite operator \( \tau \) as [30] \( D(\omega\|\tau) \equiv \text{Tr}[\omega \log \omega - \log \tau] \), whenever \( \text{supp}(\omega) \subseteq \text{supp}(\tau) \), and by convention, it is defined to be \( +\infty \) otherwise. It is monotone with respect to quantum channels [7,8] in the following sense: \( D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq 0 \). We refer to the quantity on the right-hand side of (3.2) as a ‘relative entropy difference’.

For \( \alpha = \frac{1}{2} \), observe that

\[
\Delta_{1/2}(\rho, \sigma, \mathcal{N}) = -\log \|([\mathcal{N}(\rho)]^{1/2}[\mathcal{N}(\sigma)]^{-1/2}) \otimes I_E)U_{SE \rightarrow BE} \sigma^{1/2} \rho^{1/2}\|_1^2
\]

= -\log F(\rho, R_{\sigma, \mathcal{N}}^P(\mathcal{N}(\rho))), \tag{3.3}

where \( F(\rho, \sigma) \equiv \sqrt{\rho \sigma} \) is the quantum fidelity [31] and \( R_{\sigma, \mathcal{N}}^P \) is the Petz recovery map [12,13] (see also [32]) defined as

\[
R_{\sigma, \mathcal{N}}^P(\cdot) \equiv \sigma^{1/2} \mathcal{N}^\dagger([\mathcal{N}(\sigma)]^{-1/2}(\cdot)[\mathcal{N}(\sigma)]^{-1/2})\sigma^{1/2}. \tag{3.4}
\]

(See appendix B for a brief justification that \( R_{\sigma, \mathcal{N}}^P \) is a completely positive trace-non-increasing linear map and a quantum channel when acting on \( \text{supp}(\mathcal{N}(\sigma)) \).) From the definition, one can see that the fidelity possesses the following property:

\[
\sqrt{F}(\omega_{XB}, \tau_{XB}) = \sum_x p_X(x)\sqrt{F}(\omega_x, \tau_x), \tag{3.5}
\]

where

\[
\omega_{XB} \equiv \sum_x p_X(x)|x\rangle\langle x| \otimes \omega_x \quad \text{and} \quad \tau_{XB} \equiv \sum_x p_X(x)|x\rangle\langle x| \otimes \tau_x, \tag{3.6}
\]

\( p_X \) is a probability distribution, \( \{|x\rangle\} \) is some orthonormal basis, and \( \{\omega_x\} \) and \( \{\tau_x\} \) are sets of states.

For the upper bounds given in this paper, the situation is a bit more restrictive, and we take \( \rho, \sigma \) and \( \mathcal{N} \) in the following definition.

**Definition 3.2.** Let \( \rho_{SE} \) be a positive definite density operator and let \( \sigma_{SE} \) be a positive definite operator, each acting on a finite-dimensional tensor-product Hilbert space \( \mathcal{H}_S \otimes \mathcal{H}_E \). Let \( \mathcal{N} \) be a quantum channel given as follows: \( \mathcal{N}(\rho_{SE}) = T_{BE}(U_{SE \rightarrow BE} \rho_{SE} U_{SE \rightarrow BE}^\dagger) \), where \( U_{SE \rightarrow BE} \) is a unitary operator taking \( \mathcal{H}_S \otimes \mathcal{H}_E \) to an isomorphic finite-dimensional tensor-product Hilbert space \( \mathcal{H}_B \otimes \mathcal{H}_E \), such that \( \mathcal{N}(\rho) \) and \( \mathcal{N}(\sigma) \) are each positive definite and act on \( \mathcal{H}_B \).

Let \( \rho, \sigma \) and \( \mathcal{N} \) be as given in definition 3.2. We require this restriction for the upper bounds because in this case, we will be taking matrix inverses and need to conclude statements such as the following one:

\[
[\theta_B^{-1}\mathcal{N}^\dagger([\mathcal{N}(\sigma)]^{1/2}\theta_B^{-1}[\mathcal{N}(\sigma)]^{-1/2})]^{-1} = \theta_B^{-1}\mathcal{N}^\dagger([\mathcal{N}(\sigma)]^{1/2}\theta_B^{-1}[\mathcal{N}(\sigma)]^{-1/2})\theta_B^{-1}, \tag{3.7}
\]

where \( \theta_B \) is positive definite. The equality above follows because in this case \( \mathcal{N}^\dagger(\theta_B) = U^\dagger(\theta_B \otimes I_E)U \), with \( U \) unitary, so that \( \mathcal{N}^\dagger(\theta_B^{-1}) = \mathcal{N}^\dagger(\theta_B^{-1}) \) (this equality need not hold if \( \rho, \sigma \) and \( \mathcal{N} \) are allowed the more general form as in definition 3.1)—i.e. a matrix inverse does not commute with a partial trace. It then follows from the method of proof given in [26, proposition 29] that the following limit holds:

\[
\lim_{\alpha \to \infty} \tilde{A}_\alpha(\rho, \sigma, \mathcal{N}) = D_{\max}(\rho\|R_{\sigma, \mathcal{N}}^P(\mathcal{N}(\rho))), \tag{3.8}
\]

where \( D_{\max}(\omega\|\tau) \equiv \log \|\omega^{1/2} \tau^{-1} \omega^{1/2}\|_\infty = 2\log \|\omega^{1/2} \tau^{-1/2}\|_\infty \) is the max-relative entropy [33]. The quantity on the right-hand side of (3.8) was defined in [29], following directly from the ideas
presented in [21,26]. From the definition, one can see that the max-relative entropy possesses the following property:

$$D_{\text{max}}(\omega_{XB} \| \tau_{XB}) = \max_x D_{\text{max}}(\omega_x \| \tau_x),$$

(3.9)

where $\omega_{XB}$ and $\tau_{XB}$ are as in (3.6).

We can now state the main theorem of this paper.

**Theorem 3.3.** Let $\rho$, $\sigma$ and $\mathcal{N}$ be as given in definition 3.1. Then the following inequality holds:

$$D(\rho \| \sigma) - D(\mathcal{N}(\rho) \| \mathcal{N}(\sigma)) \geq -\log \left[ \sup_{t \in \mathbb{R}} F(\rho, \mathcal{R}_{\rho,\mathcal{N}}^P(\mathcal{N}(\rho))) \right],$$

(3.10)

where $\mathcal{R}_{\rho,\mathcal{N}}^P$ is the following rotated Petz recovery map:

$$\mathcal{R}_{\rho,\mathcal{N}}^P(\cdot) \equiv \mathcal{U}_{\rho,t} \circ \mathcal{R}_{\rho,\mathcal{N}}^P \circ \mathcal{U}_{\mathcal{N}(\sigma),-t}(\cdot),$$

(3.11)

$\mathcal{R}_{\rho,\mathcal{N}}^P$ is the Petz recovery map defined in (3.4), and $\mathcal{U}_{\rho,t}$ and $\mathcal{U}_{\mathcal{N}(\sigma),-t}$ are partial isometric maps defined from

$$\mathcal{U}_{\rho,t}(\cdot) \equiv \omega^t(\cdot)\omega^{-it},$$

(3.12)

with $\omega$ a positive semi-definite operator. If $\rho$, $\sigma$ and $\mathcal{N}$ are as given in definition 3.2, then the following inequality holds:

$$D(\rho \| \sigma) - D(\mathcal{N}(\rho) \| \mathcal{N}(\sigma)) \leq \sup_{t \in \mathbb{R}} D_{\text{max}} \left( \rho \| \mathcal{R}_{\rho,\mathcal{N}}^P(\mathcal{N}(\rho)) \right).$$

(3.13)

**Proof.** We can prove this result by employing theorem 2.1. We first establish the inequality in (3.10). Let $U_{S \rightarrow BE}$ be an isometric extension of the channel $\mathcal{N}$, which we abbreviate as $U$ in what follows. Pick

$$G(z) \equiv ([\mathcal{N}(\rho)]^{1/2} [\mathcal{N}(\sigma)]^{-1/2} \otimes I_E) U \sigma^{1/2} \rho^{1/2},$$

$$p_0 = 2, p_1 = 1$$

and $\theta \in (0, 1)$, which fixes $p_0 = 2/(1 + \theta)$. The operator valued-function $G(z)$ satisfies the conditions needed to apply theorem 2.1.\(^2\) For the choices above, we find that

$$\|G(\theta)\|_{2/(1+\theta)} = \|([\mathcal{N}(\rho)]^{1/2} [\mathcal{N}(\sigma)]^{-1/2} \otimes I_E) U \sigma^{1/2} \rho^{1/2}\|_{2/(1+\theta)},$$

