High-frequency homogenization for periodic media

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An asymptotic procedure based upon a two-scale approach is developed for wave propagation in a doubly periodic inhomogeneous medium with a characteristic length scale of microstructure far less than that of the macrostructure. In periodic media, there are frequencies for which standing waves, periodic with the period or double period of the cell, on the microscale emerge. These frequencies do not belong to the low-frequency range of validity covered by the classical homogenization theory, which motivates our use of the term ‘high-frequency homogenization’ when perturbing about these standing waves. The resulting long-wave equations are deduced only explicitly dependent upon the macroscale, with the microscale represented by integral quantities. These equations accurately reproduce the behaviour of the Bloch mode spectrum near the edges of the Brillouin zone, hence yielding an explicit way for homogenizing periodic media in the vicinity of ‘cell resonances’. The similarity of such model equations to high-frequency long wavelength asymptotics, for homogeneous acoustic and elastic waveguides, valid in the vicinities of thickness resonances is emphasized. Several illustrative examples are considered and show the efficacy of the developed techniques.

Keywords: Floquet–Bloch waves; stop bands; photonics; high-frequency long waves; homogenization

1. Introduction

Many structures are constructed from composites with periodic, or doubly periodic, variations in material parameters. The varied, and sometimes unexpected, wave propagation properties of composites (Milton 2002) have motivated a number of remarkable new applications, including, but not limited to, photonic crystals and microstructured fibres (Joannopoulos et al. 1995; Zolla et al. 2005), as well as the rapidly developing area of metamaterials (Smith et al. 2004). Thus, there is considerable interest in modelling wave propagation through media with regularly spaced defects or inhomogeneities. Direct numerical simulation using finite elements (Zolla et al. 2005) is popular, but can become

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intensive for large-scale media with many small inclusions even if calculated over a single cell with Floquet–Bloch conditions invoked. In the latter case, progress can also be made using Rayleigh’s multipole method (Rayleigh 1892) and modern extensions and refinements are possible (Movchan et al. 2002; Poulton et al. in press). Another popular procedure is the plane wave expansion method that amounts to expanding all relevant fields over the cell into Fourier series and solving the resulting infinite system of linear equations (Kushwaha et al. 1993; Andrianov et al. 2008).

Complementary to this literature is that of homogenization, which involves taking a medium with rapidly oscillatory material properties on a fine microscale and averaging these out, in some fashion, to obtain an equivalent homogeneous material with effective material parameters. This is naturally convenient as it buries the microstructure into coefficients, and computations (or analysis) are then just performed on the macroscale. The development of traditional techniques of asymptotic homogenization has been strongly focused on recovering the classical limiting behaviours in effective media (Sanchez-Palencia 1980; Bakhvalov & Panasenko 1989). This usually implies slow variation of relevant fields both on the microscale and on the macroscale, which limits the applicability of the associated expansions to low-frequency situations, e.g. Parnell & Abrahams (2006) or Andrianov et al. (2008). While the frequency range of such models can be extended by considering higher order correction terms (Sanchez-Palencia 1980; Bakhvalov & Eglit 2000; Smyshlyaev & Cherednichenko 2000), the resulting models cannot fully reproduce high-frequency dynamic behaviours characteristic of microstructured materials, such as strong dispersion, the presence of band gaps or negative refraction. The other way to describe this limitation of the traditional homogenization theory would be to say that it is only capable of describing the fundamental Bloch mode at low frequencies. When inclusions are small with respect to the period, it is possible to construct ‘wide-spacing’ approximation schemes capable of reproducing higher Bloch modes (Poulton et al. 2001; McIver 2007).

Traditional homogenization holds the frequency fixed, while the natural small parameter tends to zero, which is incompatible with the rapidly oscillating fields characteristic of higher Bloch modes. Bensoussan et al. (1978) demonstrate that in order to obtain the complete spectrum of Bloch modes, in their notation, one needs to scale frequency as the inverse of the natural small parameter, i.e. to consider the high-frequency regime. Even at leading order, fields obtained in this asymptotic limit oscillate on the microscale, hence motivating the use of the Wentzel–Kramers–Brillouin–Jeffreys (WKB) ansatz by Bensoussan et al. (1978). Intuitively, this may seem to contradict the very definition of what homogenization is usually thought to achieve. Nevertheless, for fields oscillating at the microscale, the variation of the solution from one periodicity cell to another can be very small. The high-frequency homogenization we develop is designed to exploit this situation and aims to model the modulation of the strongly oscillating field. Although very different from the physical point of view, a mathematically similar asymptotic regime is also observed in the so-called ‘double porosity limit’ of high-contrast homogenization, e.g. Arbogast et al. (1990). When the contrast in material parameters is sufficiently large, higher Bloch modes may become part of the low-frequency response; this has been recently studied in the context of wave propagation (Babych et al. 2008; Smyshlyaev 2009).
For periodic, and doubly periodic, media the interest is often in identifying the Bloch spectra and stop band structure of the solution (Zolla et al. 2005). The Bloch spectra at the edges of the irreducible Brillouin zone correspond to standing waves (Brillouin 1953). Our aim is to develop a high-frequency asymptotic procedure based upon perturbing about these standing wave solutions occurring at particular frequencies across a periodic structure. Taking advantage of the scale separation between micro- and macroscales, the standing waves can be considered upon the microscale and then ‘averaged’ to get a problem just upon the macroscale. The existence of a differential operator of this type has been recently proved rigorously (Birman 2004; Birman & Suslina 2006). The upshot of our analysis is that an effective partial differential equation is constructed on the macroscale that is valid for frequencies in the vicinity of the standing wave. The coefficients of this equation involve integrals of the standing wave solutions and an auxiliary solution along an elementary cell on the microscale. The problem is therefore homogenized in the sense that the microscale plays no explicit role in the effective model. However, in contrast to classical homogenization, the asymptotic macroscale equation is strongly dispersive and can be used to model a range of dynamic phenomena characteristic for composite media. The final form of the effective model bears remarkable similarity to high-frequency long-wave equations in asymptotic theories, for elastic and acoustic waveguides, valid in the vicinity of thickness resonances (Berdichevski 1983; Kaplunov et al. 1998; Le 1999; Gridin et al. 2005).

This paper is structured as follows. In §2, we take a two-dimensional structure and develop an asymptotic procedure where we perturb away from standing wave solutions. The exposition of the two-dimensional case is simplified by considering standing waves with period of exactly one cell size. Very similar expansions can be constructed in the vicinity of standing waves with the period of two cell sizes along one or both coordinates; motivated by the form of corresponding boundary conditions, we term such waves anti-periodic. The point is explicitly demonstrated for one-dimensional examples treated in §2a. Illustrative examples aimed at showing the accuracy of the asymptotics for Floquet–Bloch problems are then chosen. A piecewise homogeneous medium, §3a, illustrates the influence of double roots in the Bloch spectra. The piecewise homogeneous string on a Winkler foundation, §3b, has a lowest curve in the spectra that does not pass through the origin, and a low-frequency band gap, allowing us to contrast the asymptotics developed here with the classical theory. A string with periodically varying density considered in §3c leads us to a Mathieu equation for which the leading order solutions can have a degeneracy that must be overcome, and is a non-trivial example for which the asymptotics are pursued using numerical methods. Some concluding remarks are drawn together in §4.

