Oscillating flames: multiple-scale analysis

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A complete multiple-scale solution is constructed for the one-dimensional problem of an oscillating flame in a tube, ignited at a closed end, with the second end open. The flame front moves into the unburnt mixture at a constant burning velocity relative to the mixture ahead, and the heat release is constant. The solution is based upon the assumption that the propagation speed multiplied by the expansion ratio is small compared with the speed of sound. This approximate solution is compared with a numerical solution for the same physical model, assuming a propagation speed of arbitrary magnitude, and the results are close enough to confirm the validity of the approximate solution. Because ignition takes place at the closed end, the effect of thermal expansion is to push the column of fluid in the tube towards the open end. Acoustics set in motion by the impulsive start of the column of fluid play a crucial role in the oscillation. The analytical solution also captures the subsequent interaction between acoustics and the reaction front, the effect of which does not appear to be as significant as that of the impulsive start, however.

Keywords: flame acceleration; deflagration-to-detonation transition; oscillating flame; galloping

1. Introduction

Flames accelerating over narrow flow passages, either with obstacles (Lee et al. 1985; Gamezo et al. 2007) or even in smooth ducts (Kuznetsov et al. 2005), are an important contributor to deflagration-to-detonation transition. Mallard & Le Châtelier (1883) first observed oscillating flames in tubes. When ignition occurs at the closed end of a tube that is open at the other end, these oscillations can result in high velocities. A review by Guénoche (1964) identifies the crucial role played by acoustics. Newer investigations include Leyer (1969), Leyer & Manson (1970), Jones (1977), Jones & Thomas (1991) and Kerampran et al. (2000). A one-dimensional analysis by Jost (1939) determines the slowly varying spectrum, but stops short of finding amplitudes. Crucially, Jost (1939) recognized that the flame front moves slowly compared with the acoustics. The current work considers the same physical model, including a constant propagation speed and an impulsive start. In addition to the spectrum, the amplitudes are determined, thus completing the solution to the problem studied...
by Jost and Guénoche. According to Guénoche (1964), Jost (1939) only considered the case of ignition near an open end. Among the four possible combinations between closed and open tube ends, Guénoche (1949) himself considered three cases, including ignition near a closed end, but he did not analyse the fourth, current case. If the propagation speed is negligible compared with the speed of sound, then, for any front location, acoustics occur as though the front location were frozen, and the spectrum is obtained for a frozen location. The front separates two regions in which temperature, hence the speed of sound, differs. Since the front evolves slowly compared with the speed of sound, the spectrum evolves slowly as the front moves. The dispersion relationship is transcendental (Lees 1929; Rott 1969).

The one-dimensional incompressible formulation assuming impulsive ignition at a closed end leads to a solution that simply consists of a front moving at a constant speed equal to the propagation speed multiplied by the expansion ratio, with fluid at rest in the burnt region and with unburned mixture being pushed out at a constant velocity equal to the expansion ratio minus unity, multiplied by the propagation speed. However, such an incompressible solution does not take into account the initial effects of acoustics, which lead to acoustic velocities comparable with the speed of the front, and do not disappear over the relatively brief travel time of the front. Indeed, when the initial process is analysed on the acoustic time scale, the entire column of fluid does not start instantaneously. Initially, the front is preceded by a weak shock wave moving into fluid at rest, separating the fluid already in motion from the open end. Thus, in the early phase of the process, the tube length is divided into three regions. From the closed end to the flame front, the fluid is at rest. From the front to its leading shock, the fluid is moving towards the open end at a speed equal to the burning velocity multiplied by the expansion ratio. Between the shock and the open end, the fluid is at rest. The current solution differs from the simple incompressible one in that when the shock reaches the open end, a reflection occurs and an expansion wave returns towards the closed end. More reflections occur at the ends and at the front. These oscillations do not disappear, but frequencies and amplitudes evolve. The spectrum varies with the location of the front because there is a jump in the speed of sound at the front. The amplitudes vary because frequencies vary and also because of interaction with the front. The theory below quantifies precisely both effects. This one-dimensional model alone leads to sufficient oscillations to overcome the steady front motion, resulting in occasional flow reversals.

Acoustic amplitudes are normally determined by the initial conditions. However, in the current situation, no initial condition is available for acoustics assuming a frozen front location, except initially when the front is at the closed end. Construction of the complete solution entails the determination of the evolution on a slow time scale associated with front location, not only of the spectrum, but also of the acoustic amplitudes. As shown below, these are determined by removal of secularity at higher order. The results thus obtained turn out to be quite irregular, not unlike experimental measurements (such as Kerampran et al. 2000). For verification, an independent numerical investigation was conducted for the same physical model, except that no assumption relating the magnitudes of the flame propagation speed and of the speed of sound was made. The results from the simulation are quite similar to the results from the analysis.
The simplicity of the physical model entails serious limitations, the most significant of which stems from its one-dimensional nature as well as the simple flame propagation model. The former precludes appearance of tulip flames (Clanet & Searby 1996) and issues associated with walls, boundary layer and early flame shape, and also the development of hydrodynamic and Rayleigh–Taylor instabilities. However, here, the goal is to obtain the far field as it results from bulk flame motion, rather than to study combustion instabilities (Mcmanus et al. 1993) or slowly oscillating laminar flames (Mcintosh 1999).

2. Physical model

The problem considers a tube closed at one end, at \( x=0 \), and open into the atmosphere at its other end. A planar flame front starts at \( t=0 \) and \( x=0 \), and propagates into the tube filled with reactive mixture at rest.

The reaction zone is negligibly thin compared with acoustic length. Then, outside of the flame zone, diffusion can be neglected, and the problem is governed by the reactive inviscid non-conductive conservation laws of gas dynamics, for mass, momentum and energy. Between locations where diffusion has become negligible, the Schvab–Zel’dovich variable is conserved. The longitudinal coordinate \( x \) is scaled by the tube length; pressure \( p \) by its atmospheric value; and time \( t \) by the tube length divided by the flame propagation speed into unburnt fluid. Density \( \rho \) and temperature \( T \) are scaled by their initial values, and velocity \( u \) by the flame propagation speed. The conservation equations are

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \tag{2.1}
\]

\[
\frac{\partial \gamma M^2 \rho u}{\partial t} + \frac{\partial (\gamma M^2 \rho u^2 + p)}{\partial x} = 0 \tag{2.2}
\]

and

\[
\frac{\partial}{\partial t} \left\{ \rho \left[ \frac{T + QZ}{\gamma (\gamma - 1)} + \frac{M^2}{2} u^2 \right] \right\} + \frac{\partial}{\partial x} \left\{ \rho u \left[ \frac{\gamma T + QZ}{\gamma (\gamma - 1)} + \frac{M^2}{2} u^2 \right] \right\} = 0, \tag{2.3}
\]

in which \( \gamma \) is the (constant) ratio of specific heats and the reference Mach number \( M \) is the ratio of the burning velocity to the initial speed of sound in unburnt fluid. The limiting reactant mass fraction \( Z \) is set to 0 on the burnt side of the front and to 1 on the unburnt side. \( Q \) is the heat release, scaled by the initial value of the internal energy of the unburnt fluid. The state variables are related by \( p = \rho T \).

The boundary conditions are \( u=0 \) at \( x=0 \) and \( p=1 \) at \( x=1 \). The initial conditions at \( t=0 \) are \( u=0, \ p=1, \ T=1 \) and \( \rho=1 \).

