Fundamental properties of surface waves in lossless stratified structures

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This paper is focused on dispersive properties of lossless planar layered structures with media having positive constitutive parameters (permittivity and permeability), possibly uniaxially anisotropic. Some of these properties have been derived in the past with reference to specific simple layered structures, and are here established with more general proofs, valid for arbitrary layered structures with positive parameters. As a first step, a simple application of the Smith chart to the relevant dispersion equation is used to prove that evanescent (or plasmonic-type) waves cannot be supported by layers with positive parameters. The main part of the paper is then focused on a generalization of a common graphical solution of the dispersion equation, in order to derive some general properties about the behaviour of the wavenumbers of surface waves as a function of frequency. The wavenumbers normalized with respect to frequency are shown to be always increasing with frequency, and at high frequency they tend to the highest refractive index in the layers. Moreover, two surface waves with the same polarization cannot have the same wavenumber at a given frequency. The low-frequency behaviours are also briefly addressed. The results are derived by means of a suitable application of Foster’s theorem.

Keywords: electromagnetic theory; surface waves; layered structures

1. Introduction

Wave excitation and propagation in planar layered structures has been one of the most intensively studied topics in electromagnetics (e.g. Sommerfeld 1949; Marcuvitz 1951; Tamir & Oliner 1963a,b). Beyond the straightforward and elegant mathematical formulation of such problems (Chew 1995; Felsen & Marcuvitz 1991; Hanson & Yakovlev 2001) and the frequent implementation of numerical approaches for their analyses (Sorrentino 1989; Collin 2001), these structures have proven to be of great interest for a very wide range of applications: from microwave to optical frequencies, both for integrated circuit design (Collin 1991; Pozar 2004) and for antennas (Jackson 2007).

In the last decade, interest in this topic has renewed, in part due to the great amount of research related to artificial materials. Multi-layered structures have successfully been used to realize metamaterials by means of periodic

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perturbations (e.g. metalizations) within the layered structure, realizing devices such as frequency selective surfaces (as described by Munk (2000) and Novotny & Hecht (2006)), metamaterial leaky-wave antennas (as in Caloz & Itoh 2006), cloaking devices (as in Kildal et al. 2007), etc. New dispersive properties have been found when some of the layers themselves are composed of artificial materials, as reported, e.g., by Prade et al. (1991), Ruppin (2000) and Alú & Engheta (2005); see Baccarelli et al. (2009) for an overview of this topic.

In order to correctly design devices embedded in a stratified environment, the propagation properties of the background layered medium need to be taken into account, to avoid spurious effects due to the excitation of several kinds of waves, as shown in Feise et al. (2002). These waves can cause interference among devices (Mesa & Jackson 2002), diffraction owing to the truncation of the substrates, coupling between array elements and loss of efficiency due to leakage of power (Baccarelli et al. 2009). Hence, the correct determination of the possible kinds of waves supported by a given background medium appears fundamental in order to both understand the physical phenomena involved and design the best structure for the required application.

On the other hand, efficient numerical analyses (to study dispersive behaviours as in Rodríguez-Berral et al. (2004), or to formulate integral equations as in Michalski & Zheng 1990) rely on the possibility to perform a fast calculation of Green’s functions and related quantities in layered media; these quantities have singularities in their (spectral) domain of definition corresponding to wave-numbers of surface waves (SWs) supported by the structure considered. Various regularizations have been proposed in the literature involving the numerical extraction of SWs (e.g. Demuynck et al. 1998; Aksun & Dural 2005), with the aim of smoothing the considered functions; this allows for their easier interpolation, thus reaching a higher accuracy with a reduced number of precalculated points. Since the wavenumbers of the SWs must be numerically determined, a significant benefit can be obtained from the knowledge of the kinds of waves supported by the considered structure, and in particular of the frequency behaviour of the SWs, which leads to a restriction of the domain needed to perform the numerical search. Such dispersive properties have usually been derived in the past for specific simple layered structures (such as a single dielectric layer) from a simple inspection of the relevant characteristic equations, as in Marcuse (1991). The conclusions have then been assumed to be valid for more complex structures.

In this paper, the most important modal properties of SWs are investigated for general lossless layered structures, and the properties that are well known for simple single-layered structures will be shown to be valid for more general arbitrary layered structures having positive parameters $\varepsilon$ and $\mu$, i.e. double-positive (DPS) media. Some fundamental differences will be outlined between DPS structures and layered structures containing media having one or both negative parameters, i.e. single negative (SNG) media, having $\varepsilon < 0$ and $\mu > 0$ (epsilon negative, ENG) or $\varepsilon > 0$ and $\mu < 0$ (mu negative, MNG), or double negative (DNG) media, having $\varepsilon < 0$ and $\mu < 0$.

In §2, the fundamental theory of planar multi-layered structures is briefly recalled with reference to the aspects needed in the present paper.

In §3, attention is given to evanescent modes, or plasmon-like modes (Barnes et al. 2003), i.e. SWs with exponential behaviour in all of the media along the direction of stratification. Such modes are shown to possibly exist for lossless
The structure analysed in this paper is a planar layered configuration as in figure 1(a). Each layer is filled with a linear, homogeneous, spatially non-dispersive and generally uniaxial anisotropic medium, i.e. its parameters are

\[
\mathbf{\varepsilon}_i = \mathbf{\varepsilon}_0 \begin{pmatrix} \varepsilon_{t,i} & 0 & 0 \\ 0 & \varepsilon_{t,i} & 0 \\ 0 & 0 & \varepsilon_{z,i} \end{pmatrix} \quad \text{and} \quad \mathbf{\mu}_i = \mathbf{\mu}_0 \begin{pmatrix} \mu_{t,i} & 0 & 0 \\ 0 & \mu_{t,i} & 0 \\ 0 & 0 & \mu_{z,i} \end{pmatrix}. \tag{2.1}
\]

where \(\varepsilon_0\) and \(\mu_0\) are the vacuum permittivity and permeability, respectively. Isotropic media can be regarded as a particular case with \(\varepsilon_{t,i} = \varepsilon_{z,i}\) and \(\mu_{t,i} = \mu_{z,i}\). A medium is said to be DPS if all the elements in the matrices (2.1) are positive. The upper layer is unbounded along the positive \(z\) direction. The lower (0th) layer can be unbounded along the negative \(z\) direction or replaced by a perfect electric conductor (PEC) or a perfect magnetic conductor (PMC) plane.
A monochromatic time dependence $e^{j\omega t}$ will be assumed and suppressed in the following. No variations are assumed along the $y$ direction.