2. General theory

For generality, we consider a regular periodic structure composed of inclusions and a matrix material. The inclusions are defined on a short scale \( l \), that of the microstructure, whereas the overall problem is considered on a macroscale \( L \) that can be thought of as a typical wavelength dominating the dynamic response, or an overall dimension of the structure (see figure 1 for an illustration of typical...
structures). The ratio of these scales, \( \epsilon \equiv l / L \), is assumed small and provides a natural small parameter. The microstructure is characterized by functions \( \hat{a}(x_1/l, x_2/l) \) and \( \hat{\rho}(x_1/l, x_2/l) \) that are periodic in \( \xi = (x_1/l, x_2/l) \) and can be smooth functions, or piecewise continuous; \( \mathbf{x} = (x_1, x_2) \) are Cartesian coordinates orientated along the edges of the cell.

We consider a model wave equation for, say, shear horizontal waves in anti-plane isotropic elasticity with periodic density \( \hat{\rho}(\xi) \) and shear modulus \( \hat{a}(\xi) \), and with time harmonic dependence \( \exp(-i \omega t) \) assumed understood, as

\[
\nabla_x \cdot [\hat{a}(\xi) \nabla_x u(x)] + \omega^2 \hat{\rho}(\xi) u(x) = 0
\]

on \(-\infty < x_1, x_2 < \infty\) with \( \nabla_x \) as the gradient operator with respect to the \( \mathbf{x} \) coordinates. The material parameters in equation (2.1) can be measured in their typical unit values, so that \( \hat{a} \equiv \hat{a}_0 a(x) \) and \( \hat{\rho} \equiv \hat{\rho}_0 \rho(x) \), with the absence of hats indicating non-dimensionality. In this case, equation (2.1) transforms to

\[
l^2 \nabla_x \cdot [a(\xi) \nabla_x u(x)] + \lambda^2 \rho(\xi) u(x) = 0 \quad \text{with} \quad \lambda = \frac{\omega l}{\hat{c}_0} \quad (2.2)
\]

with \( \hat{c}_0 = \sqrt{\hat{a}_0 / \hat{\rho}_0} \) a characteristic wave speed. We adopt a multiple scales approach treating the disparate length scales \( \mathbf{X} = x/L \), and \( \xi = x/l \) as new independent variables with the result that equation (2.2) becomes

\[
\nabla_{\xi} \cdot [a(\xi) \nabla_{\xi} u(\mathbf{X}, \xi)] + \lambda^2 a(\xi) u(\mathbf{X}, \xi) \\
+ \epsilon \left[ 2 a(\xi) \nabla_{\xi} + \nabla_{\xi} a(\xi) \right] \cdot \nabla_{X} u(\mathbf{X}, \xi) + \epsilon^2 a(\xi) \nabla_{X}^2 u(\mathbf{X}, \xi) = 0, \quad (2.3)
\]

where the separation of scales is now made explicit. The notations \( \nabla_{\xi} \) and \( \nabla_{X} \) denote \( \left( \partial_{\xi_1}, \partial_{\xi_2} \right) \) and \( \left( \partial_{X_1}, \partial_{X_2} \right) \) respectively. Classical homogenization theory is usually concerned with \( \lambda \ll 1 \), see equation (2.2); the crucial distinction of high-frequency homogenization allows for \( \lambda \sim O(1) \).

We develop the methodology for perturbations to standing wave solutions that are periodic on the cell. If we take the periodicity in \( \xi \) to be on a square cell with
−1 ≤ ξ_i ≤ 1, for i = 1, 2, then periodicity conditions are to be taken in ξ along the edges of the cell, namely
\[ u|_{ξ_1=1} = u|_{ξ_1=-1}, \quad u|_{ξ_2=1} = u|_{ξ_2=-1} \] (2.4)
and
\[ u_ξ|_{ξ_1=1} = u_ξ|_{ξ_1=-1}, \quad u_ξ|_{ξ_2=1} = u_ξ|_{ξ_2=-1}. \] (2.5)
The corresponding solution \( u(X, ξ) \) will be periodic in ξ, but not necessarily in X. One can replace these periodicity conditions along one or both spatial dimensions with special ‘out-of-phase’ boundary conditions. These anti-periodic boundary conditions lead to solutions that are periodic across two cells in the appropriate direction(s) of ξ. For the sake of clarity, we specialize the two-dimensional derivations to the case periodic in both \( ξ_1 \) and \( ξ_2 \) for which equations (2.4) and (2.5) apply. We illustrate the application and consequences of anti-periodic boundary conditions only for the one-dimensional case presented in §2a.

Next, we adopt the ansatz:
\[ u(X, ξ) = u_0(X, ξ) + ε u_1(X, ξ) + ε^2 u_2(X, ξ) + \cdots, \quad λ^2 = λ_0^2 + ε λ_1^2 + ε^2 λ_2^2 + \cdots. \] (2.6)
Each \( u_i(X, ξ) \) for \( i = 1, 2, \ldots \), is periodic in ξ, cf. equations (2.4) and (2.5). This ansatz assumes variation at both the microscale and macroscale even at leading order, as opposed to the classical homogenization theory for which \( u_0(X, ξ) \equiv u_0(X) \).

Substituting equation (2.6) into equation (2.3), and equating to zero the coefficients of individual powers of ε, we obtain a hierarchy of equations for \( u_i(X, ξ) \) and \( λ_i \) with associated boundary conditions from the periodicity in ξ. This hierarchy is resolved from the lowest order up. At the lowest order, we obtain an eigenvalue problem for
\[ \nabla_ξ \cdot [a(ξ) \nabla_ξ u_0] + λ_0^2 ι(ξ) u_0 = 0 \] (2.7)
subject to the appropriate periodicity boundary conditions. This gives rise to a discrete spectrum of eigenvalues \( λ_0^2 \) for which there is no phase shift across a period of the structure and standing wave is formed. If we fix a simple eigenvalue \( λ_0 \), then the corresponding eigenmode is of the form
\[ u_0(X, ξ) = f_0(X) U_0(ξ, λ_0), \] (2.8)
where \( U_0(ξ, λ_0) \) is a periodic function of ξ, and \( f_0(X) \) remains to be determined; we introduce \( λ_0 \) into the argument of the periodic function to emphasize that it depends upon the frequency \( λ_0 \). This leading order solution is exactly periodic on the cell, hence conforming to the commonly used notion of the unit cell resonance (or microresonance). Notably, the case of coincident eigenvalues can also arise and is discussed in the simpler context for one-dimensional structures in the following section. Generally, the leading order problem would need to be solved numerically, as is the case in classical homogenization. As we desire the corrections \( λ_1^2, λ_2^2 \) and the function \( f_0(X) \), we continue with the hierarchy.
At next order, the equation for $u_1(X, \xi)$ is