Close to the reaction zone, the flow will have a complex three-dimensional structure. However, in a long and narrow tube, the far field will remain more or less one dimensional. In a globally one-dimensional model, the reaction zone can only be represented as a planar propagation front, endowed with some prescribed propagation velocity. This propagation velocity is obviously not constant, and it is significantly larger than the laminar burning velocity. Rather than dealing...
with empirical models, here the propagation speed is simply taken to be constant. Likewise, the ignition process is typically dominated by multi-dimensional effects (Kerampran et al. 2000). Although non-impulsive ignition could be included in the analysis, the substance of the problem is captured by the impulsive model.

The flame front is located at $x=X(t)$ at time $t$, and, under the current assumptions, it propagates at unit speed into the unburnt fluid, i.e.

$$U = \frac{dX}{dt} = 1 + u_R(X, t),$$

in which $U$ is the speed of the front in the laboratory frame. Subscripts R and L denote, respectively, the unburnt and burnt sides of the front. For a known propagation speed, the Rankine–Hugoniot equations accurately express conservation across the front. Taking into account that $Z_L=0$ and $Z_R=1$,

$$\rho_L(u_L - U) = \rho_R(u_R - U),$$

$$\gamma M^2 \rho_L(u_L - U)^2 + p_L = \gamma M^2 \rho_R(u_R - U)^2 + p_R$$

and

$$2\gamma T_L + \gamma(\gamma - 1)M^2(u_L - U)^2 = 2\gamma T_R + 2Q + \gamma(\gamma - 1)M^2(u_R - U)^2.$$

This problem is solved (i) under the scaling assumptions described in §3 and then (ii) numerically in §4, with no additional assumption.

### 3. Multiple-scale solution

(a) Scaling assumptions

As in Jost (1939), on the time scale of acoustics, the front is moving very slowly, so that the acoustics can be solved by assuming a frozen front location. Indeed, motion is triggered by combustion, so that all velocities scale with the propagation speed, typically of the order of $1 \text{ m} \text{s}^{-1}$, multiplied by the expansion ratio, normally less than 10, which is quite small compared with the speed of sound in the unburnt mixture. Thus, one seeks a solution assuming that $M \ll 1$, for an expansion ratio of order unity.

Large-scale motion of the front will occur on the time scale $t$ already defined in §2. However, the formulation, in which length and time are scaled in a ratio of the order of the flow velocities, describes low Mach number, incompressible flow, but precludes acoustics, in which length and time scales are in a ratio comparable with the speed of sound. In the current problem, subsequent to ignition near the closed end, expansion of the fluid due to combustion will trigger motion towards the open end of the entire fluid column that separates the open end from the ignition location. This impulsive start will initially be described by an acoustic model. Furthermore, since there is no dissipation mechanism in the physical model, the acoustics will not disappear (Jost 1939). As a result, a complete description of the motion requires accounting for both the fast acoustic scale and the slow incompressible time scale. A multiple-scale analysis is thus required, in the limit of infinitely fast acoustics when compared with front motion. For any value of the slow time $t$, the acoustic spectrum $\omega(t)$ can then be obtained assuming a frozen...
Physically, this spectrum is the set of resonant acoustic frequencies, determined as eigenvalues for the fast time acoustic problem, but which still vary with the slow time \( t \). Additionally, when characterizing fast times, each acoustic mode requires its own scale, so that the analysis considers one single slow time \( t \) and an infinite family of fast times \( \theta_k \), defined for mode \( k \) by

\[
\theta_k = \frac{1}{M} \int_0^t \omega_k(s) ds. \tag{3.1}
\]

Next, perturbation series are introduced in the conservations, initial, boundary and jump conditions, for \( p, \rho, u \) and \( X \), such as

\[
p = p^{(0)} + M p^{(1)} + M^2 p^{(2)} + \cdots. \tag{3.2}
\]

(b) Conservation equations

Under the assumptions above, the conservation equations become

\[
\frac{\partial p}{\partial t} + \sum_k \omega_k \frac{\partial p}{\partial \theta_k} + \frac{\partial (\rho u)}{\partial x} = 0, \tag{3.3}
\]

\[
\frac{\partial \gamma M^2 \rho u}{\partial t} + \gamma M \sum_k \omega_k \frac{\partial \rho u}{\partial \theta_k} + \frac{\partial (\gamma M^2 \rho u^2 + p)}{\partial x} = 0 \tag{3.4}
\]

and

\[
\left( \frac{\partial}{\partial t} + \sum_k \frac{\omega_k}{M} \frac{\partial}{\partial \theta_k} \right) \left\{ \rho \left[ \frac{T + QZ}{\gamma (\gamma - 1)} + \frac{M^2}{2} u^2 \right] \right\} + \frac{\partial}{\partial x} \left\{ \rho u \left[ \frac{\gamma T + QZ}{\gamma (\gamma - 1)} + \frac{M^2}{2} u^2 \right] \right\} = 0. \tag{3.5}
\]

Replacing \( T = p/\rho \) in the energy equation, taking into account that \( Z \) is constant and subtracting \( QZ \) multiplied by continuity, a pressure equation is obtained,

\[
\left( \omega \frac{\partial}{\partial \theta} + M \frac{\partial}{\partial t} \right) \left[ p + M^2 \frac{\gamma (\gamma - 1)}{2} \rho u^2 \right] + \gamma M \frac{\partial}{\partial x} \left[ w p + M^2 \frac{\gamma - 1}{2} \rho u^3 \right] = 0,
\]

in which the subscript \( k \) and the summation symbol have been dropped for simplicity. The second-order equation for \( u \) is obtained by taking the space derivative of the pressure equation minus the time derivative of momentum,

\[
\frac{\partial^2 (w p)}{\partial x^2} - \omega^2 \frac{\partial^2 (\rho u)}{\partial \theta^2} = \omega M \frac{\partial}{\partial \theta} \left[ 2 \frac{\partial (\rho u)}{\partial t} - \frac{\gamma - 3}{2} \frac{\partial (\rho u^2)}{\partial x} \right] + M \frac{\partial \omega}{\partial t} \frac{\partial (\rho u)}{\partial \theta} + O(M^2). \tag{3.6}
\]

Then, introducing the asymptotic expansions, second-order equations for both leading and second-order velocity are readily obtained. Furthermore, in view of momentum, including the Rankine–Hugoniot conditions, leading-order pressure is spatially uniform and it equals unity everywhere, in view of the boundary condition at the open end. Likewise, \( \rho^{(0)} \) is independent of \( \theta \), according to
continuity. Taking the asymptotic series into account, equation (3.6) results in the usual linear second-order wave equation, both at order unity and order \( M \).