The discrete modal spectrum can be decomposed into two sets of modes with different polarizations: transverse magnetic (TM) with $H_z = 0$ and transverse electric (TE) with $E_z = 0$, where $z$ is the direction perpendicular to the layer boundaries (see Chew (1995) and Felsen & Marcuvitz (1991) for more details about the following discussion). The $z$-dependence of the transverse (horizontal) component of the electric or magnetic field is described by a transmission-line (TL) equivalent circuit referred to as the transverse equivalent network. The $i$th homogeneous layer is modelled as a piece of a uniform TL having vertical propagation wavenumbers (figure 1b) $k_{z,i}^{TM}$ and characteristic impedances $Z_{0,i}^{TM}$, depending on the polarization ($P = TM, TE$):

$$k_{z,i}^{TM} = \sqrt{\frac{\varepsilon_{z,i}}{\varepsilon_{z,i}}} \sqrt{k_0^2 n_i^{TM} - k_t^2}, \quad Z_{0,i}^{TM} = \frac{k_{z,i}^{TM}}{\omega \varepsilon_0 \varepsilon_{t,i}}$$ (2.2)

and

$$k_{z,i}^{TE} = \sqrt{\frac{\mu_{z,i}}{\mu_{z,i}}} \sqrt{k_0^2 n_i^{TE} - k_t^2}, \quad Z_{0,i}^{TE} = \frac{\omega \mu_0 \mu_{t,i}}{k_{z,i}^{TE}}$$ (2.3)

where $k_0 = \omega \sqrt{\varepsilon_0 \mu_0}$ is the vacuum wavenumber, $n_i^{TM} = \sqrt{\varepsilon_{z,i} \mu_{z,i}}$ and $n_i^{TE} = \sqrt{\varepsilon_{z,i} \mu_{z,i}}$, and $k_t$ is the transverse propagation wavenumber of the mode (i.e. the $x$ dependence of the mode is of the form $e^{-jk_t x}$). The values of the propagation wavenumbers as functions of the frequency, i.e. the functions $k_t(\omega)$ for the TM and TE cases, are solutions of the transcendental characteristic equation, based on the well-known transverse resonance technique (e.g. Sorrentino 1989)

$$\mathcal{Z}[\omega, k_t(\omega)] = -\mathcal{Z}[\omega, k_t(\omega)],$$ (2.4)

where $\mathcal{Z}$ and $\mathcal{Z}$ are the impedances seen looking downwards and upwards, respectively, at an arbitrary reference plane $z = \text{const}$. In the following, the reference plane will usually be chosen as the uppermost interface between the $N$th layer and the upper unbounded medium. In a lossless structure, the impedances in equation (2.4) are both purely reactive, i.e. $\mathcal{Z} = j X$, $\mathcal{Z} = j X$.

In the $k_t$ complex plane (figure 2), the impedances on the left- and right-hand sides of equation (2.4) have branch points located at the propagation wavenumbers in the unbounded media:

$$k_{bp,0}^P = k_0 n_0^P \quad \text{and} \quad k_{bp,N+1}^P = k_0 n_{N+1}^P.$$ (2.5)

The complex $k_t$ plane is then a Riemann surface (Felsen & Marcuvitz 1991) with one or two pairs of branch points, depending on the number of different unbounded media present. Hence, the Riemann surface has either two sheets or four sheets, depending on whether the two unbounded media (if two are present) have the same or different material indices of refraction. The proper sheet with respect to $k_{bp,i}^P, i = 0, N + 1,$ is the sheet where the propagation constant $k_{z,i}^{P}$ grants an exponential damping along the $z$-axis away from the structure, and
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The zeros of equation (2.4) can then be classified as proper or improper, depending on the Riemann sheet where they are located. Proper zeros lie on the sheet that is proper with respect to both branch points (assuming two unbounded media), and correspond to modes of the structure since they satisfy the boundary condition at infinity. Improper zeros lie on a sheet that is improper with respect to at least one branch point. They are not wavenumbers of bounded modes of the structure, since they do not satisfy the boundary condition at infinity.

A SW on a lossless layered structure is a mode whose wavenumber \( \beta \) is a proper real zero of the characteristic equation (2.4). The wavenumber \( \beta \) of a SW will henceforth be replaced by the symbol \( \beta \). By definition, every SW is exponentially attenuated in both the 0th and \((N+1)\)th layer (unbounded media): \( \beta^P > k_0 \max \{n_0^P, n_{N+1}^P\} = k_0 n_c^P \) results. According to the classification proposed in Baccarelli et al. (2005), two kinds of SWs can be identified (figure 2). Ordinary SWs, with \( k_0 n_c^P < \beta^P < k_0 \max \{n_i^P\} = k_0 n_{\max}^P \), have, at least in one layer, a sinusoidal behaviour along the \( z \) direction. Evanescent SWs, with \( \beta^P > k_0 n_{\max}^P \), have an exponential behaviour along the \( z \) direction in all the layers. The latter can be regarded as a generalization of plasmon polaritons (Barnes et al. 2003), waves concentrated near the interface of two half spaces and exponentially attenuated away from the interface. It will be proven later that such modes are not possible for the lossless structure shown in figure 1 when all material parameters are positive.
Complex solutions of equation (2.4), or complex waves (CWs), can lie either on the proper (proper CWs) or on the improper sheet (improper CWs) of the $k_t$ complex plane. For lossless open structures, as in figure 1, all proper modes must have real wavenumbers (proven later), so a CW must be improper.

Solutions of equation (2.4) move on the $k_t$ Riemann surface when the frequency varies, with typical behaviour shown in figure 2. With reference to figure 2, when an improper solution with a real wavenumber (called an improper SW, point 1) crosses a branch cut (point 2) and becomes proper real, a new SW is included in the discrete spectrum of the layered structure, and the SW is said to be above cutoff (point 3). In §5, it will be proven that in a DPS non-dispersive structure this is the only possible type of transition between sheets.

In §3, the Smith-chart formalism (Pozar 2004) will be useful for our discussion. As is well known, this is a conformal mapping between the impedance complex plane (normalized with respect to the characteristic impedance) and the reflection coefficient $G$ complex plane for points along a TL. In figure 3 some curves of constant resistance and reactance are shown in the $G$ plane. Moving along the $i$th layer corresponds to a simple evolution in the $G$ plane. In particular, if the propagation wavenumber in the layer is real ($k_{z,i}^P = |k_{z,i}^P|$), $G$ rotates around the origin (e.g. from point $A$ to point $B$ in figure 3), while if the propagation constant is imaginary ($k_{z,i}^P = -j|k_{z,i}^P|$), $G$ is exponentially damped and travels along a line toward the origin (e.g. from point $A$ to point $C$ in figure 3).

Finally, a reactance theorem that is often used in §6 (Foster’s theorem) is recalled here, formulated in the most useful way for our purposes (figure 4):

**Theorem 2.1 (by Foster 1924).** Let a source-free volume $V$ be filled with linear lossless media, completely shielded by perfect electric or magnetic walls, apart from
an input port $S$ where the transverse fields can be expressed as

$$
\mathbf{E}_t(\omega) = V(\omega)\mathbf{e}_t, \quad \mathbf{H}_t(\omega) = I(\omega)\mathbf{h}_t, \quad V(\omega) = jX(\omega)I(\omega), \quad \int_S \mathbf{e}_t \times \mathbf{h}_t^* \cdot \mathbf{z}_0 = 1,
$$

(2.6)

where $V$ and $I$ are scalar quantities, proportional to each other through the input impedance $jX(\omega)$ of the structure; $\mathbf{e}_t$ and $\mathbf{h}_t$ are frequency-independent vectors describing the transverse dependence of the fields and satisfying the orthonormal integral relation given above (this can model a metallic waveguide, where only one mode is excited, feeding the volume $V$). The following relation then holds:

$$
\frac{\partial X}{\partial \omega} = \frac{4}{|I|^2} (W_m + W_e)
$$

$$
= \frac{4}{|I|^2} \left[ \int_V \frac{1}{4} \mathbf{H}(r) \cdot \frac{\partial (\omega \mu)}{\partial \omega} \cdot \mathbf{H}^*(r) \, dV + \int_V \frac{1}{4} \mathbf{E}(r) \cdot \frac{\partial (\omega \varepsilon)}{\partial \omega} \cdot \mathbf{E}^*(r) \, dV \right]
$$

(2.7)

where $W_m$ and $W_e$ are, respectively, the magnetic and electric energies stored in the volume $V$.