$$
\nabla_\xi \cdot [a(\xi) \nabla_\xi u_1] + \lambda_0^2 \rho(\xi) u_1 = -\nabla X f_0 \cdot [2a(\xi) \nabla_\xi U_0 + U_0 \nabla_\xi a(\xi)] - f_0 \lambda_1^2 \rho(\xi) U_0 \quad (2.9)
$$

and we now invoke an orthogonality condition that involves multiplying equation (2.9) by $U_0$ and integrating over the periodic cell in $\xi$. This yields

$$
\int_S \left( U_0 \nabla_\xi \cdot [a(\xi) \nabla_\xi u_1] + \lambda_0^2 \rho(\xi) U_0 u_1 \right) dS
= -\nabla X f_0 \cdot \int_S \nabla_\xi [a(\xi) U_0'] dS - f_0 \lambda_1^2 \int_S \rho(\xi) U_0^2 dS, \quad (2.10)
$$

where $\int_S$ denotes integration over the cell. Using a corollary to the divergence theorem, the first integral on the right-hand side is converted to an integral along the edges of the cell, and thereafter we use periodicity of $a(\xi)$ and $U_0$ and it vanishes. We now subtract the integral of equation (2.7), multiplied by $u_1/f_0$, over the cell, from equation (2.10), which results in

$$
\int_S \left( U_0 \nabla_\xi \cdot [a(\xi) \nabla_\xi u_1] - u_1 \nabla_\xi \cdot [a(\xi) \nabla_\xi U_0] \right) dS = -f_0 \lambda_1^2 \int_S \rho(\xi) U_0^2 dS. \quad (2.11)
$$

Using Green’s second identity, the left-hand side becomes

$$
\int_{\partial S} a(\xi) \left( U_0 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial U_0}{\partial n} \right) ds, \quad (2.12)
$$

where $\partial S$ is the boundary of region $S$ and $n$ is the outward pointing normal. From periodicity in $\xi$ of $U_0$, $u_1$ and $a$, this integral vanishes. Thus, the only non-zero term is that multiplying $\lambda_1^2$; therefore, $\lambda_1$ must be identically zero. An explicit solution for $u_1(X, \xi)$, from equation (2.9), with $\xi$ restricted to be in the cell $S$, is found as

$$
u_1(X, \xi) = f_1(X) U_0(\xi, \lambda_0) + \nabla X f_0(X) \cdot [V_1(\xi, \lambda_0) - \xi U_0(\xi, \lambda_0)], \quad (2.13)
$$

where the coefficient, $f_1$, of the homogeneous solution does not appear, to leading order, in the final result and plays no further role here. The vector function $V_1 = (V^{(1)}_1, V^{(2)}_1)$ satisfies the leading order equation (2.7), that is,

$$
\nabla_\xi \cdot [a(\xi) \nabla_\xi] V_1 + \lambda_0^2 \rho(\xi) V_1 = 0, \quad (2.14)
$$

i.e. each component of $V_1$ is the non-doubly periodic solution of equation (2.7) linearly independent of $U_0(\xi, \lambda_0)$. Notably $u_1(X, \xi)$ must be periodic in $\xi$; however, both terms in the square brackets of equation (2.13) are not. Therefore, we select each individual component of $V_1$ to be periodic along one of the $\xi_i$ and then choose its boundary conditions along the other $\xi_j$, $j \neq i$, in such a way that conditions (2.4) and (2.5) are enforced; the solution can then be periodically
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continued to the full structure. We set \( V_1^{(1)}(\xi, \lambda_0) \) to have periodicity in \( \xi_2 \) and then periodicity of \( u_1 \) in \( \xi_1 \) results in
\[
V_1^{(1)}|_{\xi_1=1} - V_1^{(1)}|_{\xi_1=-1} = 2U_0|_{\xi_1=1}
\]  
and
\[
V_1^{(1)}|_{\xi_1=1} - V_1^{(1)}|_{\xi_1=-1} = 2U_0|_{\xi_1=1}.
\]

Thus, \( V_1^{(1)}(\xi, \lambda_0) \) is a periodic function in \( \xi_2 \) that must be found by solving the inhomogeneous boundary value problem given by equation (2.14) subject to boundary conditions consistent with equations (2.15) and (2.16), as well as equations (2.4)_2 and (2.5)_2.

Similarly, we take \( V_1^{(2)}(\xi, \lambda_0) \) to be a solution of equation (2.14) periodic in \( \xi_1 \) with boundary conditions
\[
V_1^{(2)}|_{\xi_2=1} - V_1^{(2)}|_{\xi_2=-1} = 2U_0|_{\xi_2=1}
\]  
and
\[
V_1^{(2)}|_{\xi_2=1} - V_1^{(2)}|_{\xi_2=-1} = 2U_0|_{\xi_2=1}.
\]
The second-order equation for \( u_2(\mathbf{X}, \xi) \) is
\[
\nabla_\xi \cdot [a(\xi)\nabla_\xi u_2] + \lambda_0^2 \rho(\xi) u_2 = -a(\xi) U_0 \nabla_X f_0 - [2a(\xi) \nabla_\xi + \nabla_\xi a(\xi)] \cdot \nabla_X u_1
\]  
\[\quad - \lambda_2^2 \rho(\xi)f_0 U_0
\]  
and this contains both \( f_0(\mathbf{X}) \) and the correction, \( \lambda_2^2 \), to the eigenvalue \( \lambda_0^2 \). Invoking an orthogonality condition, as at the previous order, by multiplying equation (2.19) by \( U_0 \), subtracting equation (2.7) times \( u_2/f_0 \) from it, and integrating the result over the cell yields an eigenvalue problem for \( f_0 \) and \( \lambda_2^2 \) as the homogenized partial differential equation
\[
T \frac{\partial^2 f_0}{\partial X_i \partial X_j} + \lambda_2^2 f_0 = 0, \quad \text{with} \quad T_{ij} = \frac{t_{ij}}{\int_S \rho(\xi) U_0^2 \, dS}, \quad \text{for} \ i, j = 1, 2.
\]  

The components of the matrix \( t_{ij} \) are given by
\[
t_{11} = -2 \int_{-1}^1 [a(\xi) U_0^2]_{\xi_1=1} \, d\xi_2 + \int_S (2a(\xi) V_1^{(1)}(\xi_1) + a_{\xi_1}(\xi) V_1^{(2)}(\xi_1)) U_0 \, dS,
\]  
\[
t_{12} = t_{21} = \frac{1}{2} \int_S \left( 2a(\xi)(V_1^{(1)} + V_1^{(2)}) + a_{\xi_1}(\xi) V_1^{(1)} + a_{\xi_1}(\xi) V_1^{(2)} \right) U_0 \, dS
\]  
and
\[
t_{22} = -2 \int_{-1}^1 [a(\xi) U_0^2]_{\xi_2=1} \, d\xi_1 + \int_S (2a(\xi) V_1^{(2)} + a_{\xi_2}(\xi) V_1^{(2)}(\xi_2)) U_0 \, dS.
\]
Thus, given a particular structure, we solve the leading order equation (2.7) to determine \( \lambda_0, \ U_0(\xi, \lambda_0) \). Then, solve equation (2.14), with boundary conditions (2.15)–(2.18) and periodicity, to find \( V_1(\xi, \lambda_0) \). Given these quantities, the differential eigenvalue problem (2.20) is formed, and thus \( \lambda_2 \) identified.
We now specialize to one-dimensional or quasi-one-dimensional problems for which this scheme is more readily applied and for which some explicit details are immediately apparent.