At order unity,
\[
\frac{\partial^2 u^{(0)}}{\partial x^2} - \omega^2 \rho^{(0)} \frac{\partial^2 u^{(0)}}{\partial \theta^2} = 0.
\]  

(3.7)

It might appear that leading-order density varies on the slow time scale. However, admitting values that depend upon \( t \) would lead to a contribution growing linearly with \( q \) in the solution, which is inconsistent with the multiple-scale method. Taking into account that \( r^{(0)} \) remains constant, although with different values in the burnt and unburnt regions, the leading-order acoustic equations are

\[
\omega \frac{\partial \rho^{(1)}}{\partial \theta} + \rho^{(0)} \frac{\partial u^{(0)}}{\partial x} = 0,
\]

(3.8)

\[
\gamma \omega \rho^{(0)} \frac{\partial u^{(0)}}{\partial x} + \frac{\partial p^{(1)}}{\partial x} = 0
\]

(3.9)

and

\[
\omega \frac{\partial \rho^{(1)}}{\partial \theta} + \gamma \frac{\partial u^{(0)}}{\partial x} = 0.
\]

(3.10)

In summary, \( p^{(0)} \) = 1 in the entire domain, while \( \rho^{(0)} \) and \( T^{(0)} \) are spatially uniform on both sides of the front. The leading-order acoustic problem, described by equation (3.7) or equations (3.8)–(3.10), determines \( u^{(0)} \), \( p^{(1)} \), \( \rho^{(1)} \) and \( T^{(1)} \).

(c) Continuity across the front

At the front, jumps occur in both velocity and its gradient, for \( u^{(0)} \) and \( u^{(1)} \). Additionally, leading-order temperature and density are spatially uniform on both sides, but discontinuous at the front. The conditions across the front include the Rankine–Hugoniot relationships, equations (2.5)–(2.7), and the kinematic equation (2.4). At leading order, energy conservation yields

\[
\frac{\gamma}{\rho^{(0)}_L} = \frac{\gamma}{\rho^{(0)}_R} + Q,
\]

(3.11)

which is consistent with the leading-order approximation being isobaric and kinetic energy being negligible. Since \( \rho^{(0)}_R = 1 \), its initial value is \( \rho^{(0)}_L = \gamma/(\gamma + Q) \).

Using the leading-order expansion ratio \( \alpha = \rho^{(0)}_R/\rho^{(0)}_L = (\gamma + Q)/\gamma \) instead of \( Q \) as the independent parameter, the leading-order mass conservation equation becomes

\[
u^{(0)}_L - u^{(0)} = \alpha (u^{(0)}_R - u^{(0)}).
\]

(3.12)

The kinematic equation

\[
U = \frac{\omega}{M} \frac{\partial X}{\partial \theta} + \frac{\partial X}{\partial t} = u^{(0)}_R + 1
\]

(3.13)

requires \( X^{(0)} \) to be independent of \( \theta \). Thus, \( X^{(0)} = X^{(0)}(t) \) and

\[
U^{(0)} = \omega \frac{\partial X^{(1)}}{\partial \theta} + \frac{dX^{(0)}}{dt} = u^{(0)}_R(X^{(0)}, \theta, t) + 1 = u^{(0)}_L(X^{(0)}, \theta, t) + \alpha,
\]

(3.14)

which also takes into account conservation of mass.
The solutions to the acoustic problem described by equation (3.7) are only determined up to an integration constant. However, at $x=0$, the velocity must be zero. Thus, the constant difference between the variable parts of the solutions on both sides of the front must necessarily be allocated to $u_R^{(0)}$. Since contributions linear in $\theta$ in $X^{(1)}$ are not acceptable,

$$\frac{dX^{(0)}}{dt} = \alpha \quad \text{and} \quad \omega \frac{\partial X^{(1)}}{\partial \theta} = u_L^{(0)}(X^{(0)}, \theta, t). \quad (3.15)$$

Next, focusing upon the contribution of $O(M)$, both the kinematic and the Rankine–Hugoniot equations apply at $X$, while the conditions at $X^{(0)}$ are more convenient to use. Expanding the kinematic equation in the Taylor series about $X^{(0)}$,

$$U^{(1)} = u_R^{(1)}(X^{(0)}) + X^{(1)} \frac{\partial u_R^{(0)}}{\partial x} \big|_{X^{(0)}}. \quad (3.16)$$

From leading-order acoustics, equation (3.10), given that from the first-order momentum, equation (2.6), at $X^{(0)}$, $p_R^{(1)} = p_L^{(1)}$, it is found that, at $X^{(0)}$,

$$\frac{dU_L^{(0)}}{dx} = \frac{dU_R^{(0)}}{dx}, \quad (3.17)$$

then from equation (3.8), $p_R^{(1)} = \alpha p_L^{(1)}$ at $X^{(0)}$. Finally, from the Rankine–Hugoniot equations at $X$, in which the variables have been expanded in a Taylor series about $X^{(0)}$, taking into account that leading-order thermodynamic quantities are spatially uniform on both sides of the front, that $U^{(0)} - u_R^{(0)} = 1$ and that $U^{(0)} - u_L^{(0)} = \alpha$,

$$U^{(1)} = u_R^{(1)}(X^{(0)}) + X^{(1)} \frac{\partial u_R^{(0)}}{\partial x} \big|_{X^{(0)}}, \quad (3.18)$$

so that, at $X^{(0)}$,

$$u_L^{(1)} = u_R^{(1)} = U^{(1)} - X^{(1)} \frac{dU^{(0)}}{dx}. \quad (3.19)$$

The jump in the spatial gradient of $u^{(1)}$ still remains to be determined. The second-order perturbation to the pressure equation, equation (3.6), yields

$$\omega \frac{\partial p^{(2)}}{\partial \theta} + \omega \gamma (\gamma - 1) p^{(0)} u^{(0)} \frac{\partial u^{(0)}}{\partial t} + \frac{\partial p^{(1)}}{\partial t} + \gamma \frac{\partial u^{(0)} p^{(1)}}{\partial x} + \gamma \frac{\partial u^{(1)}}{\partial x} = 0. \quad (3.20)$$

Writing this equation for both sides of $X^{(0)}$ and subtracting the left one from the right one, taking into account that $p^{(1)}$ and the leading-order velocity gradient are continuous, replacing the leading-order density by its value as found above, using equation (3.9) and $u_L^{(0)} = u_R^{(0)} - \alpha + 1$,

$$\frac{\partial u_R^{(1)}}{\partial x} - \frac{\partial u_L^{(1)}}{\partial x} = -\omega \frac{\partial (p_R^{(2)} - p_L^{(2)})}{\partial \theta} + \omega(\alpha - 1)(u_L^{(0)} + \alpha) \frac{\partial u^{(0)}}{\partial \theta}. \quad (3.21)$$

First, focusing on the first term on the r.h.s., at order $M^2$, conservation of energy, equation (2.7), yields, at $X$ (but not at $X^{(0)}$),

$$p_R^{(2)}(X) - p_L^{(2)}(X) = \alpha - 1 \quad (3.22)$$
or, equivalently, $T_L^{(1)}(X) = T_R^{(1)}(X)$. From an expansion in the Taylor series about $X^{(0)}$,
\[
p_R^{(2)}(X^{(0)}) - p_L^{(2)}(X^{(0)}) + X^{(1)} \left[ \frac{\partial p_R^{(1)}}{\partial x} \right]_{X^{(0)}} - \frac{\partial p_L^{(1)}}{\partial x} \right]_{X^{(0)}} = \alpha - 1. \tag{3.23}
\]
Finally, evaluating all space-dependent values at $X^{(0)}$, the jump in velocity is
\[
\frac{\partial u_R^{(1)}}{\partial x} - \frac{\partial u_L^{(1)}}{\partial x} = \omega \frac{\partial}{\partial \theta} \left[ X^{(1)} \left( \frac{\partial p_R^{(1)}}{\partial x} - \frac{\partial p_L^{(1)}}{\partial x} \right) \right] + \omega (\alpha - 1)(u_L^{(0)} + \alpha) \frac{\partial u^{(0)}}{\partial \theta}. \tag{3.24}
\]

