Generalized analytic functions in magnetohydrodynamics

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An approach of generalized analytic functions to the magnetohydrodynamic (MHD) problem of an electrically conducting viscous incompressible flow past a solid non-magnetic body of revolution is presented. In this problem, the magnetic field and the body’s axis of revolution are aligned with the flow at infinity, and the fluid and body are assumed to have the same magnetic permeability. For the linearized MHD equations with non-zero Hartmann, Reynolds and magnetic Reynolds numbers ($M$, $Re$ and $Re_m$, respectively), the fluid velocity, pressure and magnetic fields in the fluid and body are represented by four generalized analytic functions from two classes: $r$-analytic and $H$-analytic. The number of the involved functions from each class depends on whether the Cowling number $S = M^2 / (Re_m Re)$ is 1 or is not 1. This corresponds to the well-known peculiarity of the case $S = 1$. The MHD problem is proved to have a unique solution and is reduced to boundary integral equations based on the Cauchy integral formula for generalized analytic functions. The approach is tested in the MHD problem for a sphere and is demonstrated in finding the minimum-drag spheroids subject to a volume constraint for $S < 1$, $S = 1$ and $S > 1$. The analysis shows that as a function of $S$, the drag of the minimum-drag spheroids has a minimum at $S = 1$, but with respect to the equal-volume sphere, drag reduction is smallest for $S = 1$ and becomes more significant for $S \gg 1$.

Keywords: generalized analytic function; magnetohydrodynamics; boundary integral equations; drag

1. Introduction

The steady flow of an electrically conducting viscous incompressible fluid in the presence of a magnetic field and with neglected thermal effects can be characterized by three independent parameters: the Hartmann number $M$, Reynolds number $Re$ and magnetic Reynolds number $Re_m$. $M$ is the ratio of the Lorentz force to the viscous force in the Navier–Stokes equations (when $M = 0$, the velocity field is uncoupled from the electromagnetic field), whereas $Re_m$ is interpreted as the ratio of magnetic advection to magnetic diffusion (when $Re_m = 0$, the magnetic field is uncoupled from the velocity field). The ratio of

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The magnetic forces to the inertial forces is characterized by the Cowling number $S$ (Cowling 1957), which can be expressed as $M^2/(Re_m Re)$ when $Re \neq 0$ and $Re_m \neq 0$.

A magnetohydrodynamic (MHD) problem that has attracted much theoretical interest is the one of an electrically conducting flow past a non-magnetic sphere in the presence of a uniform magnetic field being aligned with the undisturbed flow at infinity (Chester 1957; Blerkom 1960; Gotoh 1960a,b; Yosinobu 1960; Goldsworthy 1961; Ludford & Singh 1963). For this problem, figure 1 shows the drag for the unit sphere normalized to the Stokes drag (when $Re = Re_m = M = 0$) of the sphere as a function of $S$ in five cases: (i) $Re = Re_m = 1$; (ii) $Re = 1, Re_m = 3$; (iii) $Re = Re_m = 2$; (iv) $Re = 3, Re_m = 1$; and (v) $Re = Re_m = 3$. At $S = 1$, the drag attains minimum and is non-smooth (Gotoh 1960a; Goldsworthy 1961). Namely, the fact that the sphere’s drag at $S = 1$ is non-smooth is remarkable. It poses the following research questions. If the sphere is replaced by an arbitrary non-magnetic body of revolution, what is the body’s shape that has the smallest drag subject to a volume constraint and how does it depend on $S$? Are there any qualitative differences in the minimum-drag shapes for $S < 1$, $S = 1$ and $S > 1$? How does drag reduction depend on $S$? Answering these questions is the subject of this paper and the follow-up work (Zabarankin 2011).

The challenge in addressing the posed questions is that the traditional partial differential equation (PDE)-constrained optimization approach coupled with the finite-element method (FEM) is slowly converging and inaccurate in general. The second deficiency is attributed to the fact that being applied to external problems, the FEM truncates and discretizes an external domain, and minimum-drag shapes are found point-wise. Our approach consists of two parts: (i) reducing the MHD problem to boundary integral equations and (ii) deriving the optimality condition for the minimum-drag shapes analytically and obtaining minimum-drag shapes in a function series form (see Zabarankin & Molyboha (2010, 2011)).