(3.14)

$$M_0 = \sup_{t \in \mathbb{R}} \|G(it)\|_2 = \sup_{t \in \mathbb{R}} \|([\mathcal{N}(\rho)]^{it/2} [\mathcal{N}(\sigma)]^{-it/2} \otimes I_E) U \sigma^{it} \rho^{1/2}\|_2 \leq \|\rho^{1/2}\|_2 = 1,$$

(3.15)

$$M_1 = \sup_{t \in \mathbb{R}} \|G(1 + it)\|_1 = \sup_{t \in \mathbb{R}} \|([\mathcal{N}(\rho)]^{(1+it)/2} [\mathcal{N}(\sigma)]^{-(1+it)/2} \otimes I_E) U \sigma^{(1+it)/2} \rho^{1/2}\|_1$$

$$= \sup_{t \in \mathbb{R}} \|([\mathcal{N}(\rho)]^{it/2} [\mathcal{N}(\sigma)]^{-it/2} \otimes I_E) U \sigma^{it/2} \rho^{1/2}\|_1$$

$$= \sup_{t \in \mathbb{R}} \sqrt{F}(\rho, (\mathcal{U}_{\rho,-it} \circ \mathcal{R}_{\rho,\mathcal{N}}^P \circ \mathcal{U}_{\mathcal{N}(\sigma),it/2})(\mathcal{N}(\rho)))$$

$$= \left[ \sup_{t \in \mathbb{R}} F(\rho, \mathcal{R}_{\rho,\mathcal{N}}^P(\mathcal{N}(\rho))) \right]^{1/2}.$$  

(3.16)

Then we can apply (2.3) to conclude that

$$\|([\mathcal{N}(\rho)]^{\theta/2} [\mathcal{N}(\sigma)]^{-\theta/2} \otimes I_E) U \sigma^{\theta/2} \rho^{1/2}\|_{2/(1+\theta)} \leq \left[ \sup_{t \in \mathbb{R}} F(\rho, \mathcal{R}_{\rho,\mathcal{N}}^P(\mathcal{N}(\rho))) \right]^{\theta/2}.$$  

(3.17)

\(^2\)Note that boundedness follows from the finite-dimensional assumption—however, the stronger bound $\|G(z)\|_\infty \leq 1$ holds for all $z \in S$, where $S$ is defined in (2.1) (this is a consequence of (3.17), given that the quantum fidelity does not exceed one).
Taking a negative logarithm gives
\[
- \log \left[ \sup_{t \in \mathbb{R}} F(\rho, R_{\sigma, N}^P(\mathcal{N}(\rho))) \right] \leq -\frac{2}{\theta} \log \| (\mathcal{N}(\rho))^{\theta/2} [\mathcal{N}(\sigma)]^{-\theta/2} \otimes I_E \| U\sigma^{\theta/2} \rho^{1/2} \|_{2/(1+\theta)}. \tag{3.18}
\]

Letting \( \theta = (1 - \alpha)/\alpha \), we see that this is the same as
\[
- \log \left[ \sup_{t \in \mathbb{R}} F(\rho, R_{\sigma, N}^P(\mathcal{N}(\rho))) \right] \leq \tilde{\Delta}_\alpha(\rho, \sigma, N). \tag{3.19}
\]

Since the inequality in (3.18) holds for all \( \theta \in (0, 1) \) and thus (3.19) holds for all \( \alpha \in (\frac{1}{2}, 1) \), we can take the limit as \( \alpha \rightarrow 1 \) and apply (3.2) to conclude that (3.10) holds.

We now establish the inequality in (3.13) for \( \rho, \sigma \) and \( \mathcal{N} \) as given in definition 3.2. Note that in this case, \( U \) is a unitary. Pick
\[
G(z) \equiv (\mathcal{N}(\rho))^{-z/2} [\mathcal{N}(\sigma)]^{z/2} \otimes I_E \| U\sigma^{-z/2} \rho^{1/2}, \tag{3.20}
\]

\( p_0 = 2, p_1 = \infty \) and \( \theta \in (0, 1) \), which fixes \( p_0 = 2/(1 - \theta) \). The operator valued-function \( G(z) \) satisfies the conditions needed to apply theorem 2.1. We then find that \( M_0 = 1 \) as before, and
\[
\| G(\theta) \|_{2/(1-\theta)} = \| (\mathcal{N}(\rho))^{\theta/2} [\mathcal{N}(\sigma)]^{\theta/2} \otimes I_E \| U\sigma^{\theta/2} \rho^{1/2} \|_{2/(1-\theta)} \tag{3.21}
\]

\[
M_1 = \sup_{t \in \mathbb{R}} \| G(1 + it) \|_{\infty} = \sup_{t \in \mathbb{R}} \| (\mathcal{N}(\rho))^{-(1+it)/2} [\mathcal{N}(\sigma)]^{(1+it)/2} \otimes I_E \| U\sigma^{-(1+it)/2} \rho^{1/2} \|_{\infty}
\]

\[
= \sup_{t \in \mathbb{R}} \| (\mathcal{N}(\rho))^{-it/2} [\mathcal{N}(\sigma)]^{-1/2} [\mathcal{N}(\sigma)]^{-it/2} \otimes I_E \| U\sigma^{-1/2} \sigma^{-it/2} \rho^{1/2} \|_{\infty}
\]

\[
= \sup_{t \in \mathbb{R}} \| (\mathcal{N}(\rho))^{-1/2} [\mathcal{N}(\sigma)]^{it/2} [\mathcal{N}(\sigma)]^{-1/2} \otimes I_E \| U\sigma^{-1/2} \sigma^{-it/2} \rho^{1/2} \|_{\infty}
\]

\[
= \left[ \exp \sup_{t \in \mathbb{R}} D_{\max}(\rho \| U_{\sigma,-t} \circ R_{\sigma, N}^P \circ U_{\mathcal{N}(\sigma), t} \mathcal{N}(\rho)) \right]^{1/2} \tag{3.22}
\]

Then we can apply (2.3) to conclude that
\[
\| (\mathcal{N}(\rho))^{\theta/2} [\mathcal{N}(\sigma)]^{-\theta/2} \otimes I_E \| U\sigma^{-\theta/2} \rho^{1/2} \|_{2/(1-\theta)} \leq \left[ \exp \sup_{t \in \mathbb{R}} D_{\max}(\rho \| R_{\sigma, N}^{P,t} \mathcal{N}(\rho)) \right]^{\theta/2}. \tag{3.23}
\]

Taking a logarithm gives
\[
\frac{2}{\theta} \log \| (\mathcal{N}(\rho))^{\theta/2} [\mathcal{N}(\sigma)]^{-\theta/2} \otimes I_E \| U\sigma^{-\theta/2} \rho^{1/2} \|_{2/(1-\theta)} \leq \sup_{t \in \mathbb{R}} D_{\max}(\rho \| R_{\sigma, N}^{P,t} \mathcal{N}(\rho)). \tag{3.24}
\]

Letting \( \theta = (\alpha - 1)/\alpha \), we see that this is the same as
\[
\tilde{\Delta}_\alpha(\rho, \sigma, N) \leq \sup_{t \in \mathbb{R}} D_{\max}(\rho \| R_{\sigma, N}^{P,t} \mathcal{N}(\rho)). \tag{3.25}
\]

Since the inequality in (3.24) holds for all \( \theta \in (0, 1) \) and thus (3.25) holds for all \( \alpha \in (1, \infty) \), we can take the limit as \( \alpha \rightarrow 1 \) and apply (3.2) to conclude that (3.13) holds.

**Remark 3.4.** We cannot necessarily conclude which value of \( t \) is optimal in theorem 3.3. However, it is clear that the partial isometric map \( U_{\omega,t} \) preserves the density operator \( \omega \) (i.e. that the partial isometry \( \omega^it \) is diagonal in the eigenbasis of \( \omega \)). Furthermore, the optimal value of \( t \) could have a dependence on the state \( \rho \), which is undesirable for some applications such as approximate quantum error correction.
Remark 3.5. When \( \mathcal{N}(\sigma) \) is positive definite, any recovery map of the form \( R_{\sigma,\mathcal{N}}^{P,t} \) perfectly recovers \( \sigma \) from \( \mathcal{N}(\sigma) \), in the sense that \( (U_{\sigma,t} \circ R_{\sigma,\mathcal{N}}^{P,t} \circ U_{(\sigma,-t)})(\mathcal{N}(\sigma)) = \sigma \), because \( U_{\mathcal{N}(\sigma),-t}(\mathcal{N}(\sigma)) = \mathcal{N}(\sigma) \), \( R_{\sigma,\mathcal{N}}^{P}(\mathcal{N}(\sigma)) = \sigma \) and \( U_{\sigma,t}(\sigma) = \sigma \). This answers an open question discussed in [34]. In particular, we can say that there is a map \( R_{\sigma,\mathcal{N}}^{P,t} \) that perfectly recovers \( \sigma \) from \( \mathcal{N}(\sigma) \), while having a performance limited by (3.10) when recovering \( \rho \) from \( \mathcal{N}(\rho) \).

