Relative equilibria of point vortices and the fundamental theorem of algebra

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Relative equilibria of identical point vortices may be associated with a generating polynomial that has the vortex positions as its roots. A formula is derived that relates the first and second derivatives of this polynomial evaluated at a vortex position. Using this formula, along with the fundamental theorem of algebra, one can sometimes write a general polynomial equation. In this way, results about relative equilibria of point vortices may be proved in a compact and elegant way. For example, the classical result of Stieltjes, that if the vortices are on a line they must be situated at the zeros of the \( N \)th Hermite polynomial, follows easily. It is also shown that if in a relative equilibrium the vortices are all situated on a circle, they must form a regular \( N \)-gon. Several other results are proved using this approach. An ordinary differential equation for the generating polynomial when the vortices are situated on two perpendicular lines is derived. The method is extended to vortex systems where all the vortices have the same magnitude but may be of either sign. Derivations of the equation of Tkachenko for completely stationary configurations and its extension to translating relative equilibria are given.

Keywords: point vortices; relative equilibria; fundamental theorem of algebra

1. The fundamental theorem of algebra

The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients, \( p(z) \), has at least one complex root. Alternatively and equivalently, every non-zero single-variable polynomial with complex coefficients has exactly as many complex roots as its degree, if each root is counted with its multiplicity. This follows by polynomial division as

\[ p(z) = (z - z_0)q(z), \]

where \( q(z) \) is a polynomial of degree one less than the degree of \( p(z) \), if and only if \( z_0 \) is a root of \( p(z) \). We shall be interested primarily in the case where all the roots are simple and assume this henceforth.

If we add a normalization condition, e.g. that the highest order coefficient of \( p(z) \) is 1, we get the result that a polynomial of a given degree is uniquely determined by its roots. In other words, if \( p_1(z) \) and \( p_2(z) \) are polynomials with the same roots, say \( z_1, z_2, \ldots, z_N \), both with highest order coefficient

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Dedication: it is a pleasure to dedicate this paper to Russell J. Donnelly, a lifelong student of vortices, on the occasion of his 80th birthday.
(i.e. the coefficient of $z^N$) equal to 1, then $p_1(z) = p_2(z)$ for all $z$. This follows by considering the polynomial $p_1(z) - p_2(z)$, which has degree at least one less than the common degree of $p_1(z)$ and $p_2(z)$ (since the highest order terms cancel), but still has the $N$ common roots of $p_1(z)$ and $p_2(z)$. The only polynomial that can have more roots than its degree is the null polynomial.

The theorem has a long history going back to about 1600. Several well-known mathematicians—and mechanicians—attempted proofs, including d’Alembert in 1746, Euler in 1749, Lagrange in 1772 and Laplace in 1795. The proven theorem is usually attributed to Gauss who published a proof in 1799. There was, however, also a proof by James Wood a year earlier, but his contribution has been ignored. The first rigorous proof was published by Argand in 1806. Gauss returned to the theorem several times and produced two other proofs in 1816, and a different version of his original proof in 1849. The first textbook containing a proof of the theorem was Cauchy’s *Cours d’Analyse de l’École Royale Polytechnique*, published in 1821. The theorem has been a mainstay of advanced mathematics courses at the high school and college level.

These historical remarks, extracted from the entry available in *Wikipedia*, are meant to set the stage from the mathematical side for what is to follow. We shall not be concerned here with proofs of the theorem but rather with its application in point vortex dynamics. It is somewhat rare that a fundamental theorem is of direct utility in fluid dynamics, although existence and uniqueness theorems for solutions to various partial differential equations (PDEs) would be a different example of this kind. We shall see that use of the fundamental theorem of algebra allows one to derive various ordinary differential equations (ODEs) for the generating polynomials of point vortex relative equilibria and thus to prove properties about such patterns. This is the main theme of the paper, which focuses primarily on methodology. Most of the results presented are already known by other means, but use of the fundamental theorem unifies them and shows them in a more transparent light. It also provides valuable consistency to a number of results that otherwise appear rather disconnected.

Khavinson & Neumann (2008) describe the problem of microlensing by a mass distribution where the governing equation is reminiscent of a problem of advection by a system of point vortices. They discuss rigorous results based on extensions of the fundamental theorem of algebra. There is a substantial literature on this problem some of which we shall have occasion to mention in §2f.

### 2. Relative equilibria of point vortices

The instantaneous vortex positions in a relative equilibrium of $N$ identical point vortices may be thought of as a set of distinct points, $z_\alpha = x_\alpha + iy_\alpha$, $\alpha = 1, \ldots, N$, in the complex plane that satisfy the relations

$$
\bar{z}_\alpha = \sum_{\beta=1}^N \frac{1}{z_\alpha - z_\beta}, \quad \alpha = 1, \ldots, N. \tag{2.1}
$$

The overbar on the left-hand side (l.h.s.) represents complex conjugation. The prime on the summation sign implies $\beta \neq \alpha$. There is a convention in the choice of units of length and time implicit in writing the equations in this form (see
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below), but there is no loss of generality. The general solution of the system of equation (2.1), formally a problem of algebraic geometry, determines all relative equilibria of the $N$ identical vortices.