1This number is also known as the ratio of the magnetic pressure to the dynamic pressure and is denoted by $\beta$ in Blerkom (1960) and by $S$ in Yosinobu (1960) and Gotoh (1960a).
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for a discussion of boundary integral equation-constrained optimization versus FEM-based PDE-constrained optimization). This paper addresses the first part, whereas Zabarankin (2011) addresses the second.

In the proposed approach, the MHD equations are linearized, and the velocity, pressure and magnetic fields in the fluid and body of revolution are represented by four generalized analytic functions. Under the assumption that the fluid and body are both non-magnetic with the same magnetic permeability, the axially symmetric MHD problem is reduced to integral equations for the boundary values of the two involved generalized analytic functions based on the generalized Cauchy integral formula. The Hartmann number is assumed to be non-zero since when $M = 0$, the velocity field is uncoupled from the magnetic field. With $M \neq 0$, three cases are studied separately: (i) $Re_m \neq 0$, $Re_m Re \neq M^2$ ($S \neq 1$); (ii) $Re_m = 0$; and (iii) $Re_m Re = M^2$ ($S = 1$). The reason for this is that in each case, solving the MHD problem involves different numbers of generalized analytic functions from two classes: $r$-analytic and $H$-analytic. This is the mathematical explanation of why the case $S = 1$ is special.

(a) $r$-analytic and $H$-analytic functions

A function $G(x, y) = U(x, y) + i V(x, y)$ with $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$ is called generalized analytic or pseudo-analytic if its real and imaginary parts, $U$ and $V$, respectively, satisfy the so-called Carleman or Bers–Vekua system (Bers 1953; Vekua 1962)

$$\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} + aU + bV = 0 \quad \text{and} \quad \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} + cU + dV = 0, \quad (1.1)$$

where $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$ and $d = d(x, y)$ are real-valued functions. For example, for $a \equiv b \equiv c \equiv d \equiv 0$, the system (1.1) defines ordinary analytic functions of a complex variable $x + iy$. Generalized analytic functions arise in various areas of applied mathematics including hydrodynamics, gas dynamics, theory of elasticity, heat transfer, electromagnetism, quantum mechanics, etc. (Bers 1958; Vekua 1962; Polozhii 1973; Alexandrov & Soloviev 1978). Their theory has been extensively developed since the mid-twentieth century and extends the majority of results for ordinary analytic functions, e.g. the formal powers and the Cauchy integral formula (Bers 1953, 1956; Vekua 1962; Polozhii 1973; Chemeris 1995).

Several important classes of generalized analytic functions arise from the relationship

$$\text{curl } \Phi + [a \times \Phi] = -\text{grad } \Psi \quad \text{and} \quad \text{div } \Phi = 0 \quad (1.2)$$

for a vector field $\Phi$ and scalar field $\Psi$, where $a$ is a known vector function. This relationship is frequently encountered in problems of applied mathematics. For example, for $\Psi = 0$ and $a = 0$, equation (1.2) simplifies to an irrotational solenoidal field $\Phi$ found in electrostatics and an ideal fluid, whereas for $a = 0$, it defines the so-called related potentials $\Psi$ and $\Phi$, e.g. the pressure and vorticity in the Stokes equations, electric potential and magnetic field in conductive materials, etc. (Zabarankin & Krokhmal 2007). For $a \neq 0$ and $\Psi \neq 0$, equation (1.2) arises in the Maxwell equations for quasi-stationary electromagnetic fields, in the Oseen equations for viscous incompressible fluid (Zabarankin 2010), and in the linearized MHD equations as will be shown in this paper.
Let \((r, \varphi, z)\) be a cylindrical coordinate system with the basis \((e_r, e_\varphi, k)\). In the axially symmetric case with the \(z\)-axis being the axis of revolution, let \(\Phi = \Phi_r(r, z)e_r + \Phi_z(r, z)k\), \(\Psi = 0\) and \(a = -2\lambda k\), where \(\lambda\) is a real-valued constant. Then, for \(U = e^{-\lambda z}\Phi_z\) and \(V = e^{-\lambda z}\Phi_r\), equation (1.2) reduces to

\[
\frac{\partial U}{\partial r} = \left(\frac{\partial}{\partial z} - \lambda\right) V \quad \text{and} \quad \left(\frac{\partial}{\partial z} + \lambda\right) U = -\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) V,
\]

which is a special case of equation (1.1) and implies that

\[
(\Delta_0 - \lambda^2) U = 0 \quad \text{and} \quad (\Delta_1 - \lambda^2) V = 0,
\]