Remark 3.6. From (3.19) in the proof given above, we can conclude that

\[
\tilde{\Delta}_a(\rho, \sigma, \mathcal{N}) \geq -\log \left[ \sup_{t \in \mathbb{R}} F(\rho, R_{\sigma,\mathcal{N}}^{P,t}(\mathcal{N}(\rho))) \right],
\]

(3.26)

for all \( \alpha \in (\frac{1}{2}, 1) \). This inequality improves upon a previous result from [29], which established that \( \tilde{\Delta}_a \) is non-negative for the same range of \( \alpha \). One also sees that \( \tilde{\Delta}_a(\rho, \sigma, \mathcal{N}) = 0 \) implies that the channel \( \mathcal{N} \) is sufficient for \( \rho \) and \( \sigma \), in the sense that this condition implies the existence of a recovery map which perfectly recovers \( \rho \) from \( \mathcal{N}(\rho) \) and \( \sigma \) from \( \mathcal{N}(\sigma) \).

4. Functoriality

For a fixed \( t \), recovery maps \( R_{\sigma,\mathcal{N}}^{P,t} \) of the form in (3.11) satisfy several desirable ‘functoriality’ properties stated in [20], in addition to the property stated in remark 3.5. These include normalization, parallel composition and serial composition, which we discuss in the following subsections.

(a) Normalization

If there is in fact no noise, so that \( \mathcal{N} = \text{id} \), then we would expect the recovery map to be equal to the identity channel as well. This property is known as normalization [20], and we confirm it below for all maps \( R_{\sigma,\mathcal{N}}^{P,t} \) of the form in (3.11) when \( \mathcal{N} = \text{id} \):

\[
R_{\sigma,\text{id}}^{P,t}(\cdot) = (U_{\sigma,t} \circ R_{\sigma,\text{id}}^{P} \circ U_{\text{id},-t})(\cdot)
\]

\[= \sigma^{it} \sigma_{1/2}^{1/2} \text{id}^{1/2}(\sigma)^{-1/2} \text{id}(\sigma)^{-1/2} [\sigma^{it} \sigma_{1/2}^{1/2} \text{id}^{1/2} \text{id}(\sigma)]^it \text{id}(\sigma)^{1/2} \sigma^{-1/2} \sigma^{-it} = \sigma^{it} \sigma^{1/2} \sigma^{-1/2} \sigma^{-1/2} \sigma^{-it} = \Pi_{\sigma}(\cdot) \Pi_{\sigma}.
\]

(4.1)

Thus, when \( \sigma \) is positive definite, the recovery map is the identity channel.

(b) Parallel composition

If the \( \sigma \) operator is a tensor product \( \sigma_1 \otimes \sigma_2 \) and the channel \( \mathcal{N} \) is as well \( \mathcal{N}_1 \otimes \mathcal{N}_2 \) (respecting the same tensor-product structure), then it would be desirable for the recovery map to be a tensor product respecting this structure. This property is known as parallel composition [20], and we confirm it below for all maps \( R_{\sigma,\mathcal{N}}^{P,t} \) of the form in (3.11) when \( \sigma = \sigma_1 \otimes \sigma_2 \) and \( \mathcal{N} = \mathcal{N}_1 \otimes \mathcal{N}_2 \). In fact, this property is a consequence of the following:

\[
U_{\sigma_1 \otimes \sigma_2,t}(\cdot) = [\sigma_1 \otimes \sigma_2]^{-it}(\cdot)[\sigma_1 \otimes \sigma_2]^{it} = [\sigma_1]^{it} \otimes [\sigma_2]^{it}\]

(4.2)

and

\[
R_{\sigma_1 \otimes \sigma_2,\mathcal{N}_1 \otimes \mathcal{N}_2}^{P,t}(\cdot) = (R_{\sigma_1,\mathcal{N}_1}^{P} \otimes R_{\sigma_2,\mathcal{N}_2}^{P})(\cdot),
\]

(4.3)

where (4.3) follows because

\[
[\sigma_1 \otimes \sigma_2]^{1/2} = \sigma_{1/2}^{1/2} \otimes \sigma_{2/2}^{1/2}, \quad (\mathcal{N}_1 \otimes \mathcal{N}_2)^{1/2} = \mathcal{N}_1^{1/2} \otimes \mathcal{N}_2^{1/2}\]

(4.4)

and

\[
((\mathcal{N}_1 \otimes \mathcal{N}_2)(\sigma_1 \otimes \sigma_2))^{-1/2} = [\mathcal{N}_1(\sigma_1)]^{-1/2} \otimes [\mathcal{N}_2(\sigma_2)]^{-1/2}.
\]

The following equality results from similar reasoning as in (4.2):

\[
U_{(\mathcal{N}_1 \otimes \mathcal{N}_2)(\sigma_1 \otimes \sigma_2),t}(\cdot) = (U_{\mathcal{N}_1(\sigma_1),t} \otimes U_{\mathcal{N}_2(\sigma_2),t})(\cdot).
\]

(4.5)

Putting everything together, parallel composition holds: \( R_{\sigma_1 \otimes \sigma_2,\mathcal{N}_1 \otimes \mathcal{N}_2}^{P,t}(\cdot) = (R_{\sigma_1,\mathcal{N}_1}^{P} \otimes R_{\sigma_2,\mathcal{N}_2}^{P})(\cdot) \).
(c) Serial composition

If the channel \( \mathcal{N} \) consists of the serial composition of two channels \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \), so that \( \mathcal{N} = \mathcal{N}_2 \circ \mathcal{N}_1 \), then it would be desirable for the recovery map to consist of recovering from the last channel first and then from the first channel. This property is known as serial composition [20], and we confirm it below for all maps \( \mathcal{R}_{\mathcal{N}}^{P_f} \) of the form in (3.11) when \( \mathcal{N} = \mathcal{N}_2 \circ \mathcal{N}_1 \). It is a consequence of the fact that \( (\mathcal{N}_2 \circ \mathcal{N}_1)^\dagger = \mathcal{N}_1^\dagger \circ \mathcal{N}_2^\dagger \), so that

\[
\mathcal{R}_{\mathcal{N} \circ \mathcal{N}_1}^{P_f}(\cdot) = \sigma^{1/2}(\mathcal{N}_1^\dagger \circ \mathcal{N}_2^\dagger)(\mathcal{N}_2 \circ \mathcal{N}_1)(\sigma)^{-1/2} = \sigma^{1/2}(\mathcal{N}_1^\dagger (\mathcal{N}_1(\sigma)^{-1/2}[\mathcal{N}_2(\mathcal{N}_1(\sigma))^{-1/2}]\mathcal{N}_2(\mathcal{N}_1(\sigma))^{-1/2})\mathcal{N}_1(\sigma))^{-1/2} \sigma^{1/2}
\]

Then the serial composition property follows because

\[
\mathcal{R}_{\mathcal{N}_2 \circ \mathcal{N}_1}^{P_f}(\cdot) = (\mathcal{R}_{\mathcal{N}_2}^{P_f} \circ \mathcal{R}_{\mathcal{N}_1}^{P_f})(\cdot)
\]

where \( \mathcal{R}_{\mathcal{N}_1}^{P_f} \) is the map that recovers from \( \mathcal{N}_2 \) and \( \mathcal{R}_{\mathcal{N}_1}^{P_f} \) is the map that recovers from \( \mathcal{N}_1 \).

5. Consequences and applications of theorem 3.3

Theorem 3.3 leads to a strengthening of many entropy inequalities, including strong subadditivity of quantum entropy, concavity of conditional entropy, joint convexity of relative entropy, non-negativity of quantum discord, the Holevo bound and multipartite information inequalities. We list these as corollaries and give brief proofs for them in the following subsections. Furthermore, there are potential applications to approximate quantum error correction as well, which we discuss. Some of the observations given below have been made before in previous papers as either conjectures or concrete results [18,19,21,22,26] (with many of them becoming concrete after the posting of [18]), but in many cases, theorem 3.3 allows us to make more precise statements due to the structure of the recovery map \( \mathcal{R}_{\mathcal{N}}^{P_f} \) in (3.11) and its functoriality properties discussed in the previous section.