(a) One-dimensional periodic media

Assuming \( a(\xi) \) is constant, then for one-dimensional structures equation (2.2) simplifies to

\[
\frac{d^2 u}{d x^2} + \frac{\lambda^2}{c^2(\xi)} \frac{u}{c^2(\xi)} = 0, \quad \text{with} \quad \lambda = \frac{\omega l}{c_0},
\]

(2.24)

where we define \( c^2(\xi) = a/\rho(\xi) \). The assumption that \( a(\xi) \) is constant in equation (2.24) is not essential as any homogeneous second-order differential equation can be transformed into this form by an appropriate change of variables.

The independent variables are now \( X = x/L, \xi = x/l \), and the solution \( u(X, \xi) \) will be periodic, or anti-periodic, in \( \xi \), but not necessarily in \( X \). The two-scales approach yields

\[
\frac{\partial^2 u}{\partial \xi^2} + 2\epsilon \frac{\partial^2 u}{\partial \xi \partial X} + \epsilon^2 \frac{\partial^2 u}{\partial X^2} + \frac{\lambda^2}{c^2(\xi)} u = 0.
\]

(2.25)

This equation is solved subject to either \( \xi \)-periodicity conditions \( u(X, 1) = u(X, -1) \) and \( u_\xi(X, 1) = u_\xi(X, -1) \) or, what we have named, anti-periodicity conditions \( u(X, 1) = -u(X, -1) \) and \( u_\xi(X, 1) = -u_\xi(X, -1) \), which actually result in the solutions being \( \xi \)-periodic across two cells. As we shall see shortly, these two cases correspond to standing waves that are either in-phase or completely out-of-phase at the end of each cell and, in Floquet–Bloch theory, to wavenumbers situated at the opposite ends of the Brillouin zone.

The separation of scales is loosely analogous to the long-wave high-frequency asymptotics outlined for infinite, deformed, waveguides (Berdichevski 1983; Kaplunov et al. 1998; Le 1999) and this analogy can be pursued to obtain a physical interpretation of the results and this is discussed in §§2b and 4. As in the two-dimensional theory, we form a hierarchy of equations to be solved order-by-order. To leading order

\[
u_0_{\xi\xi} + \frac{\lambda_0^2}{c^2(\xi)} u_0 = 0,
\]

(2.26)

and if \( \lambda_0 \) is a simple eigenvalue of the problem with an appropriate set of boundary conditions, then

\[
u_0(\xi, X) = f_0(X) U_0(\xi, \lambda_0).
\]

(2.27)

In the periodic case, \( U_0(1, \lambda_0) = U_0(-1, \lambda_0) \), \( U_0 \xi(1, \lambda_0) = U_0 \xi(-1, \lambda_0) \) and in the anti-periodic case, \( U_0(1, \lambda_0) = -U_0(-1, \lambda_0) \), \( U_0 \xi(1, \lambda_0) = -U_0 \xi(-1, \lambda_0) \). From Floquet theory, all band gaps present in the Bloch spectra of equation (2.26) are bound by the eigenvalues corresponding to periodic or anti-periodic solutions. Hence, the high-frequency asymptotics developed here are expected to provide accurate description of the solution’s band gap structure in a wide range of problems.
To next order
\[ u_{1\xi\xi} + \frac{\lambda_0^2}{c^2(\xi)} u_1 = -2u_0\xi X - \frac{\lambda_1^2}{c^2(\xi)} u_0 \] (2.28)
with the compatibility condition giving \( \lambda_1 = 0 \). The solution is
\[ u_1(X, \xi) = f_0X [AW_1(\xi, \lambda_0) - \xi U_0(\xi, \lambda_0)] + f_1(X) U_0(\xi, \lambda_0), \] (2.29)
where \( W_1(\xi, \lambda_0) \) is a non-periodic solution of the leading order equation and
\[ A = \frac{2U_0(1, \lambda_0)}{W_1(1, \lambda_0) \mp W_1(-1, \lambda_0)} \] (2.30)
with the upper (lower) sign being for the periodic (anti-periodic) case and the constant \( A \) is chosen to enforce that behaviour. At next order,
\[ u_{2\xi\xi} + \frac{\lambda_0^2}{c^2(\xi)} u_2 = -\frac{\lambda_2^2}{c^2(\xi)} u_0 - u_{0XX} - 2u_{1\xi X} \] (2.31)
with the compatibility condition giving
\[ T f_{0XX} + \lambda_2^2 f_0 = 0, \] (2.32)
where the coefficient \( T \) is
\[ T = 2 \left( -\frac{U_0^2(1, \lambda_0)}{U_1^2(1, \lambda_0) \mp U_1^2(-1, \lambda_0)} + A \int_{-1}^{1} U_0 W_1 d\xi \right). \] (2.33)
The differential eigenvalue problem (2.32) then encapsulates the essential physics on the macroscale and the correction to the frequency squared, \( \lambda_2^2 \), is its eigenvalue.

If the leading order solution has \( U_0(\pm1, \lambda_0) = 0 \), then equation (2.29) is not valid; however, this is easy to overcome with equation (2.33) replaced by
\[ T = 2A \int_{-1}^{1} U_0 W_1 \frac{d\xi}{U_0^2(1, \lambda_0) \mp U_1^2(-1, \lambda_0)}, \] (2.34)
where \( A = 2U_0(1, \lambda_0)/[W_1(1, \lambda_0) \mp W_1(-1, \lambda_0)] \) and the non-periodic function \( W_1 \) satisfies the Dirichlet conditions, \( W_1(\pm1, \lambda_0) = 1 \) (or any non-zero constant).

A degeneracy occurs whenever \( \lambda_0 \) is not a simple eigenvalue, as the leading order solution (2.26) must consist of two linearly independent periodic solutions
\[ u_0(\xi, X) = f_0^{(1)}(X) U_0^{(1)}(\xi, \lambda_0) + f_0^{(2)}(X) U_0^{(2)}(\xi, \lambda_0). \] (2.35)
The compatibility condition for the first-order term then gives two coupled ordinary differential equations (ODEs) for \( f_0^{(1,2)} \) from which the eigenvalue correction \( \lambda_1^2 \) is obtained; the correction to the eigenvalue is now linear rather than quadratic. The coupled equations are
\[ f_0^{(i)} \int_{-1}^{1} U_0^{(j)} U_0^{(i)} d\xi + \lambda_i^2 \int_{-1}^{1} \left( f_0^{(j)} U_0^{(j)} + f_0^{(i)} U_0^{(i)} U_0^{(j)} \right) \frac{d\xi}{c^2(\xi)} = 0 \] (2.36)
for \( i, j = 1, 2 \) and \( j \neq i \).
The asymptotic model (2.32) is not uniform in the sense that when material parameters are continuously varied, such that two eigenvalues approach one another, one has then to switch to the model equations (2.36). Examples from the waveguide theory suggest that it is possible to construct uniformly valid composite expansions (Moukhomodiarov et al. in press) to overcome this.