Comment 2.2. If the volume $V$ is filled with media such that $W_m, W_e > 0$, then $\partial X/\partial \omega > 0$. This comment obviously applies to every lossless DPS medium, but also to realistic DNG, ENG and MNG media when losses are neglected, as explained by Engheta (2003).

Comment 2.3. The theorem holds even though perfect walls do not completely shield the structure, provided that the fields are attenuated strongly enough (e.g. exponentially) at infinity.
3. Evanescent modes

As stated previously, proper evanescent SWs are waves with exponential behaviour along $z$ in all the layers (i.e. $\beta^P > k_0 \max_i (n^P_i) = k_0 n^P_{\max}$, figure 2). They are the generalization to the multi-layered case of plasmon polaritons waves, present at the interface between two half spaces as described by Barnes et al. (2003). In this section, we will show that DPS lossless layered structures cannot support evanescent SWs. Following the notation in figure 1, the characteristic impedance of the $i$th layer is $Z^P_{0,i}$, the impedance seen looking downward in the $i$th layer is $Z^P_{-,i}$ at its bottom boundary and $Z^P_{+,i}$ at its top boundary. Normalized impedances are identified with a bar: $\bar{Z}^P_{-i} = Z^P_{-,i} / Z^P_{0,i}$.

For an evanescent SW, we can assume, e.g., $k^P_{z,i} = -j|k^P_{z,i}|$ in all layers. All the characteristic impedances are thus purely imaginary, and the inequality

$$\bar{Z}^P_{+,i+1} > 0,$$  \hspace{1cm} (3.1)

holds for all $i$. Moreover, since the structure is lossless, all the impedances $Z^P_{\pm,i}$ are imaginary, while all the normalized impedances $\bar{Z}^P_{\pm,i}$ are real.

The impedance seen looking downwards at the top boundary plane, i.e. $\bar{Z}^P_{+,N}$, is calculated recursively. The steps can be visualized on the Smith chart. At the bottom plane $\bar{Z}^P_{+,1} = 0$ ($= 0$ for a PEC grounded structure). The shift along the first layer corresponds to an exponential damping of the reflection coefficient, since the propagation constant $k^P_{z,1}$ is imaginary. The evolution of the relevant point on the Smith chart shows that the sign of a positive normalized resistance is never changed by such a damping (figure 3). When the discontinuity between the first and the second layer is crossed, the impedance is renormalized through the positive factor $\xi^P_{1,2}$, defined in equation (3.1), and $\bar{Z}^P_{+,2} = \bar{Z}^P_{+,1} \xi^P_{1,2} > 0$. At the end of the recursive calculation, the inequality $\bar{Z}^P_{+,N} > 0$ holds. The characteristic equation (2.4), written as

$$Z^P_{+,N} Z^P_{0,N} = -Z^P_{0,N} \xi^P_{N+1,N},$$  \hspace{1cm} (3.2)

can never be satisfied because the left-hand side is non-negative, while the right-hand side is strictly negative from equation (3.1). If DNG or SNG media were present, some of the renormalization coefficients $\xi$ would be negative and the characteristic equation could be satisfied by evanescent SWs, as shown, e.g., in Prade et al. (1991), Ruppin (2000), Alù & Engheta (2005) and Baccarelli et al. (2005).

4. Complex modes

This section is focused on an analysis of the proper modes supported by a DPS-layered structure, i.e. solutions of equation (2.4) associated with fields that are proper, and hence decay at infinity along the $z$ direction. Hence, the
modal wavenumbers can lie only on the proper sheet of the $k_t$ complex plane in figure 2; the values $k_t = \beta < k_0 n_c$ and $k_t = \pm j|k_t|$ (i.e. the Sommerfeld branch cuts in figure 2) are also excluded for the time being.

A modal analysis can be formulated through a transverse Sturm–Liouville problem. In fact, the proper zeros of equation (2.4) are the eigenvalues of the operator $L$ defined as

$$Lu = \left[-s(z) \frac{d}{dz} \frac{1}{s(z)} \frac{d}{dz} - k^2(z)\right] u, \quad (4.1)$$

where $s(z) = \epsilon(z)$ or $\mu(z)$ is assumed piecewise constant, $u = H_y$ or $E_y$ in the TM or TE cases, respectively, and $k(z)$ is the medium wavenumber (see Hanson & Yakovlev 2001 for a detailed discussion). If the layered medium is bounded both at the top and at the bottom by perfect conductors, the operator $L$ is self-adjoint with respect to the inner product

$$\langle a | b \rangle_{s^{-1}(z)} = \int_{-\infty}^{+\infty} \frac{a(z) b^*(z)}{s(z)} dz \quad (4.2)$$

and then the absence of complex modes easily follows. In fact, since $Lu = -k^2_t u$,

$$-k^2_t \langle u | u \rangle_{s^{-1}(z)} = \langle Lu | u \rangle_{s^{-1}(z)} = \langle u | Lu \rangle_{s^{-1}(z)} = -k_{t*}^2 \langle u | u \rangle_{s^{-1}(z)}, \quad (4.3)$$

where the second identity expresses the self-adjointness of $L$. Since $\langle u | u \rangle_{s^{-1}(z)} \neq 0$ if $u \neq 0$, it follows that $k^2_t = k^2_{t*}$.

For the unbounded structure as shown in figure 1, the domain of the operator $L$ is taken to be those functions that are proper, and hence decaying away from the structure. This ensures that the inner product in equation (4.2) exists. This assumption, together with the boundary conditions on the ground plane (if present), ensures the self-adjointness of the operator with respect to this defined domain of functions. The proof fails for improper modes since the inner product equation (4.2) no longer exists, and, indeed, it is well known that complex improper modes (leaky modes) do exist on lossless structures.

A limiting case is where the field in the unbounded region is at the boundary between being proper and improper, and thus has a vertical wavenumber in the unbounded region that is real. This corresponds to either a real propagation wavenumber that is less than $k_0$ ($k_t = \beta < k_0$) or a purely imaginary one ($k_t = \pm j|k_t|$). While the above proof fails for these two cases, the first case may be ruled out from conservation of energy, since it corresponds to power flowing vertically away from the structure, with no power loss in the propagation direction. The second case also violates conservation of energy, since power flows away from the structure vertically, while there is no power flow in the direction of propagation in any of the layers.