(a) Strong subadditivity

The conditional quantum mutual information of a tripartite state \( \rho_{ABC} \) is defined as

\[
I(A;B|C)_\rho \equiv H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho,
\]

where \( H(F)_\sigma \equiv -\text{Tr}[\sigma \log \sigma] \) is the von Neumann entropy of a density operator \( \sigma_F \). Strong subadditivity is the statement that \( I(A;B|C)_\rho \geq 0 \) for all tripartite states \( \rho_{ABC} \) [9,10].

Corollary 5.1 gives an improvement of strong subadditivity, in addition to providing a bound on conditional mutual information. It is a direct consequence of theorem 3.3 after choosing \( \rho = \rho_{ABC} \), \( \sigma = \rho_{AC} \otimes I_B \), and \( \mathcal{N} = \text{Tr}_A \), so that \( \mathcal{N}(\rho) = \rho_{BC} \), \( \mathcal{N}(\sigma) = \rho_C \otimes I_B \), \( N^\dagger(\cdot) = (\cdot) \otimes I_A \), and

\[
D(\rho||\sigma) - D(\mathcal{N}(\rho)||\mathcal{N}(\sigma)) = D(\rho_{ABC}||\rho_{AC} \otimes I_B) - D(\rho_{BC}||\rho_C \otimes I_B) = I(A;B|C)_\rho,
\]

where \( \mathcal{R}_{C \rightarrow AC}^{P} \) is a special case of the Petz recovery map [12,13] (see also [20]).
Corollary 5.1. Let $\rho_{ABC}$ be a density operator acting on a finite-dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Then the following inequality holds:

$$I(A;B|C)_\rho \geq -\log \left[ \sup_{t \in \mathbb{R}} F(\rho_{ABC}, \mathcal{R}_{C \rightarrow AC}^P(\rho_{BC})) \right],$$

(5.4)

where $\mathcal{R}_{C \rightarrow AC}^P$ is the following rotated Petz recovery map:

$$\mathcal{R}_{C \rightarrow AC}^P(\cdot) \equiv (U_{\rho_{AC},t} \circ \mathcal{R}_C^P \circ U_{\rho_{C,-t}})(\cdot),$$

(5.5)

the Petz recovery map $\mathcal{R}_C^P$ is defined in (5.3), and the partial isometric maps $U_{\rho_{AC},t}$ and $U_{\rho_{C},-t}$ are defined from (3.12). If $\rho_{ABC}$ is positive definite, then the following inequality holds as well:

$$I(A;B|C)_\rho \leq \sup_{t \in \mathbb{R}} D_{\text{max}}(\rho_{ABC} \| \mathcal{R}_{C \rightarrow AC}^P(\rho_{BC})).$$

(5.6)

Remark 5.2. A lower bound on $I(A;B|C)_\rho$, similar to that in corollary 5.1 has already been identified (O. Fawzi, R. Renner, V. Scholz and D. Sutter 2015, private communication) by putting together various statements from [18,19]. One needs to examine the discussion surrounding eqns (120)–(122) in [19] and make several further arguments in order to arrive at this conclusion. See [19, remark 2.5].

Remark 5.3. We note that $\hat{\Delta}_\alpha(\rho, \sigma, N)$ for the choices above reduces to the Rényi conditional mutual information from [26]:

$$\hat{I}_\alpha(A;B|C)_\rho \equiv \frac{2\alpha}{\alpha - 1} \log \| \rho_{BC}^{(1-\alpha)/2\alpha} \rho_C^{(\alpha - 1)/2\alpha} \rho_{AC}^{(1-\alpha)/2\alpha} \rho_{ABC}^{1/2} \rho_{ABC}^{2\alpha} \|_2,$$

(5.7)

as observed in [21]. Thus, from the inequality in (3.19), we can conclude that

$$\hat{I}_\alpha(A;B|C)_\rho \geq -\log \left[ \sup_{t \in \mathbb{R}} F(\rho_{ABC}, \mathcal{R}_{C \rightarrow AC}^P(\rho_{BC})) \right],$$

(5.8)

for all $\alpha \in (\frac{1}{2}, 1)$. This inequality improves upon a previous result from [26], which established that $\hat{I}_\alpha$ is non-negative for the same range of $\alpha$.

Remark 5.4. A statement similar to the first two sentences of remark 3.4 applies as well to the partial isometric maps in corollary 5.1. Furthermore, the parameter $t$ has a dependence on the global state $\rho_{ABC}$, so that the recovery map given above does not possess the universality property discussed in [19]. Regardless, the structure of the unitaries is sufficient for us to conclude that any recovery map of the form $U_{\rho_{AC},t} \circ \mathcal{R}_C^P \circ U_{\rho_{C,-t}}$ always perfectly recovers the state $\rho_{AC}$ from $\rho_C$: $(U_{\rho_{AC},t} \circ \mathcal{R}_C^P \circ U_{\rho_{C},-t})(\rho_C) = \rho_{AC}$, because $U_{\rho_{C},-t}(\rho_C) = \rho_C$, $\mathcal{R}_C^P(\rho_C) = \rho_{AC}$ and $U_{\rho_{AC},t}(\rho_{AC}) = \rho_{AC}$.

(b) Concavity of conditional quantum entropy

The ideas in this section follow a line of thought developed in [34]. Let $\mathcal{E} \equiv \{p_X(x), \rho_{AB}^x\}$ be an ensemble of bipartite quantum states with expectation $\tilde{\rho}_{AB} \equiv \sum_x p_X(x) \rho_{AB}^x$. Concavity of conditional entropy is the statement that

$$H(A|B)_{\tilde{\rho}} \geq \sum_x p_X(x) H(A|B)_{\rho^x},$$

(5.9)

where the conditional quantum entropy $H(A|B)_{\sigma}$ is defined for a state $\sigma_{AB}$ as $H(A|B)_{\sigma} \equiv H(AB)_{\sigma} - H(B)_{\sigma} = -D(\sigma_{AB} \| I_A \otimes \sigma_B)$. 


Let $\omega_{XAB}$ denote the following classical-quantum state in which we have encoded the ensemble $\mathcal{E}$:

$$
\omega_{XAB} \equiv \sum_x p_X(x)|x\rangle_X \otimes \rho_{AB}^x.
$$

(5.10)

We can rewrite

$$
H(A|B)_{\tilde{\rho}} - \sum_x p_X(x)H(A|B)_{\rho^x} = H(A|B)_{\omega} - H(A|BX)_{\omega} = I(A;X|B)_{\omega}
$$

$$
= H(X|B)_{\omega} - H(X|AB)_{\omega} \leq D(\omega_{XAB}||I_X \otimes \omega_{AB}) - D(\omega_{XAB}||I_X \otimes \omega_B).
$$

(5.11)

We can see the last line above as a relative entropy difference (as defined in the right-hand side of (3.2)) by picking $\rho = \omega_{XAB}$, $\sigma = I_X \otimes \omega_{AB}$ and $\mathcal{N} = \text{Tr}_A$. Applying theorem 3.3, (3.5) and (3.9), we find the following improvement of concavity of conditional entropy.