(d) Boundary conditions

At the closed end $x=0$, the boundary condition is $u^{(0)} = 0$ and $u^{(1)} = 0$. At $x=1$, the boundary condition is $p^{(1)} = 0$ and $p^{(2)} = 0$. From equations (3.10) and (3.20),
\[
\frac{\partial u^{(0)}}{\partial x} = 0 \tag{3.25}
\]
and
\[
\gamma \frac{\partial u^{(1)}}{\partial x} = -\omega \gamma (\gamma - 1) \rho^{(0)} u^{(0)} \frac{\partial u^{(0)}}{\partial \theta} - \gamma \frac{\partial u^{(0)} p^{(1)}}{\partial x}. \tag{3.26}
\]
Expanding, using the leading-order value of $\rho^{(0)}$ and taking equation (3.8) into account,
\[
\frac{\partial u^{(1)}}{\partial x} = \omega u^{(0)} \frac{\partial u^{(0)}}{\partial \theta}. \tag{3.27}
\]

(e) Leading-order problem: acoustics in a domain with a temperature discontinuity

The leading-order acoustic problem includes the equation of acoustics, equation (3.7), in each region, burnt and unburnt. Furthermore, velocity $u_L^{(0)}$ is purely oscillatory, while at the front, $u_R^{(0)} - u_L^{(0)} = \alpha - 1$. Introducing $u$, the purely oscillatory velocity component, which is continuous and has a continuous gradient at the front, $u = u_L^{(0)}$ in the region $0 \leq x < X$ and $u = u_R^{(0)} - \alpha + 1$ for $X \leq x \leq 1$. Equation (3.7) becomes
\[
\rho^{(0)} \omega^2 \frac{\partial^2 u}{\partial \theta^2} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{3.28}
\]
with $\rho^{(0)} = 1/\alpha$ for $0 \leq x < X$ and $\rho^{(0)} = 1$ for $X < x \leq 1$, with boundary conditions $u=0$ at $x=0$ and $\partial u/\partial x=0$ at $x=1$. Taking these into account,
\[
u_L = U_L \exp i(\theta + \omega x/\sqrt{\alpha}) - U_L \exp i(\theta - \omega x/\sqrt{\alpha}) \tag{3.29}
\]
and
\[
u_R = U_R \exp i[\theta - \omega(1-x)] + U_R \exp i[\theta + \omega(1-x)], \tag{3.30}
\]
on the burnt and unburnt sides, respectively. At the front, continuity leads to
\[ i U_L \sin \frac{\omega x}{\sqrt{\alpha}} = U_R \cos \omega (1-X) \quad \text{and} \quad i U_L \cos \frac{\omega x}{\sqrt{\alpha}} = \sqrt{\alpha} U_R \sin \omega (1-X); \] (3.31)
hence, the dispersion relationship originally due to Lees (1929)
\[ \sqrt{\alpha} \tan \frac{\omega x}{\sqrt{\alpha}} \tan \omega (1-X) = 1. \] (3.32)
The coefficients remain to be found, which entails considering the next order.

(f) Second-order problem

Removal of secularity in the second-order problem leads to a differential equation for the evolution of the leading-order coefficients on the slow time \( t \). Beyond obtaining that condition, no further attempt is made to solve the second-order problem. Taking equations (3.8)–(3.10) into account, the perturbation equation obtained from equation (3.6), ignoring (non-resonant) terms quadratic in the oscillatory components of the leading-order solution in the forcing on the r.h.s., becomes
\[ \frac{\partial^2 u^{(1)}}{\partial x^2} - \omega^2 \rho^{(0)} \frac{\partial^2 u^{(1)}}{\partial \theta^2} = \rho^{(0)} \frac{\partial}{\partial \theta} \left[ 2 \omega \frac{\partial u^{(0)}}{\partial t} + 2 \omega \dot{U} \frac{\partial u^{(0)}}{\partial x} + u^{(0)} \frac{d \omega}{dt} \right], \] (3.33)
in which \( \dot{U} = 0 \) for \( 0 \leq x < X \) and \( \dot{U} = \alpha - 1 \) for \( X < x \leq 1 \). Then, taking the leading-order solution into account, respectively, for \( 0 \leq x < X \) and \( X < x \leq 1 \),
\[ \omega^2 \frac{\partial^2 u^{(1)}}{\partial \theta^2} - \alpha \frac{\partial^2 u^{(1)}}{\partial x^2} = A_{L1} \exp i \left( \theta + \frac{\omega x}{\sqrt{\alpha}} \right) + A_{L2} \exp i \left( \theta - \frac{\omega x}{\sqrt{\alpha}} \right) \] (3.34)
and
\[ \omega^2 \frac{\partial^2 u^{(1)}}{\partial \theta^2} - \frac{\partial^2 u^{(1)}}{\partial x^2} = A_{R1} \exp i [\theta - \omega (1-x)] + A_{R2} \exp i [\theta + \omega (1-x)], \] (3.35)
with
\[ A_{L1} = -i \left[ 2 \omega \frac{d U_L}{dt} + U_L \frac{d \omega}{dt} \right], \] (3.36)
\[ A_{L2} = i \left[ 2 \omega \frac{d U_L}{dt} + U_L \frac{d \omega}{dt} \right], \] (3.37)
\[ A_{R1} = -i \left[ 2 \omega \frac{d U_R}{dt} + 2 i \omega^2 (\alpha - 1) U_R + (1 - 2 i \omega) \frac{d \omega}{dt} U_R \right] \] (3.38)
and
\[ A_{R2} = -i \left[ 2 \omega \frac{d U_R}{dt} - 2 i \omega^2 (\alpha - 1) U_R + (1 + 2 i \omega) \frac{d \omega}{dt} U_R \right], \] (3.39)
in which the eigenmode \( \omega \) is a solution to the Lees/Rott dispersion relationship, equation (3.32). Since both the leading-order and the perturbation equations have the same kernel, this forcing is at a resonant frequency. Since terms such as \( x \exp i (\theta \pm \omega x) \) do not lead to resonances, contributions proportional to \( x \) in \( A_{L1}, A_{L2}, A_{R1} \) and \( A_{R2} \) need not be taken into consideration. They have been omitted in the above expressions.
Replacing into the equations and solving gives

\[ \frac{\partial u^{(1)}}{\partial x} = \omega u^{(0)} \frac{\partial u^{(0)}}{\partial \theta} = \omega(\alpha - 1) \frac{\partial u^{(0)}}{\partial \theta}. \]  

(3.40)

As the forcing is resonant, particular solutions on both sides have the form

\[ u^{(1)} = \theta V(x) \exp i\theta + W(x) \exp i\theta. \]  

(3.41)

Then, replacing in the equations and equating the same functions of \( \theta \), using the symbol ‘\( \ddot{\prime} \)’ to indicate second derivatives with respect to \( x \), in the unburnt region,

\[ \omega^2 V_L + \alpha V''_L = 0 \]  

(3.42)

and

\[ \omega^2 W_L + \alpha W''_L = 2i\omega^2 V_L - A_{L1}\exp i\omega x/\sqrt{\alpha} - A_{L2}\exp -i\omega x/\sqrt{\alpha} \]  