We elaborate a bit as we shall want to generalize equation (2.1) later on. The dynamical equations for a system of point vortices on the unbounded plane are well known to be

$$\frac{dz_\alpha}{dt} = \frac{1}{2\pi i} \sum_{\beta=1}^{N} \frac{\Gamma_\beta}{z_\alpha - z_\beta}, \quad \alpha = 1, \ldots, N. \quad (2.2)$$

In general, the vortices may have different strengths or circulations $\Gamma_1, \ldots, \Gamma_N$. These are constant during the motion. In particular, for the case of identical vortices, where all the $\Gamma_\alpha$ are equal to some common value $\Gamma$, there are configurations that rotate without change of shape or size about the origin. For these relative equilibria, $z_\alpha(t) = z_\alpha(0) e^{i\Omega t}$ with some constant angular velocity $\Omega$. For such a configuration, the l.h.s. of equation (2.2) becomes

$$-i\Omega z_\alpha(t) = -i\Omega z_\alpha(0) e^{-i\Omega t}. \quad \text{Thus, from equation (2.2), the } z_\alpha(0) \text{ must solve the algebraic equations}$$

$$\frac{2\pi \Omega}{\Gamma} z_\alpha(0) = \sum_{\beta=1}^{N} \frac{1}{z_\alpha(0) - z_\beta(0)}, \quad \alpha = 1, \ldots, N.$$

Equation (2.1) results from assuming that length and time are scaled such that $2\pi \Omega/\Gamma = 1$.

We note from equation (2.1) that the $z_\alpha$ must satisfy

$$\sum_{\alpha=1}^{N} z_\alpha = 0, \quad (2.3a)$$

and

$$\sum_{\alpha=1}^{N} |z_\alpha|^2 = \sum_{\alpha=1}^{N} z_\alpha \sum_{\beta=1}^{N} \frac{1}{z_\alpha - z_\beta} = \sum_{\alpha,\beta=1}^{N} \frac{z_\alpha}{z_\alpha - z_\beta} = \frac{1}{2} N(N - 1). \quad (2.3b)$$

There is an entire hierarchy of such ‘moment relations’ (Aref & van Buren 2005), of which equations (2.3a,b) are the two simplest, but we shall not pause to develop them here.

The review by Aref et al. (2003) of the theory of relative equilibria of point vortices provides a useful summary of the variety of results known (as of 2003) concerning these configurations. For additional perspectives on the problem, see also Newton & Chamoun (2009).

(a) The generating polynomial and its ODE

The generating polynomial of the configuration, $P_N(z)$, is defined as

$$P_N(z) = \prod_{\alpha=1}^{N} (z - z_\alpha), \quad (2.4)$$
where $z_\alpha$, $\alpha = 1, \ldots, N$, the roots of $P(z)$, solve equation (2.1). This polynomial plays an important role in the theory. Obviously, it is of degree $N$ and the coefficient of its highest order term is 1. We consider the determination of $P_N(z)$ as equivalent to finding the vortex configuration although, in general, the roots would not be known explicitly.

The polynomial $P_N(z)$ obeys an ODE that arises as follows: first, differentiating with respect to $z$, we get

\[ P_N' = P_N \sum_{\alpha=1}^{N} \frac{1}{z - z_\alpha}. \]

A second differentiation gives

\[ P_N'' = P_N' \sum_{\alpha=1}^{N} \frac{1}{z - z_\alpha} - P_N \sum_{\alpha=1}^{N} \frac{1}{(z - z_\alpha)^2} = P_N \sum_{\alpha, \beta=1}^{N} \frac{1}{(z - z_\alpha)(z - z_\beta)}. \]

(Here, the prime on the summation sign still denotes $\beta \neq \alpha$ although there is no singular term to be avoided.) Since $z_\alpha \neq z_\beta$, the summand can be re-written as

\[ \frac{1}{z - z_\alpha} \frac{1}{z - z_\beta} = \left[ \frac{1}{z - z_\alpha} - \frac{1}{z - z_\beta} \right] \frac{1}{z_\alpha - z_\beta}. \]

The double sum may then be re-written as

\[ \sum_{\alpha, \beta=1}^{N} \frac{1}{z - z_\alpha} \frac{1}{z - z_\beta} = 2 \sum_{\alpha, \beta=1}^{N} \frac{1}{z - z_\alpha} \frac{1}{z_\alpha - z_\beta}. \]

Thus,

\[ P_N''(z) = 2P_N(z) \sum_{\alpha, \beta=1}^{N} \frac{1}{z - z_\alpha} \frac{1}{z_\alpha - z_\beta}, \tag{2.5a} \]

which holds for any polynomial with simple roots $z_1, \ldots, z_N$. If in this equation, we let $z$ tend to one of the roots, say $z \to z_\lambda$, all terms with $\alpha \neq \lambda$ on the right-hand side (r.h.s.) tend to 0. From the term $\alpha = \lambda$, we find

\[ P_N''(z_\lambda) = \lim_{z \to z_\lambda} 2 \frac{P_N(z) - P_N(z_\lambda)}{z - z_\lambda} \sum_{\beta=1}^{N} \frac{1}{z_\lambda - z_\beta} = 2P_N'(z_\lambda) \sum_{\beta=1}^{N} \frac{1}{z_\lambda - z_\beta}, \tag{2.5b} \]

This is, again, a general relation for a polynomial with simple roots. If we use equation (2.5b) in equation (2.5a), we get

\[ P_N''(z) = P_N(z) \sum_{\alpha=1}^{N} \frac{P_N''(z_\alpha)/P_N'(z_\alpha)}{z - z_\alpha}. \tag{2.5c} \]

In the case at hand, the sum in equation (2.5b) equals $\bar{z}_\lambda$ and we have the important relations

\[ P_N''(z_\lambda) = 2\bar{z}_\lambda P_N'(z_\lambda), \quad \lambda = 1, \ldots, N. \tag{2.6} \]

This result is key to the applications of the fundamental theorem of algebra that we have in mind.

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Conversely, assume a polynomial of degree $N$ has simple roots, $z_1, \ldots, z_N$, and that equation (2.6) is satisfied at each of them. Then by equation (2.5b), and since $P'_N(z_i) \neq 0$ because the roots are simple, we see that equation (2.1) is satisfied. Equation (2.6) is thus both necessary and sufficient for $z_1, \ldots, z_N$ to form a relative equilibrium of identical vortices.