**Corollary 5.5.** Let an ensemble $\mathcal{E}$ be as given above. Then the following inequality holds:

$$
H(A|B)_{\tilde{\rho}} - \sum_x p_X(x)H(A|B)_{\rho^x} \geq - \log \sup_{t \in \mathbb{R}} \sum_x p_X(x)\sqrt{T(\rho_{AB}, R_{\rho_{AB|\mathcal{N}}}^x(\rho_B))},
$$

(5.12)

where the recovery map $R_{\rho_{AB|\mathcal{N}}}^x$ is defined from (3.11) and perfectly recovers $\tilde{\rho}_{AB}$ from $\tilde{\rho}_B$. If the states in the ensemble are positive definite, then the following inequality holds:

$$
H(A|B)_{\tilde{\rho}} - \sum_x p_X(x)H(A|B)_{\rho^x} \leq \sup_{t \in \mathbb{R}} D_{\text{max}}(\rho_{AB}^x || R_{\sigma_{AB|\mathcal{N}}}^x(R_B^x)).
$$

(5.13)

**Corollary 5.6.** Let ensembles be as given above. Then the following inequalities hold:

$$
\sum_x p_X(x)D(\rho_x || \sigma_x) - D(\tilde{\rho} || \tilde{\sigma}) \geq - \log \sup_{t \in \mathbb{R}} F(\rho_{XB}, R_{\sigma_{XB|\mathcal{N}}}^x(\tilde{\sigma})),
$$

(5.15)

where the recovery map $R_{\sigma_{XB|\mathcal{N}}}^x$ is defined from (3.11) and perfectly recovers $\sigma_{XB}$ from $\sigma_B$. If the operators are positive definite, then

$$
\sum_x p_X(x)D(\rho_x || \sigma_x) - D(\tilde{\rho} || \tilde{\sigma}) \leq \sup_{t \in \mathbb{R}} D_{\text{max}}(\rho_{XB}^x || R_{\sigma_{XB|\mathcal{N}}}^x(R_B^x)).
$$

(5.16)

**Remark 5.7.** The recovery map $R_{\sigma_{XB|\mathcal{N}}}^x$ from corollary 5.6 can be understood as a rotated ‘pretty good’ measurement [35–37]. By using the facts that

$$
\rho_{XB} = \bigoplus_x p_X(x)\rho_x \quad \text{and} \quad \sigma_{XB} = \bigoplus_x p_X(x)\sigma_x,
$$

(5.17)

we have for all $z \in \mathbb{C}$ that

$$
\tilde{\rho}_{XB}^{z} = \bigoplus_x [p_X(x)\rho_x]^z \quad \text{and} \quad \tilde{\sigma}_{XB}^{z} = \bigoplus_x [p_X(x)\sigma_x]^z.
$$

(5.18)
This allows us to write the recovery map as the following trace-non-increasing instrument (a trace-non-increasing linear map with a classical and quantum output):

$$R_{σ_X,Tr_X}^{P,f}(\cdot) = \sum_x |x⟩⟨x|_X \otimes p_X(x)[p_X(x)σ_X]_f^{1/2}⟨\tilde{σ}⟩^{-1/2}⟨\tilde{σ}⟩^{-i\cdot}⟨\tilde{σ}⟩^{-1/2}σ_X^{1/2}[p_X(x)σ_X]^{-i\cdot}. \quad (5.19)$$

Tracing over the quantum system then gives the following rotated pretty good measurement:

$$⟨\cdot⟩ \to \sum_x Tr((\tilde{σ})^{-i\cdot}p_X(x)σ_X(\tilde{σ})^{-1/2}σ^{-i\cdot})|x⟩⟨x|_X, \quad (5.20)$$

which should be compared with the measurement map corresponding to the pretty good measurement:

$$⟨\cdot⟩ \to \sum_x Tr((\tilde{σ})^{-i\cdot}p_X(x)σ_X(\tilde{σ})^{-1/2}σ^{-i\cdot})|x⟩⟨x|_X. \quad (5.21)$$

### (d) Non-negativity of quantum discord

The ideas in this section follow a line of thought developed in [21,22]. Let $ρ_{AB}$ be a bipartite density operator and let $|φ_x⟩⟨φ_x|_A$ be a rank-one quantum measurement on system $A$ (i.e. the vectors $|φ_x⟩_A$ satisfy $∑_x |φ_x⟩⟨φ_x|_A = I_A$). It suffices for us to consider rank-one measurements for our discussion here because every quantum measurement can be refined to have a rank-one form, such that it delivers more classical information to the experimentalist observing the apparatus. Then the (unoptimized) quantum discord is defined to be the difference between the following mutual informations [38,39]: $I(A;B)_\rho - I(X;B)_ω$, where $I(A;B)_\rho ≡ D(ρ_{AB}∥ρ_A ⊗ ρ_B), \ M_{A→X}(\cdot) ≡ \sum_x |φ_x⟩⟨φ_x|_A|x⟩⟨x|_X$ and $ω_{XB} ≡ M_{A→X}(ρ_{AB})$. The quantum channel $M_{A→X}$ is a measurement channel, so that the state $ω_{XB}$ is the classical-quantum state resulting from the measurement. The set $|x⟩_X$ is an orthonormal basis so that $X$ is a classical system. The quantum discord is known to be non-negative, and by applying theorem 3.3 we find the following improvement of this entropy inequality.

**Corollary 5.8.** Let $ρ_{AB}$ and $M_{A→X}$ be as given above. Then the following inequalities hold:

$$I(A;B)_\rho - I(X;B)_ω ≥ - \log sup_{t \in \mathbb{R}} F(ρ_{AB},(U_{ρ_A,t} \circ E_A)(ρ_{AB})), \quad (5.22)$$

where $E_A$ is an entanglement-breaking map of the following form:

$$⟨\cdot⟩_A → \sum_x ⟨φ_x|A(\cdot)|φ_x⟩_A \frac{ρ_A^{1/2}|φ_x⟩⟨φ_x|_Aρ_A^{1/2}}{|φ_x⟩_A⟨φ_x|_A}, \quad (5.23)$$

and the partial isometric map $U_{ρ_A,t}$ is defined from (3.12). The recovery map $U_{ρ_A,t} \circ E_A$ perfectly recovers $ρ_A$ from $M_{A→X}(ρ_A)$.

**Proof.** We start with the rewriting $I(A;B)_\rho - I(X;B)_ω = D(ρ_{AB}∥ρ_A ⊗ I_B) - D(ω_{XB}∥ω_X ⊗ I_B)$, and follow by picking $ρ = ρ_{AB}, σ = ρ_A ⊗ I_B$, and $N = M_{A→X},$ and applying theorem 3.3. This then shows the corollary with a recovery map of the form $R_{ρ_A,M_{A→X}}^{P,f} \circ M_{A→X}$.

As observed in [21,22], the concatenation $R_{ρ_A,M_{A→X}}^{P,f} \circ M_{A→X}$ is an entanglement-breaking channel [40] because it consists of a measurement channel $M_{A→X}$ followed by a preparation. We now work out the form for the recovery map given in (5.22). Consider that $M_{A→X}(ρ_A) = ∑_x |φ_x⟩⟨φ_x|_A|x⟩⟨x|_X$, so that

$$U_{M_{A→X}(ρ_A),−t}⟨\cdot⟩ = \left[ \sum_x [⟨φ_x|Aρ_A|φ_x⟩_A]^{-i\cdot}|x⟩⟨x|_X \right] (\cdot) \left[ \sum_x [⟨φ_x′|Aρ_A|φ_x⟩_A]^{-i\cdot}|x′⟩⟨x′|_X \right]. \quad (5.24)$$
Thus, when composing \( M_{A \rightarrow X} \) with \( U_{M_{A \rightarrow X}(\rho_A) \rightarrow t} \), the phases cancel out to give the following relation: \( U_{M_{A \rightarrow X}(\rho_A) \rightarrow t}(M_{A \rightarrow X}(\cdot)) = M_{A \rightarrow X}(\cdot) \). One can then work out that

\[
R_{\rho_A M_{A \rightarrow X}}^P (M_{A \rightarrow X}(\cdot)) = \rho_A^{1/2} M_{A \rightarrow X}(\cdot) \left[ M_{A \rightarrow X}(\rho_A) \right]^{-1/2} \left[ M_{A \rightarrow X}(\cdot) \right]^{-1/2} \rho_A^{1/2}
\]

(5.25)

\[
= \sum_x \langle \varphi_x | A(\cdot) \rangle \varphi_x \rho_A^{1/2} \left( \varphi_x \rho_A | \varphi_x \rangle \right) A
\]

(5.26)

**Remark 5.9.** If the discord is equal to zero, then by corollary 5.8 there exists an entanglement-breaking channel acting on system \( A \) for which \( \rho_{AB} \) is a fixed point. By applying [41, theorem 5.3], we can conclude in such a case that \( \rho_{AB} \) is a classical-quantum state, so that we recover the known result that a state with zero discord is classical-quantum.

(e) **Holevo bound**

The ideas in this section follow a line of thought developed in [21]. The Holevo bound [42] is a special case of the non-negativity of quantum discord in which \( \rho_{AB} \) is a quantum-classical state, which we write explicitly as

\[
\rho_{AB} = \sum_y p_Y(y) \rho_A^y \otimes |y\rangle \langle y|,
\]

(5.27)

where each \( \rho_A^y \) is a density operator, so that \( \rho_A = \sum_y p_Y(y) \rho_A^y \). The Holevo bound states that the mutual information of the state \( \rho_{AB} \) in (5.27) is never smaller than the mutual information after system \( A \) is measured. By applying corollary 5.8 and (3.5), we find the following improvement.