It is interesting to explicitly note why complex proper modes can exist in structures where both DPS and DNG or SNG are present, as stated in Baccarelli et al. (2005). In this case, the weight function $s(z)$ changes its sign, and equation (4.2) defines a pseudo-inner product, rather than an inner product. Even though $Lu = -k^2_t u$, from the identity (4.3) the result $k^2_t = k^2_{t*}$ does not follow anymore. In fact $\langle u | u \rangle_{s^{-1}(z)}$ can be zero for non-zero $u$ in these cases, so that a complex $k^2_t$ can satisfy equation (4.3). A physical interpretation of this
result is possible by recalling that $\langle u|u\rangle_{s^{-1}(z)}$ is proportional to the net flux of power carried by the mode. As is well known, complex modes carry no net power flux, owing to the balancing of the power fluxes travelling in opposite directions in media having parameters with different sign.

5. Behaviour of $\beta(\omega)/k_0$

In this section, a DPS non-dispersive lossless layered structure will be assumed. In §5a, it is shown that $\hat{\beta}' = [\beta(\omega)/k_0]' > 0$, where the hat denotes normalization with respect to the vacuum wavenumber $k_0$, and the prime denotes differentiation with respect to $\omega$. In §5b, the high-frequency limit of $\hat{\beta}(\omega)$ is found. In §5c, the low-frequency behaviour of $\hat{\beta}(\omega)$ is investigated. In §5d, numerical results are shown confirming the theoretical analysis. All the results extend the well-known behaviour of the SW wavenumbers for a single slab to this more general class of layered structures.

(a) Monotonic behaviour of $\beta(\omega)/k_0$

In order to prove the monotonic property of $\hat{\beta}(\omega)$, Foster’s theorem is quite useful, but it cannot be applied directly to the impedances $Z_i^{p,\pm}$. In fact, the reactances $\hat{X}_i$ and $\hat{X}_r$ at an arbitrary $z$-point do not necessarily behave according to Foster’s theorem: this is due to the lack of attenuation along the transverse $(x,y)$ directions, and to the frequency dependency of the modal vectors $e_t$ and $h_t$ in equation (2.6). This conclusion is evidenced by the example of the impedance seen looking toward the air at the dielectric/air interface in a single isotropic dielectric slab. In that case, $\hat{\beta}(\omega)$ increases with the frequency, and hence $d\hat{X}(\omega)/d\omega < 0$ easily follows in both the TM and TE cases. For this reason, the proof given below requires the application of Foster’s theorem to auxiliary reactances, rather than to those mentioned above. It is based on a known graphical method to find the solution of equation (2.4) for a single isotropic dielectric slab, presented, e.g., in Balanis (1989), modified in various ways both by Wu et al. (2003) and by Baccarelli et al. (2005) to treat a single DNG slab. The method is presented here with a different formulation, to treat the case of a general layered DPS structure, by means of auxiliary normalized variables.

If two unbounded media are present, $n_{N+1}^p \geq n_0^p$ will be assumed in the following, i.e. the cutoff is determined by the $(N + 1)$th layer. Two positive real-valued new variables are then defined as follows:

$$\hat{u} = \sqrt{A \max_{j=0,\ldots,N} \left\{ \left( n_j^p \right)^2 \right\} - \hat{\beta}^2} \quad \text{and} \quad \hat{v} = \sqrt{\hat{\beta}^2 - \left( n_{N+1}^p \right)^2}, \quad (5.1)$$

with

$$\hat{u}^2 + \hat{v}^2 = A \max_{j=0,\ldots,N} \left\{ \left( n_j^p \right)^2 \right\} - \left( n_{N+1}^p \right)^2 > 0. \quad (5.2)$$

The factor $A$ in the above expressions is an arbitrary positive real number greater than unity. The reason for requiring this factor is that the term $\xi_i^p$ (defined...
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Figure 5. Evolution of the curves in the $\hat{u}$ and $\hat{v}$ planes for different values of $\omega$. The functions $\mathcal{F}$ are the right-hand side of equation (5.18) or equation (5.21), depending on the polarization. The values $\hat{u}_a$ are the asymptotes of the curves $\mathcal{F}$, and the values $\hat{u}_b$ are the intersections of $\mathcal{F}$ with the circle defined by equation (5.2).

below) will then always be greater than zero, a necessary requirement for the development of the equivalent waveguide model, explained below. Even though different values are obtained for $\hat{u}$ and $\hat{v}$ depending on the polarization, for the sake of simplicity this polarization dependence will not be shown explicitly in the symbols. The solutions $\hat{\beta}(\omega)$ of the characteristic equation (2.4) are equivalent to the intersection in the $(\hat{u}, \hat{v})$ plane between the circle (5.2) and the characteristic equation expressed in terms of $\hat{u}$ and $\hat{v}$ (figure 5). The electric length $h_{i,k_{z,i}}^\text{P}$ of each layer can be expressed as

$$h_{i,k_{z,i}}^\text{P} = h_{i,k_0}\sqrt{(n_i^\text{P})^2 - \beta^2} = h_{i,k_0}\sqrt{\hat{u}^2 + (n_i^\text{P})^2 - A \max_{j=0,\ldots,N} \left\{ (n_j^\text{P})^2 \right\}}$$

$$= h_{i,k_0}\sqrt{\hat{u}^2 - (\xi_i^\text{P})^2}$$

with

$$(\xi_i^\text{P})^2 = A \max_{j=0,\ldots,N} \left\{ (n_j^\text{P})^2 \right\} - (n_i^\text{P})^2 > 0,$$

$$(\xi_i^\text{P})^2 = A \max_{j=0,\ldots,N} \left\{ (n_j^\text{P})^2 \right\} - (n_i^\text{P})^2 > 0,$$

$$(\xi_i^\text{P})^2 = A \max_{j=0,\ldots,N} \left\{ (n_j^\text{P})^2 \right\} - (n_i^\text{P})^2 > 0,$$

$$r_i^{\text{TM}} = \sqrt{\varepsilon_{t,i}/\varepsilon_{z,i}} \quad \text{and} \quad r_i^{\text{TE}} = \sqrt{\mu_{t,i}/\mu_{z,i}}.$$
The expression in equation (5.3) can be modified by multiplying and dividing by an appropriate positive factor $\chi_i^P$:

$$h_i k_{z,i} = \frac{h_i r_i P_i}{\chi_i^P} \sqrt{\tilde{u}^2 \left( \chi_i^P \right)^2 - \left( \tilde{\xi}_i^P \chi_i^P \right)^2} = \frac{\omega h_i r_i P_i}{\chi_i^P c} \sqrt{\tilde{u}^2 \left( \chi_i^P \right)^2 - w^2}, \quad (5.5)$$

where $c = 1/\sqrt{\mu_0 \varepsilon_0}$ is the velocity of light in the vacuum and $\chi_i^P$ are chosen such that

$$\tilde{\xi}_i^P \chi_i^P = w \quad (5.6)$$

with $w$ a positive number independent of the layer. Note that this is possible since $\tilde{\xi}_i^P > 0$. The propagation constant in the top unbounded medium can be written as

$$k_{z,N+1} = -j k_0 r_{N+1} P \sqrt{\beta^2 - \left( n_{N+1}^P \right)^2} = -j k_0 r_{N+1} \tilde{u} \quad (5.7)$$

