The Periodogram and its Optical Analogy.

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I have recently applied the periodogram method to the investigation of several fluctuating quantities, and the experience thus gained has led me to modify slightly the original definition.* Having always laid stress on the fact that the periodogram supplies by calculation the transformation which the spectroscope instrumentally impresses on a luminous disturbance, I may now enter a little more closely into this optical analogy, and thus lead up to what I hope will be the final definition.

Consider a parallel beam of light falling on a grating, the reflected light being collected at the focus of an observing telescope in the usual way. For simplicity of calculation I assume that the grating considered is of a particular type, which, in a former paper, I have called a simple grating. Such a grating only gives two spectra of the first order.

If $\phi\,(\mathrm{V}t+x)$ be the velocity at any point of the incident beam, the displacement at the focus of the observing telescope is†

$$\frac{hl\cos\beta}{2\pi f\mathrm{NV}\lambda}\mathrm{R},$$

where

$$\mathrm{R} = \int_{-\frac{1}{2}\mathrm{N}\lambda}^{+\frac{1}{2}\mathrm{N}\lambda}\cos nx\,\phi\,(\mathrm{V}t+x)\,dx. \tag{1}$$

In these equations f denotes the focal length of the telescope, h is the length of the lines ruled on the grating, l the width of ruled space measured at right angles to the lines, N gives the number of lines, and β the angle between the direction of the optic axis of the observing telescope and the normal to the grating. For the sake of shortness n is written for $2\pi/\lambda$. The quantity denoted by λ is the wave-length of homogeneous light which would have its first principal maximum at the focus of the telescope. It may be said to be the wave-length towards which the telescope points, and its strict definition is given by the relation

$$l\,(\sin\alpha-\sin\beta) = \mathrm{N}\lambda,$$

where α is the angle of incidence.

In order not to complicate needlessly the calculations, I shall assume that the resolving power is sufficient to ensure that at any point of the spectrum

* 'Cambridge Phil. Soc. Trans.,' vol. 18, p. 107.

† 'Phil. Mag.,' vol. 37, p. 545 (1894).

the vibrations are nearly homogeneous; this involves that the average squares of the velocities are sensibly equal to the average squares of the displacements multiplied by $4\pi^2 V^2/\lambda^2$. The average square of the velocity at the focus of the telescope is in that case—

$$\frac{h^2 l^2 \cos^2 \beta}{f^2 N^2 \lambda^4} R^2.$$

where for R^2 we must put its average value. This expression represents the measure of the intensity at the point considered. Its line and surface integrals may be called the total linear intensity and the total intensity respectively.

In observing a spectrum, we associate with a particular wave-length all the light which lies in a straight line parallel to the rulings of the grating. The distribution of light along a vertical line for nearly homogeneous light takes place according to the law $\alpha^{-2} \sin^2 \alpha$, where $\alpha = \pi hy/f\lambda$, y being the vertical distance. Multiplying by dy, and integrating from minus to plus infinity, the total intensity in a vertical line is found to be $\lambda f/h$ when the intensity at the central maximum is unity. With the value for the central intensity previously found, we now obtain the total linear intensity associated with λ to be

$$\frac{h l^2 \cos^2 \beta}{f N^2 \lambda^3} R^2. \qquad (2)$$

Changing the variable, the expression for R takes the form

$$\int_{x_0}^{x_0+N\lambda} \cos n (Vt - x)\, \phi(x)\, dx,$$

where x_0 is put for $Vt - \frac{1}{2}N\lambda$.

Write

$$A = \int_{x_0}^{x_0+N\lambda} \cos nx\, \phi(x)\, dx; \qquad B = \int_{x_0}^{x_0+N\lambda} \sin nx\, \phi(x)\, dx.$$

The mean value of R^2 is then equal to the mean value of

$$\tfrac{1}{2}(A^2 + B^2).$$

A grating such as that to which the above equations apply forms two spectra and absorbs part of the light; we must now estimate what fraction of the incident beam is utilised to form the spectrum under consideration. For this purpose we imagine homogeneous light to fall on the grating, and put $\phi(x) = \cos nx$. The mean value of $\frac{1}{2}(A^2 + B^2)$ is then easily found to be $\frac{1}{8} N^2\lambda^2$. By substitution into (2) we find that the total linear intensity in the central line is now

$$\frac{h l^2 \cos^2 \beta}{8 f \lambda} R^2. \qquad (3)$$

To either side of the principal maximum the intensity varies according to the law $\alpha^{-2}\sin^2\alpha$, where α is equal to $\pi\xi l\cos\beta/f\lambda$, ξ representing a distance measured at right angles to the spectroscopic line. The total energy measured in the focal plane of the telescope is obtained by multiplying (3) with $\alpha^{-2}\sin^2\alpha d\xi$, and integrating. This gives $\frac{1}{8}hl\cos\beta$.

If the incident light is normal to the grating, its total energy is $\frac{1}{2}hl$, the factor $\frac{1}{2}$ representing the fact that we have taken the average square of the velocity which is half the square of the greatest velocity as the measure of intensity. We conclude that $\frac{1}{4}\cos\beta$ is the fraction of light utilised to form the spectrum.

Taking account in (2) of this, we find that the type of spectroscope considered estimates the intensity of light passing through its central meridian as being

$$\frac{2hl^2\cos\beta}{fN^2\lambda^3}(A^2+B^2),$$

where for A^2 and B^2 their mean values are to be substituted.