**Corollary 5.10.** Let \( \rho_{AB} \) be as in (5.27), and let \( M_{A \rightarrow X} \) and \( \omega_{XB} \) be as in §5d, respectively. Then the following inequality holds:

\[
I(A;B)_\rho - I(X;B)_\omega \geq -2 \log \sup_{t \in \mathbb{R}} \sum_y p_Y(y) \sqrt{\langle \rho_A^y (\mathcal{U}_{\rho_{A,t}} \circ \mathcal{E}_A)(\rho_A^y) \rangle},
\]

(5.28)

where \( \mathcal{E}_A \) is an entanglement-breaking map of the form in (5.23) and the partial isometric map \( \mathcal{U}_{\rho_{A,t}} \) is defined from (3.12).

**Remark 5.11.** If the Holevo bound is saturated, then by corollary 5.10 there exists an entanglement-breaking channel acting on system \( A \) for which \( \rho_{AB} \) is a fixed point. By applying [41, theorem 5.3], we can conclude in such a case that \( \rho_{AB} \) is a classical–classical state, so that we recover the known result that the Holevo bound is saturated when all states in the ensemble are commuting.

(f) **Differences of quantum multipartite informations**

The quantum multipartite information of a multipartite state \( \omega_{B_1 \cdots B_l} \) is defined as [43,44]:

\[
I(B_1 : \cdots : B_l)_\omega = \sum_{i=1}^l H(B_i)_\omega - H(B_1 \cdots B_l)_\omega = D(\omega_{B_1 \cdots B_l} || \omega_{B_1} \otimes \cdots \otimes \omega_{B_l}).
\]

(5.29)

One can then use this to define a difference of multipartite informations for a state \( \rho_{A_1 A'_1 \cdots A_l A'_l} \) as

\[
I(A_1 A'_1 : \cdots : A_l A'_l)_\rho - I(A_1 : \cdots : A'_l)_\rho,
\]

(5.30)

using which multipartite entanglement [45] and discord-like [46] measures can be constructed. By picking \( \rho = \rho_{A_1 A'_1 \cdots A_l A'_l}, \sigma = \rho_{A_1 A'_1} \otimes \cdots \otimes \rho_{A_l A'_l} \) and \( \mathcal{N} = \text{Tr}_{A_1 \cdots A_l} \) and applying theorem 3.3, we establish the following corollary, which solves in the affirmative the open question posed in eqn (7.8) of [23]:
Corollary 5.12. Let \( \rho_{A_1 A_2 \cdots A_l} \) be a multipartite quantum state. Then the following inequality holds:

\[
I(A_1 A_2 \cdots A_l | C)_\rho - I(A_1 \cdots A_l | C)_\rho \\
\geq - \log \sup_{t \in \mathbb{R}} F \left( \rho_{A_1 A_2 \cdots A_l}, (R^{P,f}_{\rho_{A_1 A_2 \cdots A_l} P, A_1 A_2 \cdots A_l} \otimes \cdots \otimes R^{P,f}_{\rho_{A_1 A_2 \cdots A_l} P, A_1 A_2 \cdots A_l}) (\rho_{A_1 A_2 \cdots A_l}) \right). \tag{5.31}
\]

If the state is positive definite, then the following inequality holds as well:

\[
I(A_1 A_2 \cdots A_l | C)_\rho - I(A_1 \cdots A_l | C)_\rho \\
\leq \sup_{t \in \mathbb{R}} D_{\max} \left( \rho_{A_1 A_2 \cdots A_l} \left\| R^{P,f}_{\rho_{A_1 A_2 \cdots A_l} P, A_1 A_2 \cdots A_l} \otimes \cdots \otimes R^{P,f}_{\rho_{A_1 A_2 \cdots A_l} P, A_1 A_2 \cdots A_l} \right) (\rho_{A_1 A_2 \cdots A_l}) \right). \tag{5.32}
\]

(g) General quantum information measures

We remark here that the method given in the proof of theorem 3.3 can be applied quite generally, even to information quantities which cannot be written as a difference of relative entropies. To do so, one needs to follow the recipe outlined in [47] for obtaining a Rényi generalization of the entropic quantity of interest and then apply the same methods of complex interpolation used in the proof of theorem 3.3. The bounds that one ends up with might not necessarily have a physical interpretation in terms of recoverability, but it does happen in some cases.

One example of an information quantity for which this does happen and for which it is not clear how to write it in terms of a relative entropy difference is the conditional multipartite information of a state \( \rho_{A_1 \cdots A_l C} \):

\[
I(A_1 : \cdots : A_l | C)_\rho \equiv \sum_{i=1}^{l} I(A_i | C)_\rho - I(A_1 \cdots A_l | C)_\rho. \tag{5.33}
\]

This quantity can be used to define squashed-like entanglement measures [48,49]. It is not clear how to write this as a relative entropy difference, but one can follow the recipe given in [47] to find the following Rényi generalization:

\[
\tilde{I}_\alpha(A_1 : \cdots : A_l | C)_\rho \equiv \frac{2}{\alpha'} \log \parallel \rho_{A_1 \cdots A_l C}^{- \alpha'/2} \rho_{A_1 \cdots A_l C}^{- \alpha'/2} \cdots \rho_{A_1 \cdots A_l C}^{- \alpha'/2} \rho_{A_1 \cdots A_l C}^{- \alpha'/2} \parallel_{2\alpha'}, \tag{5.34}
\]

where \( \alpha' = (\alpha - 1)/\alpha \). The quantity \( \tilde{I}_\alpha(A_1 : \cdots : A_l | C) \) converges to \( I(A_1 : \cdots : A_l | C) \) in the limit as \( \alpha \to 1 \). For \( \alpha = 1/2 \) and in the limit as \( \alpha \to \infty \), \( \tilde{I}_\alpha(A_1 : \cdots : A_l | C) \) reduces to

\[
\tilde{I}_{1/2}(A_1 : \cdots : A_l | C)_\rho = - \log F(\rho_{A_1 \cdots A_l C} (R^P_{C \to A_1 C} \circ \cdots \circ R^P_{C \to A_2 C}) (\rho_{A_1 C})). \tag{5.35}
\]

and

\[
\tilde{I}_\infty(A_1 : \cdots : A_l | C)_\rho = D_{\max}(\rho_{A_1 \cdots A_l C} \left\| (R^P_{C \to A_1 C} \circ \cdots \circ R^P_{C \to A_2 C}) (\rho_{A_1 C}) \right), \tag{5.36}
\]

where \( R^P_{C \to A_1 C} \) is a Petz recovery map of the form \( \rho \to \rho_{A_1 C}^{-1/2} \rho_{A_1 C}^{-1/2} \rho_{A_1 C}^{-1/2} \rho_{A_1 C}^{-1/2} \rho_{A_1 C}^{-1/2} \) and the latter equality requires the state to be positive definite. The quantities above have an interpretation in terms of sequential recoverability, that is where one attempts to use the system \( C \) repeatedly in order to retrieve all of the \( A_1 \) systems one-by-one, for \( i \in \{2, \ldots, l\} \). One can exploit the method of proof for theorem 3.3 to obtain the following bounds.

Theorem 5.13. Let \( \rho_{A_1 \cdots A_l C} \) be a density operator. Then the following inequalities hold:

\[
I(A_1 : \cdots : A_l | C)_\rho \geq - \log \sup_{t \in \mathbb{R}} F(\rho_{A_1 \cdots A_l C} (R^P_{C \to A_1 C} \circ \cdots \circ R^P_{C \to A_2 C}) (\rho_{A_1 C})), \tag{5.37}
\]

where \( R^P_{C \to A_1 C} \) is a rotated Petz recovery map of the form in (5.5). If the state is positive definite, then

\[
I(A_1 : \cdots : A_l | C)_\rho \leq \sup_{t \in \mathbb{R}} D_{\max}(\rho_{A_1 \cdots A_l C} \left\| (R^P_{C \to A_1 C} \circ \cdots \circ R^P_{C \to A_2 C}) (\rho_{A_1 C}) \right). \tag{5.38}
\]

Remark 5.14. An advantage of the bound given above is that there is no prefactor depending on the number of parties involved, which one obtains by applying the chain rule and the bound
from [18] multiple times (cf. [20, appendix B]). Furthermore, it is not known how to apply the methods of Fawzi & Renner [18] and Sutter et al. [19] in order to arrive at a bound of the form above (which does not have a prefactor depending on the number of parties).