The TM impedances of the various layers have the expressions

$$Z_{0,i}^{TM} = \frac{k_0 \sqrt{\tilde{u}^2 \left( \chi_i^TM \right)^2 - w^2}}{\sqrt{\tilde{u}^2 \left( \chi_i^TM \right)^2 - w^2}} \quad (5.8)$$

while the appropriate TE expressions are

$$Z_{0,i}^{TE} = \frac{\omega \mu_0 \sqrt{\mu_i \mu_{z,i} \chi_i^{TE}}} {k_0 \nu \left( \chi_i^{TE} \right)^2 - w^2} = \frac{\sqrt{\nu^2 \left( \chi_i^{TE} \right)^2 - w^2}}{\sqrt{\nu^2 \left( \chi_i^{TE} \right)^2 - w^2}} \quad (5.9)$$

The electric lengths (5.5) can be regarded as those corresponding to an auxiliary guided-wave structure describing the propagation of a mode along a cascade of waveguides, each with the same cross section. This new structure, referred to in the following as the equivalent waveguide structure (EWS), is defined by appropriate effective parameters, here indicated with a prime sign. The working frequency is $\nu'$. The $i$th layer corresponds to a section of waveguide of length $h_i'$ filled with an isotropic medium with relative index of refraction $n_i'$. The expressions for these quantities can easily be given in terms of the parameters of the actual layered structure, from inspection of equation (5.5):

$$\tilde{u} = \tilde{\nu}, \quad h_i' = \omega \frac{h_i r_i P_i}{\chi_i^P c} \quad \text{and} \quad n_i' = \chi_i^P c \quad (5.10)$$

All the waveguides share the same cross section and the same eigenvalue $w$. The variable $\tilde{u}$ plays the role of frequency in the EWS, and the parameter $w$ plays the role of cutoff wavenumber. The impedances (5.8) and (5.9) are slightly modified to become the characteristic impedances in the EWS. The TM (TE) expressions can be all divided (multiplied) by the same positive factor $\tilde{u}$. It is easy to realize that the impedance at the interface with the unbounded medium in the original layered structure is then the same as the impedance at the input of the EWS, apart from a division (multiplication) by the same factor $\tilde{u}$.
The TM characteristic impedances of the EWS are then

\[ Z_{0,i}^{\text{EWS,TM}} = \frac{\sqrt{\hat{u}^2 (\chi_i^{\text{TM}})^2 - w^2}}{\hat{u} \varepsilon_0 (\sqrt{\varepsilon_i e z_{i,z} \chi_i^{\text{TM}}})}, \] (5.11)

where the effective-medium parameters \( \varepsilon_{r,i}' \) and \( \mu_{r,i}' \) of the \( i \)th waveguide section can now be identified:

\[ \varepsilon_{r,i}' = \sqrt{\varepsilon_i e z_{i,z} \chi_i^{\text{TM}}} > 0 \quad \text{and} \quad \mu_{r,i}' = \frac{\chi_i^{\text{TM}} c}{\varepsilon_i e z_{i,z}} > 0. \] (5.12)

The TE characteristic impedances of the EWS are

\[ Z_{0,i}^{\text{EWS,TE}} = \frac{\hat{u} \mu_0 (\sqrt{\mu_i e z_{i,z} \chi_i^{\text{TE}}})}{\sqrt{\hat{u}^2 (\chi_i^{\text{TE}})^2 - w^2}}, \] (5.13)

where the effective-medium parameters \( \varepsilon_{r,i}' \) and \( \mu_{r,i}' \) of the \( i \)th waveguide section can now be identified:

\[ \mu_{r,i}' = \sqrt{\mu_i e z_{i,z} \chi_i^{\text{TE}}} c > 0 \quad \text{and} \quad \varepsilon_{r,i}' = \frac{\chi_i^{\text{TE}} c}{\mu_i e z_{i,z}} > 0. \] (5.14)

Let \( F^P(\hat{u}, \omega) \) be the reactance seen looking downward at the top of the \( N \)th section of the EWS with polarization \( P \). All the media filling the EWS are DPS, and therefore, from comment 2.2,

\[ \frac{\partial F^P(\hat{u}, \omega)}{\partial \hat{u}} > 0 \] (5.15)

since, for constant \( \omega \), \( \hat{u} \) plays the role of the frequency in the EWS.

On the other hand, if \( \hat{u} \) is constant and \( \omega \) varies, the expressions (5.5), (5.11) and (5.13) can be regarded as defining an auxiliary TL (as opposed to a waveguide): the frequency is \( \omega \), the propagation wavenumbers vary linearly with the frequency and the characteristic impedances are constants. This describes the propagation of a transverse electromagnetic (TEM) wave, e.g. along a multi-conductor TL as in a cascade of coaxial cables. However, some of the sections may be filled with temporal-non-dispersive SNG media, corresponding to the fact that the vertical wavenumber \( k_{z,i}^P \) in that layer is imaginary. Theorem 2.1 still holds for this auxiliary TL, but comment 2.2 does not, since some of the stored energies could be negative. However, in the appendix it is proven that

\[ \frac{\partial F^\text{TM}(\hat{u}, \omega)}{\partial \omega} > 0 \quad \text{if} \quad F^\text{TM} \geq 0 \quad \text{and} \quad \frac{\partial F^\text{TE}(\hat{u}, \omega)}{\partial \omega} > 0 \quad \text{if} \quad F^\text{TE} \leq 0. \] (5.16)

Let us now write the characteristic equation (2.4) in terms of \( \hat{u}, \hat{v} \) and \( F^P(\hat{u}, \omega) \).

The TM reactance \( Z_{0,N+1}^{\text{TM}} \) from equation (2.2) is written in terms of \( \hat{v} \) using equation (5.7). This characteristic impedance, along with \( X_N^{\text{TM},+} \), are

\[ Z_{0,N+1}^{\text{TM}} = -j \frac{\hat{v}}{\varepsilon_0 \sqrt{\varepsilon_{N+1} e z_{N+1} c}} \quad \text{and} \quad X_N^{\text{TM},+} = \hat{u} F^\text{TM}(\hat{u}, \omega). \] (5.17)
The \( \hat{u} \) factor in \( X_{N}^{\text{TM},+} \) restores the correct impedance \( F^{\text{TM}} \) after the division by \( \hat{u} \) that occurred in all the characteristic impedances (5.11) of the EWS. The TM characteristic equation is then

\[
\hat{v} = \left( \epsilon_0 \sqrt{\epsilon_{t,N+1} \epsilon_{z,N+1}} c \right) \hat{u} F^{\text{TM}}(\hat{u}, \omega). \tag{5.18}
\]

Since both \( \hat{u} \) and \( \hat{v} \) are positive, the intersection points are on the \( F^{\text{TM}} > 0 \) branches of \( F^{\text{TM}} \). Then, from equation (5.15) we have that

\[
\frac{\partial(\hat{u} F^{\text{TM}})}{\partial \hat{u}} = F^{\text{TM}} + \hat{u} \frac{\partial F^{\text{TM}}}{\partial \hat{u}} > 0 \tag{5.19}
\]

and the relevant branches of the \( \hat{u} \) and \( \hat{v} \) curves (5.18) have positive slope in the \( \hat{u} \) and \( \hat{v} \) planes (figure 5). The same property holds with respect to \( \omega \) from equation (5.16).