(3.43)

and, in the burnt region,

\[ \omega^2 V_R + V''_R = 0 \]  

(3.44)

and

\[ \omega^2 W_R + W''_R = 2i\omega^2 V_R - A_{R1}\exp -i\omega(1-x) - A_{R2}\exp i\omega(1-x), \]  

(3.45)

which yields

\[ V_L = V_{L1}\sin \omega x/\sqrt{\alpha} + V_{L2}\cos \omega x/\sqrt{\alpha} \]  

(3.46)

and

\[ V_R = V_{R1}\sin \omega(1-x) + V_{R2}\cos \omega(1-x). \]  

(3.47)

Since in the problem for \( W_L \) and \( W_R \), there is resonant forcing in \( x \),

\[ W_L = (W_{L1} + xW_{L3})\sin \frac{\omega x}{\sqrt{\alpha}} + (W_{L2} + xW_{L4})\cos \frac{i\omega x}{\sqrt{\alpha}} \]  

(3.48)

and

\[ W_R = (W_{R1} + xW_{R3})\sin \omega(1-x) + (W_{R2} + xW_{R4})\cos \omega(1-x). \]  

(3.49)

Replacing into the equations and solving gives

\[ \sqrt{\alpha} W_{L4} = -i\omega V_{L1} + \left( 2 \frac{dU_L}{dt} + U_L \frac{d\omega}{dt} \right), \quad \sqrt{\alpha} W_{L3} = i\omega V_{L2}, \]  

(3.50)

\[ W_{R4} = i\omega V_{R1} + 2i \left[ \omega(\alpha - 1) U_R - \frac{d\omega}{dt} U_R \right], \quad W_{R3} = -i\omega V_{R2} - i \left[ 2 \frac{dU_R}{dt} + \frac{d\omega}{dt} U_R \right]. \]  

(3.51)

Next, the solution is made to satisfy the boundary conditions. At \( x=0 \), \( u=0 \), thus \( V_L(0) = 0 \) and \( W_L(0) = 0 \). So, \( V_{L2} = 0 \) and \( W_{L2} = 0 \).

At \( x=1 \), introducing the leading-order solution, the boundary condition given by equation (3.40) and comparing with equation (3.41) leads to \( V'_R = 0 \); hence, \( V_{R1} = 0 \) and

\[ W'_R = 2i\omega(\alpha - 1) U_R, \]  

(3.52)
which results in
\[ W_{R1} = i \omega V_{R2} + 2i \left( \frac{dU_R}{dt} - \frac{U_R}{2\omega} \frac{d\omega}{dt} \right). \] (3.53)

Out of the 12 integration constants, 8 have been determined at this point, leaving
\( V_{L1}, V_{R2}, W_{L1} \) and \( W_{R2} \) yet unknown. However, the solution still must satisfy the
jump conditions at the front. From equations (3.19) and (3.24), at \( x = X^{(0)} \),
\[ u_R^{(1)} - u_L^{(1)} = 0 \quad \text{and} \quad \frac{\partial u_R^{(1)}}{\partial x} - \frac{\partial u_L^{(1)}}{\partial x} = \omega (\alpha - 1) \frac{\partial u^{(0)}}{\partial \theta}, \] (3.54)
which, in view of equations (3.47) and (3.48), requires \( V_L(X) = V_R(X), \) \( V_L'(X) = V_R'(X) \) and \( W_R(X) = W_L(X) \), but
\[ W_R'(X) - W_L'(X) = 2i \omega (\alpha - 1) U_R \cos \omega (1 - X), \] (3.55)
which provides the last four conditions. Taking into account the values already
obtained for \( V_{L2}, V_{R1}, W_{L2} \) and \( W_1 \), using equation (3.31) and also their derivatives
with respect to the slow time \( t \), these four equations become
\[
\begin{bmatrix}
\cos \omega(1 - X) & -\sin \omega \frac{X}{\sqrt{\alpha}} & 0 & 0 \\
\sin \omega(1 - X) & \frac{-1}{\sqrt{\alpha}} \cos \omega \frac{X}{\sqrt{\alpha}} & 0 & 0 \\
ad_1 & ad_2 & \cos \omega(1 - X) & -\sin \omega \frac{X}{\sqrt{\alpha}} \\
ad_3 & ad_4 & \sin \omega(1 - X) & \frac{-1}{\sqrt{\alpha}} \cos \omega \frac{X}{\sqrt{\alpha}}
\end{bmatrix}
\begin{bmatrix}
V_{R2} \\
V_{L1} \\
W_{R2} \\
W_{L1}
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
c_1 \\
c_2 \end{bmatrix}, \quad (3.56)
\]
in which
\[ a_1 = i(1 - X) \omega \sin \omega(1 - X), \quad a_2 = \frac{iX \omega}{\sqrt{\alpha}} \cos \omega \frac{X}{\sqrt{\alpha}}, \] (3.57)
\[ a_3 = -i \omega(1 - X) \cos \omega(1 - X) - i \sin \omega(1 - X), \quad a_4 = \frac{-i \omega X}{\alpha} \sin \omega \frac{X}{\sqrt{\alpha}} + \frac{i}{\sqrt{\alpha}} \cos \omega \frac{X}{\sqrt{\alpha}}, \] (3.58)
\[ c_1 = -2i \left[ \frac{dU_R}{dt} - \frac{U_R}{2\omega} \frac{d\omega}{dt} \right] \sin \omega(1 - X) - \frac{2iX^2 U_R}{\alpha} (1 - \alpha) \frac{d\omega}{dt} \cos \omega(1 - X) \] (3.59)
and
\[ c_2 = -2iX U_R \frac{\alpha - 1}{\alpha} \left[ \omega \alpha - \frac{d\omega}{dt} \right] \sin \omega(1 - X) \]
\[ -2i \left[ \frac{X + \alpha(1 - X)}{\alpha} \left( \frac{U_R}{2\omega} \frac{d\omega}{dt} - \frac{dU_R}{dt} \right) + (1 - \alpha) U_R \right] \cos \omega(1 - X). \] (3.60)
The system of equation (3.56) is separable. Since forcing is resonant, the first two equations yield equation (3.56), the dispersion relationship. Eliminating $V_{R2}$, a system is obtained for $V_{L1}$ and a linear combination between $W_{L1}$ and $W_{R2}$. The condition for the resonant term coefficient $V_{L1}$ (hence also $V_{R2}$) to vanish is $c_2 \cos \omega (1 - X) - c_1 \sin \omega (1 - X) = 0$, i.e.

$$\frac{dU_R}{dt} - \frac{U_R}{2\omega} \frac{d\omega}{dt} = \frac{(\alpha - 1) U_R X (X - 1) \tan \omega (1 - X)}{\alpha + \alpha \tan^2 \omega (1 - X) - (\alpha - 1) X} \frac{d\omega}{dt}$$
$$- \frac{\alpha (\alpha - 1) U_R [1 - \omega X \tan \omega (1 - X)]}{\alpha + \alpha \tan^2 \omega (1 - X) - (\alpha - 1) X},$$

(3.61)

Using the derivative with respect to $t$ of equation (3.32), and taking into account that $X = \alpha t$, the condition becomes

$$\frac{1}{U_R} \frac{dU_R}{dX} + \frac{1}{2\omega} \frac{d\omega}{dX} = \frac{X \tan \omega (1 - X) [1 + \alpha \tan^2 \omega (1 - X)]}{\alpha + \alpha \tan^2 \omega (1 - X) - (\alpha - 1) X} \frac{d\omega}{dX},$$

(3.62)

which characterizes the evolution of the leading-order coefficients as the front moves.