(h) Approximate quantum error correction

The goal of quantum error correction is to protect quantum information from the deleterious effects of a quantum channel \( \mathcal{N} \) by encoding it into a subspace of the full Hilbert space, such that one can later recover the encoded data after performing a recovery operation. In physical situations, one can never have perfect error correction and instead aims for approximate error correction (see [50] and references therein). In more detail, let \( \mathcal{H} \) be a Hilbert space and let \( \Pi \) be a projection onto some subspace of \( \mathcal{H} \), which is referred to as the codespace. Suppose that \( \rho \) is a density operator with support only in the subspace onto which \( \Pi \) projects. Then, by choosing \( \rho = \rho', \sigma = \Pi \) and \( \mathcal{N} = \mathcal{N}_t \), and applying theorem 3.3, we find that the following inequality holds:

\[
D(\rho || \Pi) - D(\mathcal{N}(\rho)||\mathcal{N}(\Pi)) \geq -\log \left[ \sup_{t \in \mathbb{R}} F(\rho, \mathcal{R}^{\Pi}_{\Pi,\mathcal{N}}(\mathcal{N}(\rho))) \right],
\]

where \( \mathcal{R}^{\Pi}_{\Pi,\mathcal{N}} \) is the following rotated Petz recovery map: \( \mathcal{R}^{\Pi}_{\Pi,\mathcal{N}}(\cdot) \equiv (\mathcal{U}_{\Pi,t} \circ \mathcal{R}^{\Pi}_{\Pi,\mathcal{N}} \circ \mathcal{U}_{\mathcal{N}(\Pi),-t})(\cdot) \), \( \mathcal{R}^{\Pi}_{\Pi,\mathcal{N}}(-) = \Pi \mathcal{N}^t([\mathcal{N}(\Pi)]^{-1/2}(-)[\mathcal{N}(\Pi)]^{-1/2}) \), and \( \mathcal{U}_{\Pi,t} \) and \( \mathcal{U}_{\mathcal{N}(\Pi),-t} \) are partial isometric maps defined from (3.12) and acting as the identity outside the support of \( \Pi \) and \( \mathcal{N}(\Pi) \), respectively.

The inequality above is not particularly useful in the context of approximate quantum error correction, but we have stated it to motivate further developments. In particular, the recovery map \( \mathcal{R}^{\Pi}_{\Pi,\mathcal{N}} \) has a dependence on the particular state \( \rho \) being sent through the channel. Of course, the receiver does not know which state is being transmitted through the channel at any given instant and thus cannot apply the decoder given in the above bound. Nor should we allow the encoder to know which state \( \rho \) is being transmitted and send \( t \) via a noiseless classical channel. Thus, it would be ideal if the bound stated above would hold for a recovery map that has no dependence on the input (that is, if the recovery map were to have a universality property, similar to that discussed in [19] and remark 3.4). Some possibilities for universal recovery maps are the Petz recovery map itself or an averaged channel, where \( t \) is chosen randomly according to some distribution. Conjecture 26 of [21], if true, would imply the bound given above for the choice \( t = 0 \) (the Petz recovery map).

6. Discussion

Entropy inequalities such as strong subadditivity of quantum entropy [9,10] and monotonicity of quantum relative entropy [7,8] have played a fundamental role in quantum information theory and other areas of physics. Establishing entropy inequalities with physically meaningful remainder terms has been a topic of recent interest in quantum information theory (see [18–20, 19 and references therein]). A breakthrough result from [18] established an inequality of the form in corollary 5.1, with however essentially nothing being known about the input and output unitaries. The methods of Fawzi & Renner [18] were generalized in [34] to produce an inequality of the form in (3.10), again with essentially nothing known about the input and output unitaries. Meanwhile, operational proofs for physically meaningful lower bounds on conditional mutual information have appeared as well [51,52], the latter in part based on the notion of fidelity of recovery [22]. Recent work has now established that a recovery map for the conditional mutual information can possess a universality property [19], in the sense that it need not depend on the state of the system \( \mathcal{B} \) (the system that is not ‘lost and recovered’ nor ‘used to recover’). As discussed in remark 3.4, the recovery map given in [5.4] does not possess the universality property. Also, the structure of the input and output unitaries given in (5.4) is essentially the same as that found in previous work [18,19] (see the discussion around eqns (120–(122) of [19]). However, the argument given here to
arrive at this structure is more direct than that in [18,19] and applies as well to the recovery map in (3.10) for a relative entropy difference.

An important open question is to determine whether we could take \( t = 0 \) and still have the inequalities hold, as conjectured previously [20,21,26]. More generally, it is still open to determine whether the Rényi quantities in (3.1) and (5.7) are monotone non-decreasing with respect to the Rényi parameter \( \alpha \).

In light of the efforts put into addressing the recovery question, it is pleasing that the Hadamard three-line theorem leads to simple proofs. This theorem has already been put to good use in characterizing local state transformations [53] and in obtaining chain rules for Rényi entropies [28], for example, and it should be interesting to find further applications of it in the context of quantum information theory. A recent application, related to the developments here, is in the study of swiveled Rényi entropies [54].

Data accessibility. This work does not have any experimental data.

Competing interests. I declare I have no competing interests.

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Appendix A. Convergence to the quantum relative entropy difference

**Definition A.1.** Let \( \rho, \sigma \) and \( \mathcal{N} \) be as given in definition 3.1. For \( \alpha \in (0,1) \cup (1,\infty) \), let

\[
\hat{\Delta}_\alpha(\rho,\sigma,\mathcal{N}) = \frac{1}{\alpha - 1} \log \hat{Q}_\alpha(\rho,\sigma,\mathcal{N}),
\]

where

\[
\hat{Q}_\alpha(\rho,\sigma,\mathcal{N}) \equiv \|\mathcal{N}(\rho)^{(1-\alpha)/2\alpha} \mathcal{N}(\sigma)^{(\alpha-1)/2\alpha} \otimes I_E\| U_\alpha^{(1-\alpha)/2\alpha} \rho^{1/2} \|^{2\alpha}_{2\alpha}.
\]

**Theorem A.2.** Let \( \rho, \sigma \) and \( \mathcal{N} \) be as given in definition 3.1. The following limit holds:

\[
\lim_{\alpha \rightarrow 1} \hat{\Delta}_\alpha(\rho,\sigma,\mathcal{N}) = D(\rho||\sigma) - D(\mathcal{N}(\rho)||\mathcal{N}(\sigma)).
\]

**Proof.** Let \( \Pi_\omega \) denote the projection onto the support of \( \omega \). From the condition \( \text{supp}(\rho) \subseteq \text{supp}(\sigma) \), it follows that \( \text{supp}(\mathcal{N}(\rho)) \subseteq \text{supp}(\mathcal{N}(\sigma)) \) [55, appendix B.4]. We can then conclude that

\[
\Pi_\sigma \Pi_\rho = \Pi_\rho \quad \text{and} \quad \Pi_{\mathcal{N}(\rho)} \Pi_{\mathcal{N}(\sigma)} = \Pi_{\mathcal{N}(\rho)}.
\]

We also know that \( \text{supp}(U_\rho U^\dagger) \subseteq \text{supp}(\mathcal{N}(\rho) \otimes I_E) \) [55, appendix B.4], so that

\[
(\Pi_{\mathcal{N}(\rho)} \otimes I_E)\Pi_{U_\rho U^\dagger} = \Pi_{U_\rho U^\dagger}.
\]

When \( \alpha = 1 \), we find from the above facts that

\[
\hat{Q}_1(\rho,\sigma,\mathcal{N}) = \|\Pi_{\mathcal{N}(\rho)} \Pi_{\mathcal{N}(\sigma)} \otimes I_E\| U_\sigma \rho^{1/2} \|^{2}_{2} = \|\Pi_{\mathcal{N}(\rho)} \otimes I_E\| U_\rho \rho^{1/2} \|^{2}_{2} = 1.
\]