The TE reactance \( Z_{0,N+1}^{\text{TE}} \) and \( X_{N}^{\text{TE},+} \) are

\[
Z_{0,N+1}^{\text{TE}} = j \frac{\mu_0 \sqrt{\mu_{t,N+1} \mu_{z,N+1}} c}{\hat{v}} \quad \text{and} \quad X_{N}^{\text{TE},+} = \frac{F^{\text{TE}}(\hat{u}, \omega)}{\hat{u}}. \tag{5.20}
\]

The \( \hat{u} \) factor in \( X_{N}^{\text{TE},+} \) once again restores the correct impedance \( F^{\text{TM}} \) after the multiplication by \( \hat{u} \) that occurred in all the characteristic impedances (5.13) of the EWS. The TE characteristic equation is then

\[
\hat{v} = - \left( \mu_0 \sqrt{\mu_{t,N+1} \mu_{z,N+1}} c \right) \frac{\hat{u}}{F^{\text{TE}}(\hat{u}, \omega)}. \tag{5.21}
\]

Since both \( \hat{u} \) and \( \hat{v} \) are positive, the intersection points are on the \( F^{\text{TE}} < 0 \) branches of \( F^{\text{TE}} \). Then, from equation (5.15) we have that

\[
\frac{\partial}{\partial \hat{u}} \left[ - \frac{\hat{u}}{F^{\text{TE}}(\hat{u}, \omega)} \right] = - \frac{F^{\text{TE}} - \hat{u} \left( \frac{\partial F^{\text{TE}}}{\partial \hat{u}} \right)}{(F^{\text{TE}})^2} > 0 \tag{5.22}
\]

and the relevant branches of the \( \hat{u} \) and \( \hat{v} \) curves obtained from equation (5.21) have positive slope in the \( \hat{u} \) and \( \hat{v} \) planes (see figure 5). From equation (5.16) the same property holds with respect to \( \omega \):

\[
\frac{\partial}{\partial \omega} \left[ - \frac{1}{F^{\text{TE}}(\hat{u}, \omega)} \right] = \frac{1}{(F^{\text{TE}})^2} \frac{\partial F^{\text{TE}}}{\partial \omega} > 0. \tag{5.23}
\]

The behaviour of \( \hat{\beta}(\omega) \) can be now determined, given by the intersections between the circle defined by equation (5.2) and the curves defined by equation (5.18) or equation (5.21), depending on the polarization. When \( \omega \) varies, the circle is fixed, while the other curves move. From equations (5.16) and (5.23), the behaviour of the curves for increasing \( \omega \) is as shown in figure 5: as \( \omega \) increases, more branches intersect with the circle, corresponding to new modes above cutoff. The intersections move on the circle, so that \( \hat{v} \) increases and \( \hat{u} \) decreases. From the definitions (5.1), this is equivalent to \( \hat{\beta} \) being a strictly increasing
function of $\omega$:

$$
\frac{\partial \hat{\beta}}{\partial \omega} > 0.
$$

(5.24)

Also, $\hat{\beta}$ is stationary with respect to $\omega$ (the derivative in equation (5.24) being null) only at cutoff, because of the vertical slope of the circle at $\hat{\beta} = 0$.

(b) High-frequency limit of $\beta(\omega)/k_0$

In this subsection, the high-frequency limit $\lim_{\omega \to +\infty} \hat{\beta}(\omega)$ is investigated. With this aim, the vertical asymptotes of the curves (5.18) and (5.21) in the $(\hat{u}, \hat{v})$ plane will be located for large $\omega$. For a given $\omega$, a vertical asymptote at $\hat{u} = \hat{u}_a$ corresponds to an infinite or zero input impedance (depending on the polarization) of the EWS described in §4. The value $\hat{u}_a$ is then a resonance frequency of the resonator made by the cascade of waveguides in the EWS (where $\hat{u}_a$ plays the role of frequency). The value of $\hat{u}_a$ will be a function of the actual frequency $\omega$. The EWS is closed at one end with a PEC, a PMC or a reactive impedance (depending on the kind of medium present in the 0th layer), and is closed at the other end with a PEC or a PMC. To find such a resonance frequency, a phase matching condition can be imposed, equivalent to require a standing wave resonating inside the structure:

$$
\sum_{i \in \Psi} 2 \frac{\omega h_i r_i^P}{\chi_i^P} c \sqrt{\hat{u}_a^2 (\chi_i^P)^2 - w^2} + \sum_{j=1}^{N} \text{Arg}\{ T_{j,j-1} \} = \varphi. 
$$

(5.25)

In equation (5.25), $\Psi$ is the set of indices corresponding to waveguide sections where the mode is above cutoff (otherwise there is no phase variation). From the absence of evanescent modes stated in §3 and definitions (5.1), it follows that $\Psi$ is non-empty. $\text{Arg}\{z\}$ is the principal argument of the complex number $z$, and $T_{j,j-1}$ are the factors that accounts for the change in the reflection coefficient across the waveguide sections. The term $\varphi$ is an angle that is dependent on the kind of closure at the ends of the EWS and on the order of the resonance frequency $\hat{u}_a$. For large $\omega$

$$
\sum_{i \in \Psi} 2 \frac{h_i r_i^P}{\chi_i^P} c \sqrt{\hat{u}_a^2 (\chi_i^P)^2 - w^2} = \frac{1}{\omega} \left[ - \sum_{j=1}^{N} \text{Arg}\{ T_{j,j-1} \} + \varphi \right] \xrightarrow{\omega \to +\infty} 0.
$$

(5.26)

In order for equation (5.26) to hold, since each term of the sum on the left-hand side is positive, propagation must be supported by the densest layer (i.e. with maximum refraction index) only, and its electrical length must tend to zero. Letting $n_s^P = \max_j \{n_j^P\}$, then we have

$$
\hat{u}_a^2 \xrightarrow{\omega \to +\infty} \frac{w^2}{(\chi_s^P)^2} = (\xi_s^P)^2 = A \max_j \left\{ (n_j^P)^2 \right\} = \left( n_s^P \right)^2 = (A - 1) \max_j \left\{ (n_j^P)^2 \right\}. 
$$

(5.27)

Let $\hat{u}_b$ be the intersection with the circle (5.2) corresponding to the branch having asymptote at $\hat{u}_a$. Since $\hat{\beta} < \max_j \{n_j^P\}$ (in order to have propagation in the densest...
layer) it follows from (5.1) that $\hat{u}_\beta > \sqrt{A - 1} \max_j \{n_j^P\}$. Also, from figure 5 it follows that $\hat{u}_\beta < \hat{u}_a$. Therefore, we have that

$$
\sqrt{A - 1} \max_j \{n_j^P\} < \hat{u}_\beta < \hat{u}_a. \quad (5.28)
$$

In the limit we then have

$$
\hat{u}_\beta(\omega) \xrightarrow{\omega \to +\infty} \sqrt{A - 1} \max_j \{n_j^P\} \quad (5.29)
$$

and, from equation (5.1),

$$
\hat{\beta}(\omega) \xrightarrow{\omega \to +\infty} \max_j \{n_j^P\}. \quad (5.30)
$$

(c) **Low-frequency behaviour**

In the low-frequency limit, no more than one TE and one TM SW can have a zero cutoff frequency. This can easily be proven by referring to the graphical solution discussed in the previous paragraphs. If $\omega \to 0$, from equation (5.25) any asymptote $\hat{u}_a \to \infty$, so that, if the frequency is low enough, only one branch (if any) of the curve $\mathcal{F}$ in figure 5 can intersect the circle (5.2).