where \(\Delta_k\) denotes the so-called \(k\)-harmonic operator, \(\Delta_k \equiv \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial z^2 - k^2/r^2\). The functions \(U\) and \(V\) satisfying equation (1.3) form an \(H\)-analytic function \(G = U + iV\) of a complex variable \(\zeta = r + iz\). For \(\lambda = 0\), equation (1.3) defines a so-called \(r\)-analytic function and is arguably the most studied system among various classes of generalized analytic functions. It arises in the axially symmetric theory of elasticity (Polozhii 1973; Alexandrov & Soloviev 1978) and in the axially symmetric Stokes and Oseen flows (Zabarankin & Ulitko 2006; Zabarankin 2008, 2010). Since an \(r\)-analytic function satisfies system (1.4) for \(\lambda = 0\), it is also referred to as the 0-harmonically analytic function (Zabarankin 2008). Both classes of \(r\)-analytic and \(H\)-analytic functions will be instrumental in constructing solutions to axially symmetric MHD problems.

\[(b) \text{ The generalized Cauchy integral formula and series representation}\]

Let \(G^+\) be a generalized analytic function in a bounded open region \(\mathcal{D}^+\) in the right-half \(rz\)-plane (\(\mathcal{D}^+\) may contain parts of the \(z\)-axis). The boundary of \(\mathcal{D}^+\) is assumed to be a piece-wise smooth positively oriented curve \(\ell\), which is either closed or open with the endpoints lying on the \(z\)-axis.\(^2\) Let \(\mathcal{D}^-\) be the complement of \(\mathcal{D}^+ \cup \ell\) in the right-half \(rz\)-plane (\(\mathcal{D}^-\) is unbounded), and let \(G^-\) be a generalized analytic function in \(\mathcal{D}^-\) that vanishes at infinity. For convenience, an arbitrary function \(f(r, z)\) will be denoted by \(f(\zeta)\) without assuming its analyticity. Let \(G^\pm(\zeta)\) satisfy the Hölder condition\(^3\) on \(\ell\), and let \(\ell'\) be the reflection of \(\ell\) over the \(z\)-axis. Then, under the assumption of the symmetry condition \(G^\pm(-\zeta) = \overline{G^\pm(\zeta)}\), the Cauchy integral formula for \(G^\pm\) is given by

\[
G^\pm(\zeta) = \pm \frac{1}{2\pi i} \oint_{\ell \cup \ell'} G^\pm(\tau)\mathcal{W}(\zeta, \tau) \, d\tau, \quad \zeta \in \mathcal{D}^\pm,
\]

where \(\zeta = r + iz, \tau = r_1 + iz_1\) and \(\mathcal{W}(\zeta, \tau) \equiv \mathcal{W}(r, z; r_1, z_1)\) is a generalized Cauchy kernel (Vekua 1962; Polozhii 1973; Alexandrov & Soloviev 1978). If the boundary \(\ell\) is smooth, then on \(\ell\), \(G^\pm(\zeta)\) satisfies the generalized Sokhotski–Plemelj formula,

\[
G^\pm(\zeta) = \frac{1}{2} G^\pm(\zeta) \pm \frac{1}{2\pi i} \oint_{\ell \cup \ell'} G^\pm(\tau)\mathcal{W}(\zeta, \tau) \, d\tau, \quad \zeta \in \ell.
\]

\(^2\)Open segments of the \(z\)-axis that \(\mathcal{D}^+\) may contain are not parts of \(\ell\).

\(^3\)This condition means that for some parametrization \(\zeta(t)\) of \(\ell\), the boundary value \(f(\zeta(t))\) satisfies \(|f(\zeta(t_2)) - f(\zeta(t_1))| \leq c|t_2 - t_1|\gamma\) for all \(t_1\) and \(t_2\), some \(\gamma \in (0, 1]\) and non-negative constant \(c\).
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If \( \ell \) is piece-wise smooth (has salient points), in particular if the endpoints of \( \ell \) lie on the \( z \)-axis and the angle between \( \ell \) and the \( z \)-axis at one of the endpoints is not \( \pi/2 \) (conic endpoint), then at a salient point of \( \ell \), the coefficient 1/2 at \( G^\pm(\zeta) \) in equation (1.6) is replaced by a finite function of the angle between tangents at the salient point (see Alexandrov & Soloviev (1978, systems 31.13a, 31.13b) and Zabarankin & Nir (2011, system 3.7)).