So from the definition of the derivative, this means that

\[
\lim_{\alpha \rightarrow 1} \hat{\Delta}_\alpha(\rho,\sigma,\mathcal{N}) = \lim_{\alpha \rightarrow 1} \frac{\log \hat{Q}_\alpha(\rho,\sigma,\mathcal{N}) - \log \hat{Q}_1(\rho,\sigma,\mathcal{N})}{\alpha - 1} = \frac{d}{d\alpha} \left[ \log \hat{Q}_\alpha(\rho,\sigma,\mathcal{N}) \right]_{\alpha = 1} = \frac{1}{\hat{Q}_1(\rho,\sigma,\mathcal{N})} \frac{d}{d\alpha} \left[ \hat{Q}_\alpha(\rho,\sigma,\mathcal{N}) \right]_{\alpha = 1}.
\]
Let $\alpha' \equiv (\alpha - 1)/\alpha$. Now consider that

$$\tilde{Q}_\alpha(\rho, \sigma, N) = \text{Tr}[(\rho^{1/2}(\sigma)\alpha'^{1/2}N(\sigma)^{\alpha'}N(\rho)^{-\alpha'}N(\sigma)^{\alpha'})\sigma^{-\alpha'/2}\rho^{1/2}]^\alpha.$$  (A 8)

Define the function

$$\tilde{Q}_{\alpha, \beta}(\rho, \sigma, N) \equiv \text{Tr}[(\rho^{1/2}N(\sigma)^{\alpha'}N(\rho)^{-\alpha'}N(\sigma)^{\alpha'})\sigma^{-\alpha'/2}\rho^{1/2}]^\beta,$$  (A 9)

and consider that

$$\left. \frac{d}{d\alpha} \tilde{Q}_\alpha(\rho, \sigma, N) \right|_{\alpha=1} = \left. \frac{d}{d\alpha} \tilde{Q}_{\alpha,1}(\rho, \sigma, N) \right|_{\alpha=1} + \frac{d}{d\beta} \tilde{Q}_{1,\beta}(\rho, \sigma, N) \right|_{\beta=1}.$$  (A 10)

We first compute $\tilde{Q}_{1,\beta}(\rho, \sigma, N)$ as follows:

$$\tilde{Q}_{1,\beta}(\rho, \sigma, N) = \text{Tr}[(\rho^{1/2}N(\sigma)^{1/2}N(\rho)^{1/2})^\beta] = \text{Tr}[(\rho^{1/2}N(\sigma)^{1/2}N(\rho)^{1/2})^\beta]\text{Tr}[(\rho^{1/2}N(\sigma)^{1/2}N(\rho)^{1/2})^\beta] = \text{Tr}[(\rho^{1/2}U^\dagger(\rho)(\rho^{1/2})^\beta] = \text{Tr}[(\rho^{1/2}U^\dagger(\rho)] = \text{Tr}[\rho^\beta].$$  (A 11)

So then

$$\left. \frac{d}{d\beta} \tilde{Q}_{1,\beta}(\rho, \sigma, N) \right|_{\beta=1} = \left. \frac{d}{d\beta} \text{Tr}[\rho^\beta] \right|_{\beta=1} = \text{Tr}[\rho^\beta \log \rho] = \text{Tr}[\rho \log \rho].$$  (A 12)

Now we turn to the other term $(d/d\alpha)\tilde{Q}_{\alpha,1}(\rho, \sigma, N)$. First consider that

$$\frac{d}{d\alpha} \left(-\alpha'\right) = \frac{d}{d\alpha} \left(\frac{1-\alpha}{\alpha}\right) = \frac{d}{d\alpha} \left(\frac{1}{\alpha} - 1\right) = -\frac{1}{\alpha^2}$$  (A 13)

and

$$\tilde{Q}_{\alpha,1}(\rho, \sigma, N) = \text{Tr} \left[ \rho \sigma^{-\alpha'/2}N^\dagger(\sigma)^{\alpha'/2}N(\rho)^{-\alpha'}N(\sigma)^{\alpha'/2}\right].$$  (A 14)

Now we show that $(d/d\alpha)\tilde{Q}_{\alpha,1}(\rho, \sigma, N)$ is equal to

$$\frac{d}{d\alpha} \text{Tr} \left[ \rho \sigma^{-\alpha'/2}N^\dagger(\sigma)^{\alpha'/2}N(\rho)^{-\alpha'}N(\sigma)^{\alpha'/2}\right] = \text{Tr} \left[ \rho \left( \frac{d}{d\alpha} \sigma^{-\alpha'/2}N^\dagger(\sigma)^{\alpha'/2}N(\rho)^{-\alpha'}N(\sigma)^{\alpha'/2}\right) \right]$$

$$+ \text{Tr} \left[ \rho \sigma^{-\alpha'/2}N^\dagger \left( \left[ \frac{d}{d\alpha} N(\sigma)^{\alpha'/2} \right] N(\rho)^{-\alpha'}N(\sigma)^{\alpha'/2} \right) \right]$$

$$+ \text{Tr} \left[ \rho \sigma^{-\alpha'/2}N^\dagger \left( N(\sigma)^{\alpha'/2} \left[ \frac{d}{d\alpha} N(\rho)^{-\alpha'} \right] \right) \right]$$

$$+ \text{Tr} \left[ \rho \sigma^{-\alpha'/2}N^\dagger \left( N(\sigma)^{\alpha'/2} \left[ \frac{d}{d\alpha} N(\sigma)^{\alpha'/2} \right] \right) \right]$$

$$+ \frac{1}{\alpha^2} \left[ -\frac{1}{2} \text{Tr} \left[ \rho \log \sigma \sigma^{-\alpha'/2}N^\dagger(\sigma)^{\alpha'/2}N(\rho)^{-\alpha'}N(\sigma)^{\alpha'/2}\right] \right]$$

$$+ \frac{1}{2} \text{Tr} \left[ \rho \sigma^{-\alpha'/2}N^\dagger \left( \log N(\sigma)^{\alpha'/2}N(\rho)^{-\alpha'}N(\sigma)^{\alpha'/2}\right) \right]$$

$$- \text{Tr} \left[ \rho \sigma^{-\alpha'/2}N^\dagger \left( \left[ \log N(\rho)^{-\alpha'}N(\sigma)^{\alpha'/2} \right) \right] \right]$$

$$+ \frac{1}{2} \text{Tr} \left[ \rho \sigma^{-\alpha'/2}N^\dagger \left( \left[ \log N(\sigma)^{\alpha'/2}N(\rho)^{-\alpha'}N(\sigma)^{\alpha'/2} \right) \right] \right]$$

$$- \frac{1}{2} \text{Tr} \left[ \rho \sigma^{-\alpha'/2}N^\dagger \left( \left[ \log N(\sigma)^{\alpha'/2}N(\rho)^{-\alpha'}N(\sigma)^{\alpha'/2} \right) \right] \right].$$  (A 16)
We now simplify the first three terms and note that the last two are Hermitian conjugates of the first two:

\[
\begin{align*}
\text{Tr}(\rho \log \sigma) &= \text{Tr}(\rho) \log \sigma \quad \text{and} \quad \text{Tr}(\rho \log \sigma) = \text{Tr}(\rho) \log \sigma,
\end{align*}
\]

(this then implies that the following equality holds:

\[
\frac{d}{d\alpha} \tilde{Q}_{\alpha,1}(\rho,\sigma,N) \bigg|_{\alpha=1} = -\text{Tr}[\rho \log \sigma] + \text{Tr}[\rho \log \sigma] - \text{Tr}[\rho \log \sigma].
\]

Putting together (A 7), (A 10), (A 12) and (A 21), we can then conclude the statement of the theorem.

### Appendix B. Petz recovery map

In this appendix, we explicitly show the well-known fact that the Petz recovery map is a completely positive, trace-non-increasing, linear map. Let \( \rho, \sigma \) and \( N \) be as in definition 3.1 and let \( R_{\rho,\sigma,N} \) be the Petz recovery map as defined in (3.4).

Then, \( R_{\rho,\sigma,N} \) is clearly linear, and it is completely positive because it consists of a composition of the following three completely positive maps:

\[
(\cdot) \rightarrow [N(\sigma)]^{-1/2} (\cdot) [N(\sigma)]^{-1/2}, \quad (\cdot) \rightarrow N^T(\cdot) \quad \text{and} \quad (\cdot) \rightarrow \sigma^{1/2} (\cdot) \sigma^{-1/2}.
\]

It is trace non-increasing because

\[
\begin{align*}
\text{Tr}[\sigma^{1/2} N^T([N(\sigma)]^{-1/2} X [N(\sigma)]^{-1/2})] &= \text{Tr}[\sigma N^T([N(\sigma)]^{-1/2} X [N(\sigma)]^{-1/2})] \\
&= \text{Tr}[N(\sigma) [N(\sigma)]^{-1/2} X [N(\sigma)]^{-1/2}] \\
&= \text{Tr}[N(\sigma) X] \leq \text{Tr}[X].
\end{align*}
\]

### References