As a first example of the use of equation (2.6), consider the possibility that all the vortices are on a line that we take to be the $x$-axis. We then have $P''_N(x) = 2x_N P''_N(x)$. This implies that the polynomial $P''_N(x) = 2x P''_N(x)$, which is of degree $N$, has the roots $x_1, \ldots, x_N$, i.e. the same roots as $P_N$. By the fundamental theorem of algebra, there can be only one such polynomial up to a multiplicative factor, so we must have

$$P''_N(x) - 2x P''_N(x) = -2NP_N(x),$$

where the multiplicative factor, $-2N$, follows from balancing the coefficient of $x^N$. Thus, we arrive at the ODE for Hermite polynomials and the result, owing to Stieltjes (1885), that $N$ identical vortices on a line in a relative equilibrium must be situated at the zeros of the $N$th Hermite polynomial.

If the vortices are not on a line, we have from equations (2.6) and (2.5c) the formula

$$P''_N(z) = 2P_N(z) \sum_{a=1}^N \frac{\bar{z}_a}{z - z_a}. \quad (2.7a)$$

In the summand, we may write $\bar{z}_a = \bar{z}_a - z_a + z_a - z + z$ to give

$$P''_N(z) - 2P_N(z) \sum_{a=1}^N \left[ -2iy_{a\bar{z}} - 1 + \frac{z}{z - z_a} \right] = 0,$$

which then yields

$$P''_N(z) - 2z P'_N(z) + 2NP_N(z) = -4i \sum_{a=1}^N \frac{y_{z_a}}{z - \bar{z}_a} P_N(z). \quad (2.7b)$$

Equations (2.7a,b) are, formally speaking, ODEs for $P_N(z)$, although they suffer from having in one coefficient the roots of the solution, which would generally only be known once the solution had been completed. However, for simple geometric arrangements, these ODEs can yield useful results. For example, if the vortices are all on a line, taken to be the $x$-axis, the r.h.s. of equation (2.7b) vanishes, and we return to the ODE for the Hermite polynomials. If the vortices are on a circle of radius $R$, we have

$$\sum_{a=1}^N \frac{\bar{z}_a}{z - z_a} = \frac{1}{z} \sum_{a=1}^N \frac{z\bar{z}_a}{z - \bar{z}_a} = \frac{1}{z} \sum_{a=1}^N \bar{z}_a \left[ 1 + \frac{z_{a\bar{z}}}{z - \bar{z}_a} \right] = \frac{1}{z} \sum_{a=1}^N \bar{z}_a + \frac{R^2}{z} \sum_{a=1}^N \frac{1}{z - \bar{z}_a}.$$

The first sum vanishes by equation (2.3a), and equation (2.7a) then gives

$$z P''_N(z) = 2R^2 P'_N(z), \quad (2.8)$$

an ODE that we shall derive directly from equation (2.6) in §2b. Equation (2.7b) was apparently first derived by Aref (2007).
(b) A single ring must be a regular polygon

It is a classical result going back to Thomson (1883) that \( N \) vortices placed at the vertices of a regular \( N \)-gon form a relative equilibrium. Let us now ask a question that is a partial converse to this result: if \( N \) identical vortices, all on the same circle centred at the origin, form a relative equilibrium, are there other possibilities than the regular \( N \)-gon? The answer is ‘no’ as we now show using equation (2.6).

Assume the \( N \) vortices to be all situated on a circle of radius \( R_o \). Multiplying equation (2.6) by \( z_\alpha \), we obtain

\[
    z_\alpha P_N''(z_\alpha) = 2R_o^2 P_N'(z_\alpha), \quad \alpha = 1, \ldots, N. \tag{2.9}
\]

In other words, the polynomial \( zP_N'' - 2R_o^2 P_N' \), which is of degree \( N - 1 \), has the \( N \) roots \( z_1, \ldots, z_N \). Thus, by the fundamental theorem of algebra, it must be the null polynomial, and we have for all \( z \):

\[
    zP_N''(z) - 2R_o^2 P_N'(z) = 0,
\]

which is just equation (2.8). Setting the coefficient of \( z^{N-1} \) to 0, we get

\[
    N(N-1) - 2NR_o^2 = 0, \quad \text{i.e. the radius of the circle must have the value}
\]

\[
    R_o = \sqrt{\frac{N-1}{2}}. \tag{2.10}
\]

This also follows from equation (2.3b). Then,

\[
    zP_N'(z) - (N-1)P_N(z) = (zP_N'(z))' - NP_N(z) = 0.
\]

Integrating once, we see that

\[
    zP_N'(z) - NP_N(z) = N\rho^N,
\]

where we have written the constant of integration on the r.h.s. in a suggestive form. The solution to this ODE that has the coefficient of the highest order term equal to 1 is

\[
    P_N(z) = z^N - \rho^N. \tag{2.11}
\]

This shows that the vortices form a regular \( N \)-gon, in other words that \( z_\alpha = \rho e^{i2\pi\alpha/N} \), where \( \rho \) is a complex number of modulus \( R_o \) given by equation (2.10).

Similarly, assume the \( N \) identical vortices form a relative equilibrium with \( N - 1 \) vortices on a circle of radius \( R_c \) and one at the origin. We show that the \( N - 1 \) vortices must again form a regular polygon. The polynomial \( P_N \) has the form \( P_N(z) = zQ_{N-1}(z) \), where \( Q_{N-1} \) is of degree \( N - 1 \), has highest coefficient 1, and has all its roots, \( z_1, \ldots, z_{N-1} \), on a circle of radius \( R_c \). From equation (2.6), we have that \( z_\alpha P_N''(z_\alpha) = 2R_c^2 P_N'(z_\alpha) \) for \( \alpha = 1, \ldots, N - 1 \). Now,

\[
    P_N'(z) = Q_{N-1}(z) + zQ_{N-1}'(z) \quad \text{and} \quad P_N''(z) = 2Q_{N-1}'(z) + zQ_{N-1}''(z).
\]