In this limit, the layered structure is expected to be modelled as a single slab characterized through appropriate effective parameters (a homogenization procedure analogous to that presented, e.g., by Silveirinha & Fernandes 2005), since the dominant field components tend to be piecewise constant across all the layers. This means that if the structure is grounded with a PEC (PMC), only a TM (TE) SW has a zero cutoff frequency. If the layers are surrounded by the same unbounded medium on both sides, both a TM and a TE SW have a zero cutoff frequency. If two different unbounded media surround the layers no SW has a zero cutoff frequency.

The conclusion stated above can rigorously be proven by observing that at zero frequency there is no variation in the impedance along the $z$ axis. The characteristic equation (3.2) can then be written as $Z_{1,-} = -Z_{0,N+1}$ and the above-mentioned results easily follow from the closed-form expressions of the two impedances involved.

(d) **Modal coupling**

The graphical solution of the characteristic equation (2.4) is useful for interpreting the way different modes can couple in a DPS layered structure. Each positive branch of the curve $\mathcal{F}$ in figure 5 has, at most, one intersection with the circle, corresponding to the wavenumber of a SW. Because of the positive slope of the curves in the $\hat{u}$ and $\hat{v}$ planes for any value of $\omega$, two adjacent SWs are always divided by an asymptote $\hat{u}_a$. This prevents two SWs with the same polarization from sharing the same wavenumber at the same frequency; i.e. real zeros of equation (2.4) are always single roots of that equation. This result is also valid in DPS temporal-dispersive structure, since the values of the SW wavenumbers at a given frequency only depend on the media parameters at that frequency.
(a) Normalized phase constant, numerically determined for the first six TM modes on a layered structure with $\varepsilon_{t,1} = \varepsilon_{z,1} = 1.6$, $\mu_{t,1} = \mu_{z,1} = 2.5$ ($n_1 = 2$), $\varepsilon_{t,3} = \varepsilon_{z,3} = 8$, $\mu_{t,3} = \mu_{z,3} = 2$ ($n_2 = 4$), with vacuum in the second layer and an unbounded vacuum medium on the top and on the bottom. All the layers have the same height $h = 4\text{mm}$. (b) Details of the coupling between the modes.

(e) Results

In figures 6(a) and 7(a), the typical behaviours of wavenumbers of SWs normalized with respect to $k_0$ ($\beta/k_0$) versus the frequency ($f$) proven in the previous sections are demonstrated. The reference structure is made of two unbounded media filled with air and two isotropic slabs with different refractive indices ($n_1 = 2$ and $n_3 = 4$), separated by an air layer. The modes of the two slabs couple, although all the resulting curves have a positive slope and no intersection occurs between the curves, as can be seen in the details given in figures 6(b) and 7(b). Moreover, all the curves tend to the maximum refractive index: the curves corresponding (approximately) to the modes of the less dense slab, which should approach the smaller refractive index at high frequency, are in fact formed by...
Figure 7. (a) Normalized phase constant, numerically determined for the first six TE modes on the same structure as in figure 6. (b) Details of the coupling between the modes.

pieces of curves from different modes of the complete structure, and each of them tend eventually to the highest refractive index. The coupling decreases as the distance between the two slabs increases or as the frequency increases, and consequently the nearer the curves approach each other, as shown in the detailed figures.

The coupling mechanism discussed here confirms the results by Pierce (1954), where an analysis based on linear systems is performed. It is important to recall that in lossless layered structures the operator $L$ in equation (4.1) is self-adjoint (if the structure is unbounded, the domain of the operator is restricted to proper modes as described in §4). All the modes are then orthogonal to each other and they never exchange power. However, this does not necessarily prevent two or more modes from sharing the same wavenumber while remaining orthogonal. To show that this cannot happen, the proof given in this section seems to be necessary. On the other hand, two modes can have the same wavenumber if complex constitutive parameters (Hanson & Yakovlev 1999) or complex values of the frequency (Yakovlev & Hanson 2000) are allowed.

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Different kinds of waves can be found in layers with negative electromagnetic parameters. They are shown, e.g., in figs 8, 9, 11 and 12 in Baccarelli et al. (2005) with reference to a lossless grounded DNG slab. Evanescent or plasmon SWs, absent in DPS structures (in our structure they would require $\hat{\beta} > 4$), are supported in that case. Moreover, in DNG slabs two SWs may collapse into a so-called ‘splitting-point,’ giving rise to a conjugate pair of complex proper waves. In lossless DPS structures, as proven in this paper, two SWs cannot share the same wavenumber, and proper CWs cannot be supported. Further details about DNG layers are not discussed here for the sake of brevity, and may be found in the previously referenced papers.

As a last important comment, it should be pointed out that the behaviour of $\hat{\beta}$ near cutoff and at high frequencies also has a clear physical meaning in terms of geometrical optics. In fact, it is well known that a geometrical-optics approach leads to exact results for the dispersive properties of a layer structure, as shown in Balanis (1989). In this framework, the SW is seen as a collection of plane waves (PWs), incident on, reflected by and transmitted through each planar interface; according to the definition of SW given in §2, the PWs in the unbounded regions are always transversely (i.e. along $z$) attenuated (inhomogeneous PWs). According to the value of $\hat{\beta}$, in each layer these PWs are propagating or attenuated in the transverse direction. At cutoff, all of these PWs are transversely propagating, except in those layers that have lower refractive indices than that of the densest unbounded region. On the contrary, in the high-frequency limit, the waves are transversely attenuated everywhere apart from the layers with the highest refractive index, where they behave as uniform PWs travelling along the longitudinal direction.