(b) Analogy with homogeneous waveguides

Placing equation (2.32) in dimensional form

\[ l^2 T \frac{d^2 f_0}{dx^2} + (\lambda^2 - \lambda_0^2)f_0 = 0, \]  

(2.37) 

then this equation governs the one-dimensional long-wave motion of a periodic medium near the resonance frequencies of a cell. It may be subject to macroscopic boundary conditions along the edges and may also involve inhomogeneous terms on the right-hand side corresponding to edge and surface loadings, see Kaplunov et al. (1998) and references therein.

The long-wave equation (2.37) may be interpreted as describing a homogeneous string attached to an elastic foundation. Although the microscale problem is governed by a Helmholtz equation with periodic coefficients, equation (2.24), the proposed model cannot be formally identified as a Helmholtz equation with an averaged wave speed, which contrasts with traditional homogenization theory. At the same time, the model does not contain microscale variables operating only with long-scale phenomena. It is worth noting that non-local equations are also known to occur as homogenized limits for certain high-contrast composite media (Cherednichenko et al. 2006).

As we have mentioned earlier, the model represents an analogue of the high-frequency long-wave theories for elastic and acoustic homogeneous waveguides (e.g. Kaplunov et al. 1998; Gridin et al. 2005). In fact, the problem on a cell (2.26) corresponds to one on the transverse cross section of a thin rod, plate or a shell. In this case, the cell size \( l \) in equation (2.24) corresponds to the half thickness and the cell eigenvalue \( \lambda_0 \) corresponds to a thickness resonance frequency. A similar analogy also occurs in conventional homogenization theory, which is, in a sense, a counterpart of the classical low-frequency theories for rods, plates and shells associated with the names of Euler, Bernoulli, Kirchhoff and Love (e.g. Love 1944; Landau & Lifshitz 1970; Graff 1975). The discussion above is also applicable to two-dimensional periodic media.

3. Illustrative examples

To illustrate the efficacy of this approach, we turn our attention to some examples. The first example, a periodic piecewise homogeneous material, has the advantage of being explicitly solvable and serves to illustrate typical features of Bloch waves and pass-stop bands in periodic media. For some parameters, double roots occur allowing this feature to also be explored. This example is extended by introducing a parameter leading to the formation of a low-frequency stop band, hence allowing us to compare and contrast the high-frequency theory with a
traditional low-frequency homogenization. Our last example is of a string with a periodic density that leads to Mathieu’s equation, allowing us to present a non-trivial application of our asymptotics.

(a) Periodic piecewise homogeneous media

For a material with piecewise constant variation in \( c(\xi) \),

\[
c(\xi) = \begin{cases} 
  \frac{1}{r} & \text{for } 0 \leq \xi < 1 \\
  1 & \text{for } -1 \leq \xi < 0 
\end{cases}
\]  

(3.1)

for \( r \) a positive constant, an exact solution is readily obtained. Floquet–Bloch conditions (Brillouin 1953; Kittel 1996) are set at \( \xi = \pm 1 \); so \( u(1) = \exp(i2\kappa)u(-1) \) and \( u_\xi(1) = \exp(2i\kappa)u_\xi(-1) \), the solution and its derivatives are continuous at \( \xi = 0 \), and \( \kappa \) is the Bloch parameter. The dispersion relation relating frequency, \( \lambda \), to Bloch parameter, \( \kappa \), is

\[
2r[\cos \lambda \cos r\lambda - \cos 2\kappa] - (1 + r^2) \sin \lambda \sin r\lambda = 0.
\]  

(3.2)

This dispersion relation was seemingly first obtained by Kronig & Penney (1931) for electrons in crystal lattices; it also naturally appears in many guises in one-dimensional photonic and phononic crystals (with layers of infinite height), e.g. Movchan et al. (2002) and Adams et al. (2008). There are an infinite number of discrete eigenfrequencies, \( \lambda(n) \), to the dispersion relation that we denote as \( \lambda(n) \) with \( n = 0, 1, 2 \ldots \) ordered from the lowest curve upward; typical Bloch spectra are shown in figure 2. The asymptotic technique we describe extracts the behaviour of the dispersion curves near \( \lambda(0) \), which is the frequency at the edge of the Brillouin zone where \( \kappa = 0 \). The standing waves that occur when \( \kappa = 0 \) correspond to solutions periodic across each elementary cell of width \( 2\kappa \). The other end of the Brillouin zone is at \( 2\kappa = \pi \) and corresponds to standing waves out-of-phase across the cell, the anti-periodic solutions that are periodic on a double cell and the frequencies at which these occur are denoted by \( \lambda(\pi) \).

The asymptotics we develop are valid near each \( \lambda(\pi) \) and all \( \lambda(0) \) except the low-frequency fundamental mode passing through \( \lambda(0) = 0 \). This mode is the one described by classical homogenization, whose characterization is straightforward, \( \lambda(0) \sim 2\kappa(1 + r^2) \), cf. equation (3.14), at \( \beta = 0 \), which corresponds to the substitution

\[
\left\langle \frac{1}{c^2} \right\rangle = \frac{1}{2} \int_{-1}^{1} \frac{1}{c^2(\xi)} \, d\xi
\]  

(3.3)

in equation (2.24). We compare the classical philosophy with the high-frequency theory at the end of §3b.

We now turn to the asymptotic procedure. The leading order solution is determined as

\[
U_0(\xi, \lambda(\pi)) = \begin{cases} 
  \sin r\lambda(\pi)(\xi) + p \cos r\lambda(\pi)(\xi) & \text{for } 0 \leq \xi < 1 \\
  r \sin \lambda(\pi)(\xi) + p \cos \lambda(\pi)(\xi) & \text{for } -1 \leq \xi < 0
\end{cases}
\]  

(3.4)

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for $\lambda^{(n)}_\theta = \lambda_0^{(n)}, \lambda_\pi^{(n)}$ and $p = (r \sin \lambda^{(n)}_\theta \pm \sin r \lambda^{(n)}_\theta)/(\cos \lambda^{(n)}_\theta \mp \cos r \lambda^{(n)}_\theta)$ with the upper, lower signs for $\theta = 0, \pi$, respectively. We also need a linearly independent solution, which does not satisfy periodicity or anti-periodicity conditions at $\xi = \pm 1$, $W_1(\xi, \lambda^{(n)}_\theta)$. We take it as

$$W_1(\xi, \lambda^{(n)}_\theta) = \begin{cases} 
\sin r \lambda^{(n)}_\theta \xi & \text{for } 0 \leq \xi < 1 \\
r \sin \lambda^{(n)}_\theta \xi & \text{for } -1 \leq \xi < 0.
\end{cases}$$

(3.5)

Note that $W_1$ can be taken to be any solution of the leading order equation with fixed $\lambda_0$, which is neither periodic nor anti-periodic. Once $W_1$ is found, it can be re-used in the asymptotics for either end of the Brillouin zone.