(g) Initial conditions

The form of the leading-order solution has been obtained in §3e, and the evolution on the slow time $t$ of the spectrum is given by equation (3.32). From §3f, the evolution of the amplitudes is governed by equation (3.62). To find the initial values of these coefficients, one considers the initial events on the fast time scale $\theta$, while $t = 0$: the time period when the front has just started moving and started pushing the column of fluid ahead of the front, towards the open end, while the pressure wave separating it from the fluid that is not yet moving is propagating towards the open end, which it has not yet reached. This sequence of events takes place on the fast, acoustic time scale. The set $\theta_k$ is related to the initial short time $\tilde{t}$,

$$\theta_k = \omega_k (0) \tilde{t}.$$

(3.63)

These initial conditions are projected onto the initial Fourier series. Dropping the superscript (1) in the leading-order solution, and taking into account that, on the acoustic time scale, the front has only moved a distance of order $M$, where $u_L = 0$, the leading-order velocity solution in nearly the entire tube is

$$u_R = \alpha - 1 + \sum_{k=0}^{+\infty} U_{Rk} (0) \{ \exp i \omega_k (0) [\tilde{t} + (1 - x)] + \exp i \omega_k (0) [\tilde{t} - (1 - x)] \}$$
$$+ \tilde{U}_{Rk} (0) \{ \exp - i \omega_k (0) [\tilde{t} + (1 - x)] + \exp - i \omega_k (0) [\tilde{t} - (1 - x)] \},$$

(3.64)

in which, for a tube closed at one end and open at the other end, i.e. a quarter-wave,

$$\omega_k (0) = \frac{\pi}{2} + k \pi.$$

(3.65)
Taking into account that \( \exp i\omega_k(0) = i(-1)^k \), this solution can be written as

\[
u_R = \alpha - 1 + i \sum_{k=0}^{+\infty} (-1)^k \{ U_{Rk}(0) [\exp i\omega_k(0)(\tilde{t} - x) - \exp i\omega_k(0)(\tilde{t} + x)] - \hat{U}_{Rk}(0) [\exp - i\omega_k(0)(\tilde{t} - x) - \exp - i\omega_k(0)(\tilde{t} + x)] \}.
\]

(3.66)

Introducing a left-going mode \( \mathcal{L}(z_L) \) and a right-going mode \( \mathcal{R}(z_R) \), given by

\[
\mathcal{L}(z_L) = C_k \exp i\omega_k(0)z_L + \hat{C}_k \exp - i\omega_k(0)z_L
\]

and

\[
\mathcal{R}(z_R) = -C_k \exp i\omega_k(0)z_R - \hat{C}_k \exp - i\omega_k(0)z_R,
\]

in which \( z_L = \tilde{t} + x \), \( z_R = \tilde{t} - x \) and \( C_k = -i(-1)^k U_{Rk} \), the solution is

\[
u_R = \alpha - 1 + \mathcal{L} + \mathcal{R} \quad \text{and} \quad p^{(1)} = \gamma(\mathcal{R} - \mathcal{L}). \quad (3.69)
\]

The initial conditions are \( u = 0 \) and \( p^{(1)} = 0 \), at \( \tilde{t} = 0 \) and for \( 0 \leq x \leq 1 \). For \( z_L \) and \( z_R \), \( 0 \leq z_L \leq 1 \) and \( -1 \leq z_R \leq 0 \), where \( u = 0 \) and \( p^{(1)} = 0 \), so that \( \mathcal{R}(z) = -(\alpha - 1)/2 \) for \( 0 < z < 1 \) and \( \mathcal{L}(z) = -(\alpha - 1)/2 \) for \( -1 < z < 0 \).

Boundary conditions are, respectively, at the closed end \( x = 0 \), i.e. \( z_L = z_R \), \( u = \alpha - 1 \), and at the open end \( x = 1 \), where \( z_L - z_R = 2p = \gamma(\mathcal{R} - \mathcal{L}) = 0 \). The former one, at \( x = 0 \), requires \( \mathcal{R}(z) + \mathcal{L}(z) = 0 \), while the latter requires \( \mathcal{L}(z) = \mathcal{R}(z - 2) \). These conditions result in \( \mathcal{L} \) and \( \mathcal{R} \) being periodic functions with period 4. \( \mathcal{L} = -(\alpha - 1)/2 \) for \( 0 < z < 2 \) and \( \mathcal{L} = (\alpha - 1)/2 \) for \( 2 < z < 4 \) while \( \mathcal{R} = -\mathcal{L} \), with Fourier coefficients for these functions \( C_k = -(\alpha - 1)/2i\omega_k(0) \). Thus, the coefficients \( U_{Rk} \) are

\[
U_{Rk}(0) = \frac{-(\alpha - 1)(-1)^k}{2\omega_k(0)} = \frac{-(\alpha - 1)(-1)^k}{\pi(1 + 2k)}.
\]

(3.70)

These are the initial values of the coefficients in the unburnt region. The coefficients in the burnt region are obtained from equation (3.31).

(h) Reconstruction

The leading-order solution can now be reconstructed as follows. From equations (3.64) and (3.10), and taking into consideration that the coefficients \( U_{Rk} \) are real, velocity and pressure in the unburnt region are given by

\[
u_R = \alpha - 1 + \sum_{k=0}^{+\infty} 2U_{Rk} \{ \cos[\theta + \omega_k(1-x)] + \cos[\theta - \omega_k(1-x)] \} \quad (3.71)
\]

and

\[
p_R = \sum_{k=0}^{+\infty} \frac{2\gamma U_{Rk}}{\omega_k} \{ \cos[\theta + \omega_k(1-x)] - \cos[\theta - \omega_k(1-x)] \}.
\]

(3.72)

For the burnt region, the coefficients are found from equation (3.31). Taking the latter into account, velocities and the pressure perturbation are written as
\begin{equation}
    u_L = \sum_{k=0}^{+\infty} 2\sqrt{\alpha} U_{Rk} \frac{\sin \omega_k(1-X)}{\cos \omega_k X/\sqrt{\alpha}} \left[ \sin \left( \theta + \frac{\omega_k x}{\sqrt{\alpha}} \right) - \sin \left( \theta - \frac{\omega_k x}{\sqrt{\alpha}} \right) \right]
\end{equation}

and

\begin{equation}
    p_L = \sum_{k=0}^{+\infty} 2\gamma U_{Rk} \frac{\sin \omega_k(1-X)}{\cos \omega_k X/\sqrt{\alpha}} \left[ -\sin \left( \theta + \frac{\omega_k x}{\sqrt{\alpha}} \right) - \sin \left( \theta - \frac{\omega_k x}{\sqrt{\alpha}} \right) \right].
\end{equation}

In the expressions above, the values \( \theta_k \) are related to time \( t \) by equation (3.1). The velocity at the open end and the pressure at the closed end are

\begin{align*}
    u_R &= \alpha - 1 + 4 \cos \theta \sum_{k=0}^{+\infty} U_{Rk} \quad \text{and} \quad p_L = -4\gamma \sin \theta \sum_{k=0}^{+\infty} U_{Rk} \frac{\sin \omega_k(1-X)}{\cos \omega_k X/\sqrt{\alpha}}.
\end{align*}

\section{Solution method}

Obtaining the complete solution entails solving the dispersion relationship, equation (3.32), numerically for a sufficient number of modes, then finding the evolution of the amplitudes, by numerical integration of equation (3.70), with initial conditions given by equation (3.62), and, finally, reconstructing the entire solution by adding the modes, as per equations (3.71)–(3.74). Solving the dispersion relationship is by far the most laborious part of the process. Given that there are infinitely many modes, and that the intervals between them vary, a systematic carpet search is required for each front location. To that effect, the range of frequency values that is being explored is divided into intervals equal to the desired accuracy and locating intervals in which the l.h.s. of equation (3.32) changes sign. The evolution of the amplitudes is obtained using a third-order accurate Runge–Kutta scheme, starting from the initial values provided by equation (3.70) and marching in the slow time. Implementation is easy since the frequencies are already known at the beginning and at the end of each step.