Thus,

\[
    z_\alpha P_N''(z_\alpha) - 2R_c^2 P_N'(z_\alpha) = z_\alpha[2(1 - R_c^2)Q_{N-1}'(z_\alpha) + z_\alpha Q_{N-1}''(z_\alpha)].
\]
or, in place of equation (2.9),

\[ z_\alpha Q''_{N-1}(z_\alpha) = 2(R_c^2 - 1) Q'_{N-1}(z_\alpha). \]  

(2.9')

The polynomial \( z Q''_{N-1}(z) - 2(R_c^2 - 1) Q'_{N-1}(z) \) of degree \( N - 2 \) thus has the \( N - 1 \) roots \( z_1, \ldots, z_{N-1} \) and must be the null polynomial:

\[ z Q''_{N-1}(z) - 2(R_c^2 - 1) Q'_{N-1}(z) = 0. \]

The vanishing of the coefficient of \( z^{N-2} \) gives

\[ (N - 1)(N - 2) - 2(N - 1)(R_c^2 - 1) = 0, \]

in other words, the radius of the circle containing the \( N - 1 \) vortices is

\[ R_c = \sqrt{\frac{N}{2}}. \]  

(2.12)

Again, this could have been obtained directly from equation (2.3b).

The ODE for \( Q_{N-1} \) now becomes

\[ z Q''_{N-1}(z) - (N - 2) Q'_{N-1}(z) = 0. \]

This integrates to

\[ Q_{N-1}(z) = z^{N-1} - \rho^{N-1}, \]

where the complex constant of integration, \( \rho \), must now have modulus \( R_c \) given by equation (2.12). The \( N \) vortices must form a centred, regular polygon.

One might ask if similar results hold for two rings, e.g. if we know that a relative equilibrium consists of \( n_1 \) vortices on a ring of radius \( R_1 \), and \( n_2 \) on a ring of radius \( R_2 \neq R_1 \), can we conclude that the configuration consists of a regular \( n_1 \)-gon on one ring and a regular \( n_2 \)-gon on the other? The answer to this question is ‘no’ as one sees explicitly from the degenerate case of the three-ring problem (Aref & van Buren 2005). In that problem, the configuration generally consists of three rings, of radii \( R_1, R_2 \) and \( R_3 \), all different, with a regular polygon of vortices on each. Analysis shows that the three polygons must have the same number of vortices. Most of the configurations, then, consist of three nested, regular \( n \)-gons of different radii (and there is a system of equations determining these radii as discussed extensively in the reference given). However, for every \( n \geq 3 \), there are two ‘degenerate’ configurations for which two of the radii coincide. These relative equilibria consist of \( 3n \) vortices, with \( n \) of them forming a regular \( n \)-gon on a ring of radius \( R_1 \), say, and \( 2n \) forming two regular \( n \)-gons on a ring of common radius \( R_2 = R_3 \neq R_1 \). However, the analysis shows that these \( 2n \) vortices do not form a regular \( 2n \)-gon, dashing hopes for a simple generalization of the single-ring results.

(c) Nested regular polygons

Consider a relative equilibrium in which vortices are configured on concentric regular polygons. Label the vortices \( z_j^{(p)} \), where \( p = 1, \ldots, s \), gives the polygon number and \( j = 1, \ldots, n_p \), gives the vortex number round the polygon. The total
The number of vortices in the configuration is \( N = n_1 + n_2 + \ldots + n_s \). The generating polynomial has the form

\[
P_N(z) = \prod_{p=1}^{s} \mu_p(z), \quad \mu_p(z) = z^{n_p} - (R_p e^{i\phi_p})^{n_p},
\]

(2.13)

where \( \phi_p \) is the angle through which the \( p \)th polygon is turned relative to the real axis, and \( R_p \) is its radius. We assume all the \( R_p \) to be different. From equation (2.6), we have the equations

\[
z_j^{(p)} P_N''(z_j^{(p)}) = 2R_p^2 P_N'(z_j^{(p)}), \quad p = 1, \ldots, s; \quad j = 1, \ldots, n_p.
\]

(2.14)

The l.h.s. in these equations are the values of the polynomial \( zP_N''(z) \), of degree \( N - 1 \), at the \( N \) vortex positions. We also want to write the r.h.s. as the values of one and the same polynomial (of degree \( N - 1 \)) at the vortex positions. If we call this polynomial \( \hat{P}(z) \), Lagrange’s interpolation formula gives

\[
\hat{P}(z) = P_N(z) \sum_{p=1}^{s} \frac{2R_p^2}{z - z_j^{(p)}} = 2P_N(z) \sum_{p=1}^{s} \frac{R_p^2 \mu_p'(z)}{\mu_p(z)}.
\]

It is not difficult to check that, indeed,

\[
\hat{P}(z_j^{(p)}) = 2R_p^2 \mu_p'(z_j^{(p)}) \prod_{q=1}^{s} \mu_q(z_j^{(p)}) = 2R_p^2 P_N'(z_j^{(p)}).
\]

The prime on the product sign here means \( q \neq p \). The last step follows from

\[
P_N' = P_N \sum_{p=1}^{s} \frac{\mu_p'}{\mu_p} \quad \text{and} \quad P_N'(z_j^{(p)}) = \mu_p'(z_j^{(p)}) \prod_{q=1}^{s} \mu_q(z_j^{(p)}).
\]

Since \( \hat{P} \) is a polynomial of degree \( N - 1 \), the polynomial \( zP_N''(z) - \hat{P}(z) \), which vanishes at the \( N \) vortex positions, must be the null polynomial by the fundamental theorem of algebra. We, therefore, have the equation

\[
zP_N''(z) - 2P_N(z) \sum_{p=1}^{s} R_p^2 \frac{\mu_p'(z)}{\mu_p(z)} = 0.
\]

(2.15)