Substituting these into equation (2.33) gives $T_0$ and $T_\pi$, that is, the $T$ values associated with the periodic and anti-periodic solutions. These two are written using a single formula as

$$T^{(n)}_\theta = \pm 4 \lambda^{(n)}_\theta \frac{\sin \lambda^{(n)}_\theta \sin \lambda^{(n)}_\theta r}{(r \sin \lambda^{(n)}_\theta \mp \sin r \lambda^{(n)}_\theta)(\cos \lambda^{(n)}_\theta \mp \cos r \lambda^{(n)}_\theta)},$$

(3.6)

where we append a subscript to $T$ to denote its dependence upon $\lambda^{(n)}_\theta$ and $T_0^{(n)}$; $(T^{(n)}_\pi)$ is the value with the upper (lower) sign and $\lambda^{(n)}_\theta = \lambda_0^{(n)}(\lambda^{(n)}_\pi)$. 

Figure 2. (a) The dispersion curves from equation (3.2), for $r = 1/4$, shown across the Brillouin zone. The solid line is from the numerical solution, the asymptotics for $k \to 0$ are represented by the dashed curve and that for $2\epsilon \kappa \to \pi$ are by the dotted curve. Asymptotics for the double root are shown as dot–dashed lines. (b) The detail for the Bloch spectra at the double root at $l^{(4)}_\pi = l^{(5)}_\pi = 4$ with the asymptotics shown as the dot–dashed lines. (c) Plots $\lambda^{(3)} - \lambda^{(3)}_0$ on log–log axes to illustrate the accuracy: numerics are given by the solid line and the asymptotics by the dotted line.
We now turn our attention to the ODE for $f_0$, namely equation (2.32), and note that the Floquet–Bloch conditions lead to $u(X + 2ε, ξ) = \exp(2iκ)u(X, ξ)$. For the periodic case in $ξ$, this forces $f_0(X) = \exp(ikX)$. Thus, we connect the Bloch parameter with the frequency, found from equation (2.32), via $T_{κ^2} = l_0^{(n)^2}$ and deduce that

$$\lambda^{(n)} \sim \lambda_0^{(n)} + \frac{ε^2 l_2^{(n)^2}}{2 l_0^{(n)}} + \cdots = \lambda_0^{(n)} + \frac{ε^2 T_0^{(n)\kappa^2}}{2 l_0^{(n)}} + \cdots. \quad (3.7)$$

Similarly, in the anti-periodic case, we deduce $f_0(X) = \exp(i(κ - π)X)$ and

$$\lambda^{(n)} \sim \lambda_π^{(n)} + \frac{ε^2 T_π^{(n)\kappa(κ - π)^2}}{2 l_π^{(n)}} + \cdots. \quad (3.8)$$

The same results are found directly by expanding the exact dispersion relation (3.2), thereby validating the asymptotic scheme for an example that can be explicitly analysed; the advantage of the asymptotic procedure is that it is applicable even when the dispersion relation is unwieldy or unavailable. As the derived expansion makes no specific assumptions about the form of the leading order solution, it is valid for arbitrary volume fractions, although the resulting numerical accuracy at fixed wavenumbers may vary.

We order the eigenvalues for $n = 0 \ldots ∞$ from lowest to highest in magnitude and note that the asymptotics do not apply to the lowest eigenvalue, $λ_0^{(0)}$, which passes through the origin and which have been approached separately, but is accurate for high frequencies. Figure 2 shows the band gap structure and associated asymptotics that emanate from the frequencies at the edges of the Brillouin zone. The quadratic corrections are highly accurate even for values of $κ$ far away from the edges of the zone and this is shown in figure 2c. A double root occurs at $λ_π^{(4)} = λ_π^{(5)}$ and the asymptotics shown in figure 2b are the linear corrections from equation (2.36). An essential ingredient of the asymptotic procedure is the identification of the leading order solutions $U_0(ξ, λ_0^{(n)})$ for the standing waves across the structure. Typical curves are shown in figure 3 for the $U_0$ at each edge of the Brillouin zone, and as the frequency increases the solutions gain more spatial structure.

The degenerate case of a double root is again illustrated in figure 4. In figure 4b, we note that $T_0^{(n)}$ diverges for some values of $r$ and further analysis shows that these values correspond to the cases of a double root for $λ_0^{(n)}$, which occur whenever $r = 1/m, m$ (for integer $m$). For instance, for $r = 1/3$, as in the figure, $λ_0^{(4),(5)} = 3π$, and there is no band gap (as shown in figure 4a). The asymptotics are extracted using the coupled equations (2.36) as

$$\lambda^{(3),(4)} \sim 3π \pm \frac{3εκ}{\sqrt{5}} + \cdots \quad (3.9)$$

and as noted earlier, the correction is linear in $κ$ rather than the quadratic behaviour found for distinct roots. A similar double root behaviour occurs at the other end of the Brillouin zone, and is shown in figure 2b, together with the asymptotics from equation (2.36).
Figure 3. The leading order solutions $U_0$ for $\lambda(1)$, $\lambda(2)$, $\lambda(3)$ from figure 2 shown in (a) at the $\kappa = 0$ edge of the Brillouin zone (periodic) and in (b) at the $2\kappa = \pi$ edge (anti-periodic). (a) Solid line, $U_0(\xi, \lambda_0^{(1)}$); dashed line, $U_0(\xi, \lambda_0^{(2)}$); dotted line, $U_0(\xi, \lambda_0^{(3)}$). (b) Solid line, $U_0(\xi, \lambda_\pi^{(1)}$); dashed line, $U_0(\xi, \lambda_\pi^{(2)}$); dotted line, $U_0(\xi, \lambda_\pi^{(3)}$).

Figure 4. The dispersion curves from equation (3.2), for $r = 1/3$, shown across the Brillouin zone in (a); the asymptotic solution is denoted by the dashed curve while the numerical solution of the exact dispersion relation by the solid curve. This figure shows the presence of a double root for $\lambda_0 = 3\pi \sim 9.42$ and the asymptotics for this degenerate case come from equation (3.9). (b)–(d) show $T_0^{(n)}$ versus $r$ for $n = 1 \ldots 6$. Even (odd) $n$ are positive (negative), and dotted (solid) lines that are almost indistinguishable in each pairing on the scale are shown.
The piecewise homogeneous structure has its lowest Bloch spectra curve \( \lambda^{(0)}(\kappa) \) passing through \( \lambda^{(0)}(0) = \lambda_0^{(0)} = 0 \) and as it passes exactly through zero, this low-frequency mode is not captured within our asymptotics. The classical homogenization theory describes the behaviour of this low-frequency curve both when it passes through zero and when an extra (small) parameter exists, as in the example to be considered in this section, that moves the curve upward to expose a low-frequency stop band. We now consider an adaption of the piecewise homogeneous model that allows us to contrast the classical theory with our high-frequency asymptotic approach for this lowest curve of the Bloch spectra.