\section{Numerical model}

The results from the analysis (shown below in §5) look very irregular. For independent verification, a numerical simulation was performed using the same physical model described in §2, but for arbitrary Mach numbers. A well-validated essentially non-oscillatory code was used, originally developed by Xu et al. (1997) and used in studies of detonation cells (Liang & Bauwens 2005a,b).

As in the analysis, the flame is represented as a front propagating into unburnt fluid at constant velocity relative to the unburnt mixture, accompanied by energy release. Owing to the order of magnitude difference between the speed of sound and the speed of the front, a subgrid model was implemented within the computational cell in which the front lies. Properly capturing acoustics entails satisfying a Courant–Friedrichs–Lewy condition, requiring time steps small enough to track the acoustics on the spatial grid. Over these very small time steps, the motion of the front is smaller than the cell width in a ratio of the order of the Mach number. Thus, the front stays within one computational cell for
many time steps, which would be quite diffusive if the precise location of the front within the cell was not tracked. The subgrid model entails recording and updating the precise location within the cell, and providing proper continuity conditions to the integration algorithm in the two zones on both sides of the front. To that effect, an extra artificial cell is added on each side of the front, populated with appropriate values, such that the fluid dynamics integration algorithm can handle each partial domain up to the last physical grid point, evaluated using a second-order accurate linear extrapolation, and satisfying the full nonlinear Rankine–Hugoniot conditions.

Although the open end globally consists of a subsonic outlet, there are periods during which it becomes a subsonic inlet due to velocity oscillations, hence requiring an additional boundary condition (for entropy or temperature).

### 5. Comparison between analysis and simulation

In this section, results from both the analysis and the numerical solution are compared, and the two techniques are assessed. Physical interpretation is discussed in §6. The ratio burning velocity/speed of sound in the unburnt mixture was 0.01, $\gamma=1.4$ and $\alpha=7$. Under these conditions, the steady component of the motion of the front moves at a Mach number of 0.07. The spectrum was computed at 5000 locations, and 5000 steps were used in the numerical integration determining amplitudes.

Figure 1 shows the evolution of the frequencies $\omega_j=\omega_j(X)$ with $X=\alpha t$, obtained from numerical solution of the dispersion relationship, equation (3.32). These results are similar to the spectra obtained by Jost (1939). In addition to the spectra themselves, figure 1 includes the resonant frequencies $\omega_L$ and $\omega_R$, obtained assuming a velocity node at the front, respectively, for the burnt region, with $\omega_{L_k} = (1 + 2k)\pi/2(1 - X)$, and for the unburnt region, with $\omega_{R_k} = k\pi/X\sqrt{\alpha}$. The figure shows these lines intersecting the lines representing the modes $\omega_k$ characterizing the entire tube at inflection points, at which the tangent is horizontal. Indeed, the derivative of $\omega$ becomes zero when $\cos \omega(1 - X) = 0$, which, in view of equation (3.32), also entails $\sin \omega X/\sqrt{\alpha} = 0$. The inflection points are consistent with the Rayleigh criterion (e.g. Mcmanus et al. 1993), since they occur when the front coincides with a pressure node. For $X=0$ and 1, the spectrum is that of a Fourier series. For intermediate locations, however, the eigenvalues do not differ by fixed amounts. In figure 1, in particular in the neighbourhood of inflection points, intervals between one mode and the next at given front locations clearly vary by non-monotonic amounts.

Figure 2 shows the evolution of the amplitudes $U_{R_k}$ normalized by their initial value, over the entire travel time of the front from the closed to the open end, which takes a dimensionless time of $1/\alpha$ or approximately 0.14. According to equation (3.62), $dU=0$ when $d\omega=0$. While, as mentioned above, these points are inflection points for frequencies, whose derivative does not change sign, the derivatives of the amplitudes change sign. Thus, for each inflection point in figure 1, there is an extremum in the corresponding amplitude, as the plot shows. Additionally, for the lower modes, there is an extra maximum corresponding to a zero of the coefficient of $d\omega$ in equation (3.62). The first mode experiences a single maximum, not far before the front reaches the open end. The second mode
experiences one maximum at approximately 35 per cent of the length, and one minimum at approximately 80 per cent. Higher modes become more oscillatory and the interval between peaks decreases as the front approaches the open end.

These amplitudes are obtained as the solution of equation (3.62), in which the l.h.s. equals \( \frac{d \log \sqrt{\omega_k} U_{Rk}}{dX} \). Thus, the r.h.s. represents the source of acoustic energy due to interaction with the propagating front. Because the rate at which frequencies increases suffers oscillations, as shown in figure 1, the amplitudes follow a pattern whereby higher modes are more oscillatory, particularly when close to the open end. This can be explained by conservation of acoustic energy, even before taking into account the source term. However, as figure 3 shows, the source term is significant, magnifying the oscillation. Thus, there is an interaction between the front and acoustics, with the latter serving as a source of acoustic energy.

Figure 1. Evolution of the spectrum. The thick solid curves are the actual modes \( \omega_Lk \) and thin solid curves show modes \( \omega_Uk \) obtained for the burnt side and \( \omega_Rk \) for the unburnt side of the front, assuming a velocity node at the front.

Figure 2. Amplitudes \( U_{Rk} \) normalized by their initial value. (a) First 10 modes (lower ones are less oscillatory) and (b) mode 75.
Next, the solution for velocity and pressure is reconstructed. Figures 4 and 5 show the pressure at the closed end and the velocity at the open end, respectively, from the analysis, truncating after 10 and 40 modes. Results for 100 and 200 modes are virtually indistinguishable from those with 40 modes. The narrow peaks that the solution exhibits do not become smaller with a higher number of modes.

Although resulting from the smooth amplitudes shown in figure 2, the pressure and velocity signals are quite irregular. A numerical study by Bjerkeveldt et al. (2002) showing similar behaviour motivated the independent numerical simulation, using the second-order accurate procedure described in §4. The resolution used was 1000 grid points in the tube length; results obtained with 500 and 2000 points showed no visible difference. These well-converged numerical results are very similar to results from analysis and they confirm the irregular behaviour, as shown in figure 6. The two sets of results are shown superimposed in figure 7. They are quite close, but there are differences. First, the analysis yields spurious peaks, associated with the initial reflections, and these peaks themselves reflect, leading to additional peaks later on in the plots. Indeed, initially, the order $M$ pressure at the closed end should be constant (equal to the

Figure 3. Acoustic energy $\omega_k U_k^2$. (a) First 10 modes and (b) (top to bottom) modes 25, 100 and 200.