Now,

\[
P_N''(z) = P_N(z) \left[ \sum_{p,q=1}^{s} \frac{\mu_p' \mu_q'}{\mu_p \mu_q} + \sum_{p=1}^{s} \frac{\mu_p''}{\mu_p} \right],
\]

where the prime on the summation here and in equations (2.16b,c) means \( p \neq q \), and

\[
\frac{\mu_p''(z)}{\mu_p(z)} = \frac{n_p - 1}{z} \frac{\mu_p'(z)}{\mu_p(z)}.
\]
Therefore, if we set
\[
\frac{\mu_p'(z)}{\mu_p(z)} = \frac{1}{z} v_p \left( \frac{1}{z} \right) \quad \text{and} \quad v_p(z) = \frac{n_p}{1 - (ZR_p e^{i\varphi_p} r_p)}, \tag{2.16a}
\]
we obtain from equation (2.15)
\[
\sum_{p=1}^{s} (2R_p^2 - n_p + 1)v_p = \sum_{p,q=1}^{s} v'_p v_q, \tag{2.16b}
\]
which is the basic equation used by Aref & van Buren (2005) for \( s = 2, 3 \).

For centred, nested, regular polygons, we have \( P_N(z) = zQ_{N-1}(z) \), where \( Q_{N-1}(z) \) is the generating polynomial for the polygons with a total of \( N-1 \) vortices. As in the case of the single ring, equation (2.14) is now replaced by
\[
z^{(p)}_j Q''_{N-1}(z^{(p)}_j) = 2(R_p^2 - 1) Q'_{N-1}(z^{(p)}_j), \quad p = 1, \ldots, s; \quad j = 1, \ldots, n_p, \tag{2.14'}
\]
cf. equation (2.9'). The rest of the derivation goes through as before but in equation (2.16b) \( R_p^2 \) must be replaced by \( R_p^2 - 1 \). The basic equation for centred, nested, regular polygons then is
\[
\sum_{p=1}^{s} (2R_p^2 - n_p - 1)v_p = \sum_{p,q=1}^{s} v'_p v_q. \tag{2.16c}
\]
Equation (2.16c) is consistent with equation (2.16b) in the following sense: consider adding to the \( s \) polygons in equation (2.16b) a degenerate polygon, \( p = 0 \), with one vertex \( n_0 = 1 \) and radius \( R_0 = 0 \). Such a structure corresponds to \( v_0 = 1 \) and contributes nothing to the l.h.s. of the equation. On the r.h.s., we get an additional term \( 2(v_1 + \ldots + v_s) \) that, when moved to the l.h.s., produces equation (2.16c).

For the analysis of solutions to equations (2.16b,c), we refer to Aref & van Buren (2005).

\((d)\) Vortices on two perpendicular lines

Consider the possibility that all the vortices are situated on two perpendicular lines, conveniently taken to be the real and imaginary axes. Several examples of such configurations are known analytically or to high-precision numerically. For six (seven) vortices, the open (centred) staggered triple-digon configurations (figs 6a and 12a of Aref & van Buren 2005) are of this type. For eight vortices, there is a configuration with six vortices on one axis and two on the other. For 12 vortices, the symmetric triple-square configuration (fig. 4c in Aref & van Buren 2005) has all vortices on two perpendicular lines and so on.

Let us assume that there are \( n \) vortices on the real axis and \( m \) vortices on the imaginary axis, where \( m + n = N \). We may write \( P_N(z) = p(z) q(z) \) with two polynomials \( p(z) \), of degree \( n \), highest coefficient 1, and all its simple roots real and \( q(z) \), of degree \( m \), highest coefficient 1, and all its simple roots pure imaginary.
We have

\[ P'_N = p'q + pq' \quad \text{and} \quad P''_N = p''q + 2p'q' + pq''. \]

Thus, from equation (2.6), at a (real) root \( z_a \) of \( p \),

\[ p''(z_a)q(z_a) + 2p'(z_a)q'(z_a) = 2z_a p'(z_a)q(z_a), \]

whereas, at a (pure imaginary) root \( z_\beta \) of \( q \),

\[ 2p'(z_\beta)q'(z_\beta) + p(z_\beta)q''(z_\beta) = -2z_\beta p(z_\beta)q'(z_\beta). \]

From these equations, we see that the polynomial

\[ p''q + 2p'q' + pq'' + 2zpq' - p'q, \]

which is of degree \( N \), has the same roots as \( P_N = pq \). Thus,

\[ p''q + 2p'q' + pq'' - 2z(p'q - pq') + 2(n - m)pq = 0. \quad (2.17) \]

The coefficient of the term \( pq \) follows from balancing the terms of order \( N \) in this equation. The further study of solutions to this ODE will be dealt with in a forthcoming publication by P. Beelen and M. Brøns.

In order to amplify the remark made concerning the utility of equation (2.7b), we derive equation (2.17) directly from this equation: on the r.h.s. of equation (2.7b), the vortices on the \( x \)-axis, i.e. the roots of \( p(z) \), do not contribute to the sum. The roots of \( q(z) \) along the \( y \)-axis produce a sum of the form

\[ -4i \sum_{\alpha=1}^m \frac{y_\alpha}{z - iy_\alpha} = -4 \sum_{\alpha=1}^m \frac{i y_\alpha - z + z}{z - iy_\alpha} = 4m - 4z \sum_{\alpha=1}^m \frac{1}{z - iy_\alpha}. \]

Thus, from equation (2.7b)

\[ P'_N - 2z(p'q + pq') + 2(n + m)P_N = 4mP_N - 4zpq', \]

which is seen to be equivalent to equation (2.17).

Clarkson (2009) discusses the equation

\[ p''q - 2p'q' + pq'' - 2z(p'q - pq') + 2(n - m)pq = 0, \]

which arises in the problem of relative equilibria of vortices (with \( n(m + 1) \) having one sign, and \( m(n + 1) \) the opposite sign of the circulation) placed in a ‘quadrupole’ background flow. He shows that this equation has solutions in terms of generalized Hermite polynomials. We refer the reader to Clarkson (2009) for details. Analytical solutions to equation (2.17) are not currently known. Equation (2.27) below is another ODE of the same general form, albeit pertaining to a different physical situation.