We previously mentioned that the model equations obtained during the high-frequency homogenization of one-dimensional periodic structures describe vibration of an effective homogeneous string on a Winkler foundation. In this context, it is instructive to study homogenization of a periodically piecewise homogeneous string on a Winkler foundation. This setup also describes a striped waveguide (Adams et al. 2008, 2009) of finite thickness with Dirichlet, Neumann or impedance (Robin) boundary conditions along the guide walls. In both cases, the governing equation is equivalent to

\[
\frac{\lambda^2}{c^2(x/l)} \frac{d^2 u}{dx^2} - \beta^2 u = 0. \tag{3.10}
\]

The additional term \(-\beta^2 u\) corresponds to the constant elastic restoring parameter in the Winkler model, or to the separation constant that is related to the transverse mode number for the striped waveguide, and its presence creates a low-frequency stop band. The dispersion relation is a minor adjustment of equation (3.2) to

\[
2 k_1 k_2 (\cos k_1 \cos k_2 - \cos 2\kappa) - (k_1^2 + k_2^2) \sin k_1 \sin k_2 = 0 \tag{3.11}
\]

with \(k_1^2 = \lambda^2 - \beta^2, k_2^2 = r^2 \lambda^2 - \beta^2\). For non-zero \(\beta\), there is no fundamental mode passing through the origin in the Bloch spectra for this model; hence, high-frequency asymptotics are capable of describing every Floquet–Bloch mode. The two-scales analysis goes through precisely as before with \(\lambda_0^{(n)}\) deduced from equation (3.11) with \(\kappa = 0\). The asymptotic correction \(\lambda_2^{(n)}\) is deduced from equation (2.32) by integrating the leading order solution \(U_0\) and the associated \(W_1\) to find \(T^{(n)}\) from equation (2.33). Results for typical values are shown in figure 5. Importantly, there is no curve passing through the origin and the lowest curve cuts the frequency axis at \(\lambda_0^{(0)}\). The variation of \(\lambda_0^{(0)}\) versus change in wave speed, \(r\), is shown in figure 5b for various \(\beta\). When \(\beta = 0\), \(\lambda_0^{(0)} = 0\) always and the curve then passes through the origin and is a conventional low-frequency mode. As \(\beta\) increases, the curves move away from the \(r\) axis and there is always a low-frequency band gap bounded above by \(\lambda_0^{(0)}\). If \(\beta = 0\), then the leading order solution \(U_0 = U(\xi, \lambda_0^{(0)})\) is just a flat line; that is, it has no spatial structure and is the state that one would use in classical homogenization, and as \(\beta\) increases, the mode shape varies and begins to take on more structure; the curves in figure 5c
Figure 5. The dispersion curves for the piecewise homogeneous string on a Winkler foundation. (a) is for \( r = 1/4, \beta = 1 \) showing the absence of the fundamental mode and the curves from the full numerics (solid) versus the asymptotics (dashed). (b) The variation of the lowest frequency cut-off at \( k = 0 \), namely \( \lambda_0^{(0)} \), versus the change in wave speed \( r \) for various values of \( \beta \). (c) The variation in the leading order solution, \( U_0(\xi, \lambda_0^{(0)}) \) as \( \beta \) increases for fixed \( r \): \( r = 1.5 \) in (c). (b, c) Solid line, \( \beta = 0.25 \); dashed line, \( \beta = 0.5 \); dot-dashed line, \( \beta = 0.75 \); dotted line, \( \beta = 1 \).

are normalized to have \( \max(|U_0|) = 1 \). Further details of the Bloch spectra for this example are in Adams et al. (2008, 2009) together with the details of numerical schemes and other asymptotic techniques that can be applied.

The lowest curve of the Bloch spectra can be approximated using quasi-static distributions along the cell corresponding to classical homogenization. This requires low frequencies and \( \beta \sim \epsilon \hat{\beta} \) for which the appropriate asymptotic ordering is that

\[
\lambda_0^{(0)} \text{ and } \lambda_0^{(2)} = e^{2} \lambda_0^{(2)} + \cdots.
\]

The leading order equation then gives \( u(X, \xi) = u_0(X, \xi) + \epsilon u_1(X, \xi) + \cdots \),

\[
\lambda^2 = e^{2} \lambda_2^2 + \cdots.
\] (3.12)

The leading order equation then gives \( u_0(X, \xi) = f_0(X) \), which is uniform along the cell; that is, it does not depend upon the small-scale \( \xi \) at all and this difference, and the scaling of \( \lambda \), is a major difference between the high- and low-frequency models. The function \( f_0(X) \), from the second-order equation, satisfies the differential eigenvalue problem

\[
f_0_{XX} - \hat{\beta}^2 f_0 + f_0 \lambda^2 \int_{-1}^{1} \frac{1}{c^2(\xi)} \ d\xi = 0.
\] (3.13)
High-frequency homogenization

Figure 6. A comparison of the low-frequency homogenization formula (3.14) (dotted lines) with the numerical solution (solid) and the high-frequency asymptotics (dashed). In all panels, \( r = 1/4 \) and in (a), we show \( \lambda^{(0)}_0 \) for \( \beta = 2 \) and \( \beta = 1 \). Likewise in (b) but for \( \beta = 0.5 \) and \( \beta = 0.25 \). (c) The variation in \( \lambda^{(0)}_0 \) versus \( \beta \) predicted by the low-frequency asymptotics (dotted) and from the numerics (solid).

In this classical model, the inverse-squared wave speed \( c \) is replaced in equation (3.10) by an averaged quantity \( \langle 1/c^2 \rangle \), defined in equation (3.3), and homogenization replaces the variable wave speed by this averaged quantity. The high-frequency asymptotics that we employ are fundamentally different; they are not limited by low-frequency quasi-static variations along the cell and do not simply replace periodic inverse wave speed squared by a constant. The asymptotics take the solution of the standing wave and construct an effective parameter, \( T \), as an integral over \( \xi \) that involves the wave speed and the standing wave solution.

Returning to the problem at hand, the Bloch conditions yield \( f_0(X) = \exp(ikX) \) and thus this lowest curve is given asymptotically in the low-frequency limit as

\[
\lambda^{(0)}_0^2 \sim \frac{4\beta^2 + (2\epsilon\kappa)^2}{2(1 + r^2)}
\]  

for \( \beta \ll 1 \) and small \( \kappa \). We compare this result with the high-frequency asymptotics in figure 6; from (b) and (c), we see that at low frequencies, equivalently small \( \beta \), equation (3.14) performs well in predicting the dispersion curves. The high-frequency theory is applicable for small \( \kappa \) but diverges from the numerics sooner than the low-frequency results. For large \( \beta \), figure 6a, the low-frequency theory drifts away from the solid curve, and becomes inaccurate, and the high-frequency results remain accurate and are so over a longer domain in \( \kappa \).