Figure 4. Pressure perturbation at the closed end from the analysis. Truncating after (a) 10 and (b) 40 modes.
pressure jump across the weak shock subsequent to ignition), until the return of the first reflection. Likewise, at the open end, the velocity should be zero until that shock reaches the open end, and then still constant until the next reflections return. This is indeed what the results from the simulations show: on both

Figure 5. Velocity at the open end from the analysis. Truncating after (a) 10 and (b) 40 modes.

Figure 6. (a) Pressure perturbation at the closed end and (b) velocity at the open end, from numerical simulations.

Figure 7. (a) Pressure perturbation at the closed end and (b) velocity at the open end. Thick solid curves, from numerical simulations; thin solid curves, from the analysis.
pressure and velocity plots, the signals are initially constant, and the first jump
does not exhibit any overshoot. On the results from the analysis, however, there
are already overshoots at these first reflections, both on pressure and velocity
signals. These overshoots look somewhat similar to the results of the Gibbs
phenomenon, although they may be due to a limitation of the small Mach
number assumption: in the limit, shocks travel at the speed of sound, while they
would travel faster when taking their finite magnitude into account. Thus,
including a second-order correction should alleviate this discrepancy; it is clear
from figures 4 and 5 that adding more modes does not help. Already after the first
reflection, the values of the velocity and the pressure perturbation do not quite
match. This is indeed due to the low Mach number approximation that can be
verified by looking at the velocity at the open end after the first reflection. In an
acoustic model, the velocity, which had a value equal to $a_1 = 6$ in the region
between the front and the leading-pressure wave, doubles after reflection at the
end. However, when resolving exactly the expansion wave that results from
the reflection, it is found that the velocity between the expansion wave and the
exit should have a numerical value of 11.74. Indeed, these two numbers are quite
close to the results obtained, respectively, from the theory and the simulation.
Thus, the differences are largely due to the approximation associated with
the small Mach number limit. The initial peaks, present in the results from the
analysis but absent in numerical results, are obviously spurious. However,
comparing the two sets of results, it is clear that subsequent peaks are also larger
in results from theory than in numerical results. From the times at which these
peaks occur, however, it is clear that they are associated with reflection at the
ends of the original spurious peaks.

In summary, while the comparison between the two sets of results does show
some differences, these are explained by the approximate nature of the small
Mach number theory, thus validating the method used.

6. Physical interpretation

The impulsive start of the mass of fluid separating ignition near the closed end
from the open end initially triggers a significant acoustic flow. Subsequently,
neither reflections at both ends nor the nearly inviscid and non-conducting flow in
the tube provide significant attenuation. Neglecting dissipative mechanisms
altogether, the acoustics are then not subjected to any attenuation. However, due
to the discontinuity in the temperature and density at the front, hence in the speed
of sound, the resonant acoustic frequencies are unlike those found in a homogeneous
mixture: frequencies do not differ by uniform values between modes, and they
evolve as the front moves, which also affect amplitudes. Finally, acoustic energy in
each mode is not a conserved quantity, but as equation (3.62) shows, interaction
with the front leads to a non-zero source term in the conservation equation for
acoustic energy. If the source term was neglected, results would still look similar,
but the amplitudes would not be quite as high as in the results from numerical
simulations, especially as the front approaches the open end.

The pressure signals are qualitatively similar to experimental measurements
such as in Kerampran et al. (2000). The irregularity of the signal is thus
intrinsic to the phenomenon itself, rather than, as one might
easily speculate, to difficulties associated with acoustics in the measurement set-up. The results also show that the one-dimensional acoustic model is sufficient to explain weak flow inversions, as shown in figure 8, where negative values of the front speed correspond to a front moving backward, i.e. towards the closed end. The acoustic model alone is not powerful enough, however, to explain the more violent fluctuations seen in the experimental results. Taking into account that although the reaction front is a strong density interface, and that the acoustic oscillations identified in the current analysis result in significant oscillating accelerations, the front will be subject to a powerful Rayleigh–Taylor mechanism, destabilizing an accelerating front but restabilizing a decelerating one. Based upon the second derivative of position, acoustic accelerations should have a magnitude of approximately $\pi(a - 1)S_u^2/2L$, in which $S_u$ is the burning velocity. Figure 8, however, shows velocity changing very rapidly, in thin spikes that reflect the combined action of many modes. Thus, the magnitude of acceleration is higher than that value, which assumes changes on time scales associated with the period of the lower modes. Even based upon that estimate, for typical values $a=7$, $S_u=1\text{ m s}^{-1}$ and $L=1\text{ m}$, accelerations are comparable with gravity. Thus, when the front is accelerating, light burnt combustion products are accelerating into the heavy unburnt mixture, so that the front will be strongly Rayleigh–Taylor unstable, only to see the same mechanism ‘relaminarizing’ the front when the acoustic acceleration periodically acts in the opposite direction. This is indeed clear in the experiments of Kerampran et al. (2000).

7. Conclusion

A complete one-dimensional multiple-scale acoustic/advective solution has been constructed for oscillating flames in a tube closed at one end and open at the other end, with ignition occurring at the closed end. While classical studies by Jost (1939) and Guénoche (1964) were only able to determine the spectrum and its evolution, in the multiple-scale analysis, removal of secularity at higher order
also yields the evolution of the acoustic amplitudes. The solution consists of a Fourier series in the fast time, with frequencies and amplitudes that depend upon the slow time.

This solution was verified by comparison with a fully numerical solution to the same physical model, revealing an excellent match, given the approximate nature of the low Mach number analysis. Although the fully numerical solution turns out to be faster than the numerical solution of the dispersion relationship, which the analysis requires, the analysis provides insight into the physics of the phenomenon. If the goal was only to obtain numerical values, the numerical solution would be the preferred one, given that it is easier to implement and also more accurate.

The analysis points to the mechanism leading to oscillations and its acoustic nature. Initially, impulsive start at the closed end of the tube results in a weak leading shock separating burnt mixture at rest in the space between the closed end and the front from unburnt mixture being pushed by the combustion front towards the open end, at a velocity equal to the burning velocity multiplied by the expansion ratio minus unity. However, the solution does not stay nice and steady. Indeed, the leading shock reflects at the open end, and then the reflected wave itself reflects at the front and at the closed end. This process is followed by an increasingly complex sequence of reflections, which the analysis captures.

According to both the analysis and the simulation, this sequence of increasingly complex reflections results in very irregular velocity and pressure profiles, qualitatively similar to experimental measurements (Kerampran et al. 2000). The results clearly show the acoustic nature of the process, triggered by the impulsive start of the front at the closed end. These acoustic oscillations alone are sufficient to produce reversals of the flame motion in the laboratory frame. But they also result in a large periodically oscillating acceleration, which, in addition to hydrodynamic instability, will enable periodic Rayleigh–Taylor stability/instability of the front, which the current one-dimensional model, however, is unable to deal with. The Rayleigh–Taylor mechanism explains experiments (Kerampran et al. 2000), which clearly show that, at times when the flame is accelerating, the front becomes very unstable, while a decelerating front returns to a very smooth and ‘laminar’ pattern.

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