\( (e) \) Other applications of equation (2.6)

The result (2.6) may be applied in other ways as well. Assume a relative equilibrium of \( N \) identical vortices has the property that a geometrically similar configuration augmented by a vortex at the origin is also a relative equilibrium. What can be said about the original relative equilibrium? We show that the only relative equilibria with this property are the regular vortex polygons.
Relative equilibria of point vortices

Let $P_N(z)$ be the generating polynomial of a relative equilibrium consisting of $N$ identical vortices. By assumption, $\hat{P}_{N+1}(z) = zP_N(\lambda z)$ is a generating polynomial for a relative equilibrium of $N+1$ vortices obtained by adding a vortex at the centre and re-scaling. There is no restriction in assuming $\lambda$ to be real and positive. The magnitude of $\lambda$ can be deduced from equation (2.3b): the positions of the vortices in the relative equilibrium with $N+1$ vortices are $\hat{z}_1 = \lambda^{-1}z_1, \ldots, \hat{z}_N = \lambda^{-1}z_N$ and $\hat{z}_{N+1} = 0$. Thus, by equation (2.3b)

$$
\sum_{a=1}^{N+1} |\hat{z}_a|^2 = \frac{1}{2}(N+1)N = \frac{1}{\lambda^2} \sum_{a=1}^{N} |z_a|^2 = \frac{1}{\lambda^2} \times \frac{1}{2} N(N-1)
$$

or

$$
\lambda = \sqrt{\frac{N-1}{N+1}}.
$$

Now,

$$
\hat{P}_{N+1}'(z) = P_N(\lambda z) + \lambda zP_N'(\lambda z) \quad \text{and} \quad \hat{P}_{N+1}''(z) = 2\lambda P_N'(\lambda z) + \lambda^2 zP_N''(\lambda z).
$$

The relation (2.6) for $\hat{P}_{N+1}$ holds at $\hat{z}_1, \ldots, \hat{z}_N$, and at 0 (which yields $P_N'(0) = 0$ as a necessary condition). This implies

$$
\hat{P}_{N+1}''(\hat{z}_a) = 2\lambda P_N'(z_a) + \lambda z_a P_N''(z_a) = 2\tilde{z}_a \hat{P}_{N+1}'(\hat{z}_a) = 2\lambda^{-1} \tilde{z}_a z_a P_N'(z_a).
$$

But $P_N''(z_a) = 2\bar{z}_a P_N'(z_a)$, so we have

$$
2 \left[ \lambda + \left( \lambda - \frac{1}{\lambda} \right) |z_a|^2 \right] P_N'(z_a) = 0,
$$

or, since $P_N'(z_a) \neq 0$ (because the roots are simple),

$$
(1 - \lambda^2)|z_a|^2 = \lambda^2, \quad |z_a| = \sqrt{\frac{N-1}{2}}.
$$

Thus, the vortices of the original relative equilibrium are all on a circle and the configuration must be the regular $N$-gon as given in §2b. Conversely, a regular $N$-gon clearly admits adding a vortex at the centre to produce a new relative equilibrium. The simultaneous re-scaling is due to the imposed normalization (2.1).

As one explores the relative equilibria of identical point vortices numerically, it is clear that there are ‘families’ of configurations. Certain configurations are ‘related’ as one goes from $N$ to $N+1$ to $N+2$ vortices. In some cases, one finds related configurations where the main apparent change is that a vortex has been added at the centre. This is certainly true for the analytically known doubling-ring (Havelock 1931) and triple-ring relative equilibria (Aref & van Buren 2005). As we have just seen, the only relative equilibrium for which adding a vortex at the origin results in a simple dilation are the regular vortex polygons. All other relative equilibria are, in this sense, ‘multi-scale’ configurations.

(f) Stagnation points and co-rotating points

Other significant points in the flow have characterizations in terms of the generating polynomial. These are sometimes useful in calculation. Let $z_1, \ldots, z_N$
be a relative equilibrium of \( N \) identical vortices. The velocity field, \((u,v)\), induced by these vortices at a field point \( z \) is given by

\[
u - iv = \frac{\Gamma}{2\pi i} \sum_{\alpha=1}^{N} \frac{1}{z - z_\alpha}. \tag{2.19}\]

Hence, the \textit{instantaneous stagnation points}, \( z_j^{(s)} \), of the flow are simply the roots of \( P_N'(z) \). In a system that rotates with the relative equilibrium, these points satisfy the equations

\[
\sum_{\alpha=1}^{N} \frac{1}{z_j^{(s)} - z_\alpha} = 0 \quad \text{or} \quad P_N'(z_j^{(s)}) = 0. \tag{2.20}\]

The polynomial \( P_N(z) \) has degree \( N - 1 \). Thus, by the fundamental theorem of algebra, there are at any instant \( N - 1 \) stagnation points in the flow (Aref & Brøns 1998). In other words, the index \( j \) will run from 1 to \( N - 1 \).