To obtain the total light within a small angular distance $d\beta$, we must multiply by $Fd\beta$; as $Nd\lambda = l\cos\beta d\beta$, we find that the total energy within a range $d\lambda$ is

$$\frac{2hl}{N\lambda^3}(A^2+B^2)\,d\lambda.$$

If the total energy of the light incident on the grating is unity, the energy assigned by the grating to a range dn is therefore finally—

$$\frac{A^2+B^2}{\pi N\lambda}dn. \tag{4}$$

In the application of the periodogram it is more convenient to take the time as the independent variable. Defining therefore—

$$A = \int_{t_0}^{t_0+NT}\cos\kappa t\,\phi(t)\,dt, \qquad B = \int_{t_0}^{t_0+NT}\sin\kappa t\,\phi(t)\,dt, \tag{5}$$

(4) becomes equal to $\dfrac{A^2+B^2}{\pi NT}d\kappa$.

Leaving out the constant factor, I now define

$$S = (A^2+B^2)/NT$$

to be the ordinate of the periodogram. The definition differs from the previous one by the factor NT, which occurs in the denominator instead of its square.

The present definition is not only justified by the close optical analogy which has now been formally proved, but also by the resulting convenience. I have previously shown that, in the absence of any homogeneous

periodicities, the average of A^2+B^2 increases in proportion to the time interval, NT, which occurs in the limits of the integrals for A and B. It follows that for such variations, the ordinate of the periodogram as at present defined is independent of the time limits chosen. This is an advantage. On the other hand, the former definition gave directly the amplitude of the periodic variation when it was of an absolutely homogeneous character. For such homogeneous variation the present ordinate increases proportionally to the time interval chosen.

The optical analogy explains the reason of this, and gives its justification. When homogeneous light falls on an instrument of definite resolving power the light in the central meridian does not by itself give sufficient indication of the intensity of the incident light. It is only when correction has been made for the lateral spreading that the true intensity can be deduced, the correction depending on the resolving power. It is otherwise when the spectrum is continuous, for in that case the light lost by lateral spreading is replaced by that which properly belongs to the neighbouring wave-lengths. Hence, in this case, the intensity in the central meridian is a true measure of the intensity of the incident light.

It need hardly be pointed out how constant use is made of the fact that increased resolving power (*not* increased dispersion) brings out the homogeneous lines of a spectrum by increasing their intensity beyond that of the continuous background. It is correspondingly one of the principal advantages of the periodogram method that it gives a measure of the resolving power necessary to isolate a true homogeneous period from the irregular fluctuations.

Light is thrown on parts of the previous investigation by a formula given by Lord Rayleigh for the intensity to be assigned to the homogeneous components of a disturbance. If $\phi(x)$ be the velocity at any point of a linear disturbance, so that the total intensity is

$$\int_{-\infty}^{+\infty} \{\phi(x)\}^2\, dx,$$

Lord Rayleigh shows that the energy to be assigned to a range dn, where $n = 2\pi/\lambda$, is

$$(A^2+B^2)\, dn/\pi,$$

in which

$$A = \int_{-\infty}^{+\infty} \cos \kappa v\, \phi(v)\, dv, \qquad B = \int_{-\infty}^{+\infty} \sin \kappa v\, \phi(v)\, dv.$$

The average intensity spread over a certain length L may be estimated by taking v_0 and v_0+L as lower and upper limits of the integrals, and averaging the values obtained by a change of v_0. The energy per unit

length would then be found on dividing the expression in (5) by L. We arrive in this manner at equation (4).

I might have confined myself to this simple deduction had I not wished to lay stress on the equations for instrumental resolution. This seemed all the more desirable because for absolutely homogeneous radiation a definition of the periodogram based on the average intensity per unit length would fail. If a simple periodicity exists, its amplitude may easily be derived from the ordinate of the periodogram, for, if S be that ordinate, the amplitude is $2\,(\mathrm{S}/\mathrm{NT})^{\frac{1}{2}}$.

In practical applications the function $\phi(t)$ will generally be given for successive intervals (α) of the time. The integrals occurring in (5) are then replaced by summations, unless a harmonic analyser is used. It is most convenient to write in this case

$$\mathrm{A} = \sum_{s=0}^{s=(n-1)\alpha} \Phi_s \cos\left(\frac{2\pi}{n}s\right), \qquad \mathrm{B} = \sum_{s=0}^{s=(n-1)\alpha} \Phi_s \sin\left(\frac{2\pi}{n}s\right), \tag{6}$$

$$\mathrm{S} = (\mathrm{A}^2+\mathrm{B}^2)\,\alpha^2/\mathrm{NT},$$

where ϕ_s represents the values which $\phi(t)$ takes at the successive times considered.

If we take p to be equal to the total number of separate values of $\phi(t)$ used in the calculations, we may put

$$\mathrm{S} = (\mathrm{A}^2+\mathrm{B}^2)\,\alpha/p. \tag{7}$$

If a harmonic analyser be used, and a b are the two Fourier coefficients,

$$\mathrm{S} = \tfrac{1}{4}\,(a^2+b^2)\,\mathrm{NT} = \tfrac{1}{4}\,(a^2+b^2)\,\alpha p,$$

where αp represents the complete time interval to which the Fourier analysis has been applied.