(c) A continuous periodic variation

We now consider a string with periodic variation in density that leads to the wave speed \( c^{-2}(\xi) = \alpha - 2\Theta \cos 2\xi \), where \( \alpha, \Theta \) are positive constants; this variation gives the classical Mathieu equation (Abramowitz & Stegun 1964; McLachlan 1964) and to more readily connect with the standard theory for that
equation, we choose the period to be $\pi \epsilon$ rather than $2\epsilon$. To obtain the Bloch spectrum, the Mathieu equation

$$u_{\xi\xi} + \lambda^2 (\alpha - 2\Theta \cos 2\xi) u = 0$$

(3.15)

is solved with Floquet–Bloch conditions at $\xi = 0, \pi$, namely $u(\pi) = \exp(i\pi\kappa\epsilon) u(0)$ and $u_{\xi}(\pi) = \exp(i\pi\kappa\epsilon) u_{\xi}(0)$. The resulting differential eigenvalue problem for $\lambda$ is solved numerically using a spectral collocation scheme with Chebyshev basis functions (Weideman & Reddy 2000; Adams et al. 2008), and a typical Bloch spectrum is shown in figure 7. In the language associated with Mathieu’s equation, $\kappa$ is the characteristic exponent and $\lambda^2 \alpha, \lambda^2 \Theta$ are usually, denoted as $a, q$; usually the characteristic exponent is found in terms of fixed $a, q$, whereas in the current application, this is reversed with $\kappa$ known (and $a, q$ known) but with $\lambda^2$ to be determined.

The periodic and anti-periodic leading order solutions $U_0(\xi, \lambda_0^{(n)})$ and $U_0(\xi, \lambda_0^{(n)})$ are extracted numerically for $\kappa = 0, \pi$, respectively, and $W_1$ is determined as a non-periodic solution of equation (3.15). In the periodic case, the leading order solutions fall into the two linearly independent solutions described in §20.3.
High-frequency homogenization

0.1
0.2
0.3
0.4
0.5
1.8
2.0
2.2
2.4

Figure 8. (a) and (b) show the variation of $T^{(n)}_0$ and $\lambda^{(n)}_0$ versus $\Theta$, for $n = 2, 3$. (a) Solid line, $-T^{(2)}_0$; dashed line, $T^{(3)}_0$. (b) Solid line, $\lambda^{(2)}_0$; dashed line, $\lambda^{(3)}_0$.

of Abramowitz & Stegun (1964): one being characterized by $U_0(0, \lambda^{(n)}_0) = 1$, $U_0(\pi, \lambda^{(n)}_0) = 1$, $U_0(\pi, \lambda^{(n)}_0) = 0$ (these lead to the $n$ odd cases) and the other by $U_0(0, \lambda^{(n)}_0) = 1$, $U_0(\pi, \lambda^{(n)}_0) = 1$, $U_0(\pi, \lambda^{(n)}_0) = 0$ (these lead to the $n$ even cases). The $n$ even cases have $U_0(\pi, \lambda^{(n)}_0) = 0$ and the formula for $T$ from equation (2.34) is used. The non-periodic solution $W_1$ is taken to have boundary conditions $W_1(0, \lambda^{(n)}_0) = 0$, $W_1(\pi, \lambda^{(n)}_0) = 1$ for the odd case and $W_1(0, \lambda^{(n)}_0) = W_1(\pi, \lambda^{(n)}_0) = 1$ for the even case. The anti-periodic case also naturally falls into two linear independent solutions and these too are found numerically.

Given the leading order ($U_0$) and non-periodic ($W_1$) solutions, the integrals for $T$ are evaluated numerically. The change of length of the periodic domain leads to minor changes owing to a rescaling of the domain, and after performing this change the corrections $\lambda_2^{(n)}$ to $\lambda^{(n)}_0$ are readily found and the resulting asymptotics are shown versus the complete numerics in figure 7. The asymptotics are highly accurate near the edges of the Brillouin zone, as illustrated in figure 7a. The curve for $\lambda^{(0)}$ passes through the origin and the asymptotics for this curve near zero are found using the classical low-frequency approach, as in equation (3.13), and $\lambda^{(0)} \sim \epsilon k / \sqrt{\alpha}$.

The leading order solutions for $U_0$ are shown in figure 7b with the $n = 2$ solution clearly zero at the ends of the domain, and the solutions chosen for $W_1$ are shown in figure 7c. Figure 8a,b shows that as $\Theta \to 0$, physically the material variation decreasing, the values of $T^{(n)}_0$ shown increase dramatically and the difference between the consecutive $\lambda^{(n)}_0$ decreases until the band gap disappears and one obtains a double root; the figure shows the results for $n = 2, 3$ and similar results hold for higher $n$ and at the other end of the Brillouin zone.

4. Concluding remarks

The high-frequency asymptotic theory that we present extends classical homogenization, breaking free of the static or low-frequency limitation on the solution variation along the cell. The examples chosen show that by perturbing away from the standing wave solutions, the Bloch spectra are identified through a
simple differential eigenvalue problem (2.20) in two-dimension and (2.32) in one-dimension. This differential eigenvalue problem is characterized by a constant parameter whose definition involves the integrations over the short scale of the periodic cell and this short scale plays no further role in the problem; the methodology differs from conventional homogenization theory in several critical ways, and the main one is that the basic state has spatial dependence and so the integrated quantities are not simply averaged wave speeds or simple averaged quantities.

Remarkably, the final ODE equation (2.32) is exactly that which arises in the high-frequency long-wave asymptotics in, say, a straight acoustic waveguide (Gridin et al. 2004). Near the thickness resonance frequencies for the waveguide, i.e. near the eigenvalues of the transverse resonance problem for a homogeneous waveguide, a wave bounces across the guide width, forming a near-standing wave that barely propagates along the guide. Therefore, despite being at high frequency, the wavelength is long. In the periodic situation considered in this article, the vision we have of the wave is that it bounces within a periodic cell of the structure with no phase change, or complete phase change, across the period, to leading order, again forming a standing wave barely propagating along the structure. In both situations, the transverse resonance and standing waves, the wave about which we perturb has a large wavelength, but can occur at a high frequency and so they are both amenable to similar asymptotic methods. A major benefit of having an asymptotic theory at hand is that it uncovers the physics and also complements numerical schemes. For instance, by evaluating the parameter $T$ directly from the standing wave solutions, it is possible to both determine the sign and estimate the value of group velocity of Bloch modes near the edges of the Brillouin zone.

In the long-wave high-frequency theory of waveguides, the ODE is often augmented by terms that account for curvature or geometrical variations along the guide (Gridin et al. 2005; Kaplunov et al. 2005) and which may lead to trapping or localization phenomena. Similar adjustments to the periodic theory can be undertaken and this approach leads to a theory identifying localized modes in media with weakly varying periodic behaviour. The two-scales approach outlined in this article provides a general methodology for treating doubly continuous periodic media, and can be extended to discrete periodic models consisting of point masses and springs. Such models are commonplace in solid state physics and they also exhibit band gap phenomena (Brillouin 1953; Kittel 1996). This, and other, extensions of the theory are underway.

The authors thank NSERC (Canada) and the EPSRC (UK) for support through the Discovery Grant Scheme and grant EP/H021302, respectively. J.K. thanks the University of Alberta for its hospitality and support under the Visiting Scholar programme. The authors gratefully acknowledge useful conversations with Sebastien Guenneau, Evgeniya Nolde and Valery Smyshlyaev.

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