There will also be points in the flow that rotate with the relative equilibrium as if rigidly attached to it, i.e. points at which the velocity induced by the vortices, equation (2.19), equals the velocity corresponding to the rigid body rotation of the relative equilibrium, i.e. \( u + iv = i\Omega z \), Morton (1933) called these ‘points of relative rest’, not to be confused with the aforementioned stagnation points which are, in this sense, ‘points of absolute rest’. We shall use the term \textit{co-rotating points}. With the normalization in equation (2.1) the co-rotating points, \( z_k^{(c)} \), satisfy Morton’s equation

\[
\bar{z}_k^{(c)} = \sum_{\alpha=1}^{N} \frac{1}{z_k^{(c)} - z_\alpha}. \tag{2.21a}\]

In terms of the generating polynomial of the relative equilibrium, a co-rotating point satisfies

\[
\bar{z}_k^{(c)} P_N(z_k^{(c)}) = P_N'(z_k^{(c)}). \tag{2.21b}\]

Owing to the appearance of the complex conjugation on the l.h.s., it is not immediate to count the number of co-rotating points, i.e. the number of solutions to equation (2.21a). If equation (2.21a) is written as a polynomial equation by clearing denominators, it becomes a polynomial of degree 1 in \( \bar{z} \) and of degree \( N \) in \( z \). Khavinson & Neumann (2008) discuss the problem of solving equation (2.21b) in the context of gravitational lensing that leads to similar equations. In theorem 1 of Khavinson & Neumann (2006) they prove that if \( r(z) = p(z)/q(z) \) is a rational function, where \( p \) and \( q \) are relatively prime polynomials, and if \( n \) denotes the maximum of the degrees of \( p \) and \( q \) (assumed greater than 1), then the number of solutions of \( \bar{z} = r(z) \) is less than or equal to \( 5(n - 1) \). Equation (2.21b) corresponds to a special case of the general result, where \( q = P_N \) and \( p = P_N' \). One might, therefore, expect the upper bound on the number of solutions to be less than the general bound \( 5(N - 1) \). For the regular \( N \)-gon, the number of co-rotating points is \( 3N + 1 \) for \( N \geq 3 \) (and 5 for \( N = 2 \)) as already found by Morton (1933) and shown below. Of course, \( 3N + 1 \leq 5N - 5 \) for \( N \geq 3 \), and for \( N = 2,3 \),
the number of co-rotating points hits the $5(N - 1)$ upper bound. The index $k$ in equation (2.21a), then, ranges at most from 1 to $5(N - 1)$, but could have a shorter range.

By way of example, if the relative equilibrium configuration is a regular $N$-gon of radius $R_0$, cf. equation (2.10), we have from equation (2.21a) that the $\hat{z}_k^{(c)}$ must solve

$$\bar{z}(z^N - R_0^N) = Nz^{N-1} \quad \text{and} \quad z^N(|z|^2 - N) = R_0^N|z|^2.$$  

One solution is $z = 0$. Setting this solution aside, we see that $z^N$ is real, hence $z$ may equal $R_0 r e^{i2\pi n/N}$ or $R_0 r e^{i\pi(2n+1)/N}$ for $n = 0, 1, \ldots, N - 1$, where $r > 0$ obeys

$$\rho^N - \frac{2N}{N-1}\rho^{N-2} = \pm 1.$$  

Elementary analysis shows that for $N \geq 3$, the equation with $-1$ on the r.h.s. has two positive real solutions, while the equation with $+1$ has only one. (For $N = 2$ there are just two solutions.) Together these give rise to $3N$ co-rotating points in all. When we include the origin, we find a total of $3N + 1$ co-rotating points for a regular $N$-gon (for $N \geq 3$; for $N = 2$, we find five solutions). This result, owing to Morton (1933), has been re-discovered several times, e.g. in the context of gravitational lensing by Mao et al. (1997) and Rhie (2003).

\[(g) \ Tkachenko's \ equation\]

In systems with vortices of both positive and negative strengths further solutions are possible. For example, if we place three vortices of circulation $+\Gamma$ at the vertices of an equilateral triangle, and a vortex of circulation $-\Gamma$ at its centre, we obtain a completely stationary configuration. Let us consider generalizations of this case. Thus, consider $n$ vortices of circulation $+\Gamma$ at positions $z_1, \ldots, z_n$, and $m$ vortices of circulation $-\Gamma$ at positions $\zeta_1, \ldots, \zeta_m$ (all distinct from $z_1, \ldots, z_n$) and enquire into the possibility of a configuration in which all vortices are at rest. From equation (2.2), we obtain the conditions

$$\sum_{\beta=1}^{n} \frac{1}{z_\alpha - z_\beta} = \sum_{\lambda=1}^{m} \frac{1}{z_\alpha - \zeta_\lambda}, \quad \alpha = 1, \ldots, n \quad (2.22a)$$

and

$$\sum_{\alpha=1}^{n} \frac{1}{\zeta_\lambda - z_\alpha} = \sum_{\mu=1}^{m} \frac{1}{\zeta_\lambda - \zeta_\mu}, \quad \lambda = 1, \ldots, m. \quad (2.22b)$$

Set

$$p(z) = (z - z_1) \cdots (z - z_n) \quad \text{and} \quad q(z) = (z - \zeta_1) \cdots (z - \zeta_m), \quad (2.23)$$

and calculate as before:

$$p'(z) = p(z) \sum_{\alpha=1}^{n} \frac{1}{z - z_\alpha} \quad \text{and} \quad q'(z) = q(z) \sum_{\lambda=1}^{m} \frac{1}{z - \zeta_\lambda}.$$
Notice from these equations that

\[ p'(z_\lambda) = p(z_\lambda) \sum_{\alpha=1}^{n} \frac{1}{z_\lambda - z_\alpha} \quad \text{and} \quad q'(z_\alpha) = q(z_\alpha) \sum_{\lambda=1}^{m} \frac{1}{z_\alpha - z_\lambda}. \] (2.24a)

From equation (2.5b) applied to \( p(z) \) and \( q(z) \) separately, we have

\[ p''(z_\alpha) = 2p'(z_\alpha) \sum_{\beta=1}^{n} \frac{1}{z_\alpha - z_\beta} \quad \text{and} \quad q''(z_\lambda) = 2q'(z_\lambda) \sum_{\mu=1}^{m} \frac{1}{z_\lambda - z_\mu}. \] (2.24b)

From equations (2.24a,b), we see by using equations (2.22a,b) that

\[ q(z_\alpha)p''(z_\alpha) = 2p'(z_\alpha)q'(z_\alpha) \quad \text{and} \quad p(z_\lambda)q''(z_\lambda) = 2q'(z_\lambda)p'(z_\lambda). \] (2.24c)

These relations imply that the polynomial \( qp'' + pq'' - 2p'q' \), which is of degree \( n + m - 2 \), has the \( n + m \) roots \( z_1, \ldots, z_n, \ z_1, \ldots, z_m \), and so must be the null polynomial by the fundamental theorem of algebra. Thus, we have the ODE,

\[ qp'' + pq'' - 2p'q' = 0, \] (2.25)

connecting the polynomials \( p(z) \) and \( q(z) \). We call this *Tkachenko’s equation*, since it was first derived by Tkachenko in his thesis of 1964.