6. Conclusion

Fundamental general properties of the dispersive behaviour of lossless planar multi-layered media with positive parameters $\varepsilon, \mu$ have been proven, generalizing results that are often stated for specific simpler structures (such as a single grounded slab). The propagation wavenumber of a SW normalized with respect to the vacuum wavenumber, $\hat{\beta} = \beta / k_0$, is bounded by the maximum index of refraction, and the impossibility of obtaining surface-plasmons like modes (decaying in all regions) follows. Moreover, in temporal-non-dispersive media, $\hat{\beta}$ is monotonically increasing with frequency, and at high frequency always tends to the maximum index of refraction (maximum among the layers present). At sufficiently low frequencies, the structure can be homogenized as a single-layer slab, so that no more than one SW of each type (TM or TE) is above cutoff. Although mode coupling can occur, the dispersion curves for two modes of the same type can never cross. The dispersion curves for modes of different type can cross, though the modes remain orthogonal to each other. These conclusions have been validated by numerical results. The dispersive properties proven here, often proven in the past for simple specific configurations, and only assumed to be valid for more complex structures, are useful for the numerical analyses of stratified structures, where a knowledge of the SW behaviour is important.
Appendix A

In this appendix two different proofs are given of equation (5.16), the first being a more physical proof and the second being a more mathematical proof. The relevant TL models a TEM wave along a cascade of TLs, such as coaxial cables, each one filled with either a DPS medium or a SNG one. The parameters of this TL are

\[ k_{z,i} = \omega |n_i| \quad \text{and} \quad Z_{0,i} = |Z_{0,i}| \quad (A1) \]

if propagation occurs in the \( i \)th section, and

\[ k_{z,i} = -j\omega |n_i| \quad \text{and} \quad Z_{0,i}^\mp = \mp j|Z_{0,i}| \quad (A2) \]

if attenuation occurs. The quantities \( n_i \) and \( Z_{0,i} \) are frequency independent. In equation (A2) the minus sign refers to the TM polarization in equation (5.16), and the plus sign to the TE polarization. In a TEM TL \( Z_{0,i} \) is related to \( k_{z,i} \) through the relation

\[ Z_{0,i} = \frac{\omega \mu_i}{k_{z,i}^2} = \frac{k_{z,i}}{\omega \varepsilon_i}. \quad (A3) \]

This leads to two different structures according to the sign chosen in equation (A2), to be consistent with the sign of \( k_{z,i} \). \( Z_{0,i} = -j|Z_{0,i}| \) requires in the \( i \)th section a SNG medium with \( \mu_i < 0 \) (and \( \varepsilon_i > 0 \)) (MNG). \( Z_{0,i} = j|Z_{0,i}| \) requires in the \( i \)th section a SNG medium with \( \varepsilon_i < 0 \) (and \( \mu_i > 0 \)) (ENG). We can then state the following:

**Theorem A.1.** Let the TL, whose parameters are given in equations (A1) and (A2), be closed with a PEC, a PMC or an infinite section filled with a SNG medium. Let its input reactance be \( X^\mp \), according to the \( \mp \) sign in equation (A2). Then

\[ \frac{\partial X^-}{\partial \omega} > 0 \quad \text{if} \quad X^- \geq 0 \quad \text{and} \quad \frac{\partial X^+}{\partial \omega} > 0 \quad \text{if} \quad X^+ \leq 0. \quad (A4) \]

**First proof.** The structure described here fulfills the hypotheses of theorem 2.1. Nevertheless, in each section filled with a SNG medium, one of the stored energies (magnetic or electric) is negative-valued, and the sign of the derivative of \( X^\mp \) cannot be stated in general from the comment 2.2 to Foster’s theorem. However, from Poynting’s theorem we can write

\[ X^\mp = \frac{4\omega}{|I|^2} (W_m^\mp - W_e^\mp), \quad (A5) \]

where the sign choice on \( W \) refers to the different signs in equation (A2). From the kind of SNG media present in the structure, we can state that \( W_e^- > 0 \) and \( W_m^+ > 0 \). From equation (A5), if \( X^- \geq 0 \), then \( W_m^- > 0 \) and comment 2.2 holds. Similarly, if \( X^+ \leq 0 \), then \( W_e^+ > 0 \) and comment 2.2 holds.

**Second proof.** Let us consider the same cascade of TEM TLs described at the beginning of this appendix. Let every section be filled with suitable temporal dispersive media. If the ‘−’ sign is chosen in equation (A2), the media
parameters are

\[ e_i^- (\omega) = \omega e_i \quad \text{and} \quad \mu_i^- (\omega) = \pm \frac{m_i}{\omega}, \quad (A 6) \]

with \( e_i, m_i > 0 \) frequency independent, chosen such that \( \sqrt{e_i m_i} = |n_i| \) and \( \sqrt{m_i/e_i} = |Z_{0,i}|. \) A positive \( \mu_i \) is chosen if propagation occurs in the \( i \)th layer, a negative \( \mu_i \) is chosen if attenuation occurs.

If the '+’ sign is chosen in equation (A2), the media parameters are

\[ e_i^+ (\omega) = \pm \frac{e_i}{\omega} \quad \text{and} \quad \mu_i^+ (\omega) = \omega m_i \quad (A 7) \]

with \( e_i, m_i > 0 \) and frequency independent, chosen such that \( \sqrt{e_i m_i} = |n_i| \) and \( \sqrt{m_i/e_i} = |Z_{0,i}|. \) A positive \( e_i \) is chosen if propagation occurs in the \( i \)th layer, a negative \( e_i \) is chosen if attenuation occurs.

The wavenumber and characteristic impedance constants \( k_{z,i}, \xi_i^\pm \) of the TL describing this structure are

\[ k_{z,i} = k_{z,i}, \quad \xi_i^- = \frac{Z_{0,i}^-}{\omega} \quad \text{and} \quad \xi_i^+ = \omega Z_{0,i}^+, \quad (A 8) \]

i.e. the same as in equations (A1) and (A2), apart from an \( \omega \) factor in the characteristic impedances.

From equations (A6) and (A7), it follows that

\[ \frac{\partial (\omega e_i^- (\omega))}{\partial \omega} > 0, \quad \frac{\partial (\omega \mu_i^- (\omega))}{\partial \omega} = 0 \quad (A 9) \]

and

\[ \frac{\partial (\omega e_i^+ (\omega))}{\partial \omega} = 0, \quad \frac{\partial (\omega \mu_i^+ (\omega))}{\partial \omega} > 0. \quad (A 10) \]

From equation (2.7) \( W_m^- = 0, \quad W_e^- > 0, \) and \( W_m^+ > 0, \quad W_e^+ = 0, \) where the ± sign refers to the stored energies corresponding to the two different choices of the media. In both cases \( W_m^+ + W_e^+ > 0 \) and comment 2.2 holds, even though the media in equations (A6) and (A7) are not physical (e.g. they are not causal). This means that

\[ \frac{\partial x^\pm}{\partial \omega} > 0, \quad (A 11) \]

where \( x^\pm \) is the input impedance of the TL whose parameters are given in equation (A8).

The input reactance \( X^\pm \) of the TL whose parameters are given in equations (A1) and (A2) can be expressed in terms of \( x^\pm \): 

\[ X^- = \omega x^- \quad \text{and} \quad X^+ = \frac{x^+}{\omega} , \quad (A 12) \]

where \( x^- \) and \( X^- \) share the same sign.
The derivative of $X^\pm$ are finally given as

$$
\frac{\partial X^-}{\partial \omega} = \frac{\partial}{\partial \omega} \left( \omega x^- \right) = \omega \frac{\partial x^-}{\partial \omega} \quad \text{(A13)}
$$

and

$$
\frac{\partial X^+}{\partial \omega} = \frac{\partial}{\partial \omega} \left( \frac{x^+}{\omega} \right) = \frac{1}{\omega^2} \left( \omega \frac{\partial x^+}{\partial \omega} - x^+ \right) \quad \text{(A14)}
$$

The final result easily follows.

References


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