The derivation given here is more compact than the usual one wherein one calculates general expressions for \( p'' \) and \( q'' \) and then observes that various cancellations occur to produce equation (2.25). Derivation of equation (2.25) was included as question (iii) of problem 1 in the *Mathematical Tripos* at Cambridge University, 2 June 1988 (H. K. Moffatt, 2007, personal communication).

We shall not explore the solution of Tkachenko’s equation which was found later by Bartman (1984) and, quite unexpectedly, involves the so-called *Adler–Moser polynomials* that had surfaced in the theory of rational solutions of the Korteweg–deVries equation, an equation that from the standpoint of physics is unrelated to point vortex dynamics. Clarkson (2009) gives a very useful review of the properties of these polynomials.

The degree of each term in equation (2.25) is \( n + m - 2 \). Balancing the coefficients of the highest order terms, where we recall that \( p \) and \( q \) both have highest order coefficient 1, we find

\[ n(n - 1) + m(m - 1) - 2mn = 0 \quad \text{or} \quad (n - m)^2 = n + m. \]

If we set \( n - m = k \), we see that \( n = (1/2)(n + m + n - m) = (1/2)k(k + 1) \) and, similarly, \( m = (1/2)k(k - 1) \). The number of minority and majority species of vortices in these stationary configurations are, therefore, successive triangular numbers.
For configurations that translate without change of shape or size we have in place of equation (2.22a)

$$\chi = \sum_{\alpha=1}^{n} \frac{1}{z_\alpha - z_\beta} - \sum_{\lambda=1}^{m} \frac{1}{z_\alpha - \xi_\lambda}, \quad \alpha = 1, \ldots, n \quad \text{(2.26a)}$$

and

$$\chi = \sum_{\alpha=1}^{n} \frac{1}{\xi_\lambda - z_\alpha} - \sum_{\mu=1}^{m} \frac{1}{\xi_\lambda - \xi_\mu}, \quad \lambda = 1, \ldots, m. \quad \text{(2.26b)}$$

Here, $\chi = \bar{V}/2\pi G$ is given by the ratio between the common (complex) translation velocity, $\bar{V}$ and the common magnitude of circulation of the vortices, $G$. Introducing the generating polynomials of positive and negative vortices as before, we now have in place of equations (2.24c)

$$q(z_\alpha)p''(z_\alpha) = 2p'(z_\alpha)[q'(z_\alpha) + \chi q(z_\alpha)]$$

and

$$p(\xi_\lambda)q''(\xi_\lambda) = 2q'(\xi_\lambda)[p'(\xi_\lambda) - \chi p(\xi_\lambda)]. \quad \text{(2.26c)}$$

In this case, then, the polynomials $p$ and $q$ satisfy the ODE

$$qp'' + pq'' - 2p'q' + 2\chi(pq' - p'q) = 0, \quad \text{(2.27)}$$

since the polynomial on the l.h.s. is of degree $n + m - 1$, but has the $n + m$ roots $z_1, \ldots, z_n, \xi_1, \ldots, \xi_m$. Hence, by the fundamental theorem of algebra, it must be the null polynomial.

Balancing the highest order terms proportional to $\chi$, we have the obvious necessary condition that $n = m$. It is considerably less obvious that not all values of $n$ are allowed. For example, one shows by elementary means that no translating solutions exist for $n = 2$. Indeed, solutions appear as pairs of an Adler–Moser polynomial and a modified Adler–Moser polynomial (Clarkson 2009). This result has the surprising consequence that only when $n$ is a triangular number is it possible to find translating relative equilibria of the vortex systems under consideration.

### 3. Concluding remarks

We have presented several examples from the theory of point vortex relative equilibria where the fundamental theorem of algebra can be applied directly and in a decisive way. It is interesting to see such a basic result plays a prominent role in applications. The results given are in many cases known by other means so the main purpose of the paper is to illustrate this new approach via the fundamental theorem of algebra. This also serves to unify a number of results that otherwise appear as disjoint contributions to the literature. The derivations given here are all more compact than the originals. The proofs in §2b, in particular, are considerably shorter than the original proofs given by Aref & van Buren (2005) where a more explicit calculation of the coefficients in the generating polynomial was undertaken. The ODE for the generating polynomials in §2d...
is new, as is the scaling result in §2c. A number of relations involving the values of the generating polynomials and their derivatives at the vortex positions are also new, although they would be easy enough to obtain once the ODE is derived.

The main new result, that is used repeatedly, is equation (2.6). This identity hinges on the vortices being identical. If the vortices have different circulations, equation (2.6) is not available. For example, in §2g, we introduced a generating polynomial for each of the two vortex varieties, those of positive and negative circulation. In §2c, we did something analogous for the vortices on the two lines but we still had equation (2.6) for the overall system since the vortices were identical. This was not the case in §2g. In general, if the vortices are different, we would have to introduce independent generating polynomials for each variety and then use the basic equations of motion to connect them at the values of the roots. For example, one may consider generalizing the formulae in §2c to cases where the vortices on each of the nested polygons are identical, but different from polygon to polygon. Such configurations have been studied by Lewis & Ratiu (1996).

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