Let \( \mathcal{W}_r \) and \( \mathcal{W}_H \) denote the generalized Cauchy kernels for \( r \)-analytic and \( H \)-analytic functions, respectively. For \( r \)-analytic functions, \( \mathcal{W}_r \) is given in Alexandrov & Soloviev (1978), Zabarankin (2008, theorem 2) and Zabarankin & Nir (2011, theorem 3.2). Theorem 1 and corollary 4 in Zabarankin (2010) present two forms for \( \mathcal{W}_H \) with \( \lambda \geq 0 \). The next result extends the Cauchy integral formula for \( H \)-analytic functions for arbitrary real-valued \( \lambda \).

**Theorem 1.1.** Let \( G^\pm \) be an \( H \)-analytic function in \( D^\pm \) for some real-valued \( \lambda \) (\( G^- \) vanishes at infinity) and let it satisfy the Hölder condition on \( \ell \). Suppose \( G^\pm(-\bar{\zeta}) = G^\pm(\bar{\zeta}) \). Then, the generalized Cauchy integral formula (1.5) takes the form

\[
G^\pm(\zeta) = \pm \frac{1}{2\pi i} \int_{\ell} G(\tau) \mathcal{W}_H(\zeta, \tau, \lambda) \, d\tau = \pm \frac{1}{2\pi i} \int_{\ell} U^\pm(\tau) e^{\lambda z_1} d_r K_1(\zeta, \tau, \lambda) + i V^\pm(\tau) r_1 e^{-\lambda z_1} d_r K_2(\zeta, \tau, \lambda),
\]

for \( \zeta \in D^\pm \), where \( d_r = \partial/\partial r_1 \, dr_1 + \partial/\partial z_1 \, dz_1 \),

\[
K_1(\zeta, \tau, \lambda) = r_1 e^{-\lambda z_1} \int_0^\pi \left( \frac{i(r \cos t - r_1)}{z_1 - z + \phi(\lambda) \rho(\zeta, \tau, t)} - \cos t \right) dt e^{-|\lambda| \rho(\zeta, \tau, t)} + i e^{-\lambda z} C(\tau, \zeta),
\]

\[
K_2(\zeta, \tau, \lambda) = e^{\lambda z_1} \int_0^\pi \left( \frac{i(r_1 \cos t - r)}{z - z_1 + \phi(\lambda) \rho(\zeta, \tau, t)} - 1 \right) e^{-|\lambda| \rho(\zeta, \tau, t)} dt e^{\lambda z} C(\zeta, \tau),
\]

\[
C(\zeta, \tau) = \frac{\pi}{2} (\phi(\lambda) + \text{sign}(r - r_1)(\phi(\lambda) - \text{sign}(z - z_1)))
\]

and \( \rho(\zeta, \tau, t) = \sqrt{r^2 + r_1^2 - 2rr_1 \cos t + (z - z_1)^2} \),

with \( \phi(\lambda) \) being defined as \( \phi(\lambda) = 1 \) for \( \lambda \geq 0 \) and \( \phi(\lambda) = -1 \) for \( \lambda < 0 \). The functions \( K_1 \) and \( K_2 \) are continuous provided that there is a branch cut connecting \( \tau \) and \( -\bar{\tau} \) and arg(\( \zeta - \tau \)) \( \in [0, 2\pi] \).

**Proof.** The proof is similar to that of theorem 1 in Zabarankin (2010). \hfill \square

For the region exterior to a sphere, example 5 and proposition 2 in Zabarankin (2010) provide series representations for \( r \)-analytic functions and \( H \)-analytic functions with \( \lambda > 0 \), respectively. The next proposition generalizes the series representation for \( H \)-analytic functions for any non-zero real-valued \( \lambda \).
Proposition 1.2 \((H\text{-analytic function in the region exterior to a sphere})\). Let \((R, \vartheta, \varphi)\) be a system of spherical coordinates. For the region exterior to a sphere centered at the system’s origin, an \(H\)-analytic function satisfying equation (1.3) for real-valued \(\lambda \neq 0\) and vanishing at infinity can be represented by

\[
H(R, \vartheta) = \frac{1}{\sqrt{R}} \sum_{n=1}^{\infty} A_n \left( L_n(\cos \vartheta) K_{n+\frac{1}{2}}(|\lambda| R) - (\text{sign} \lambda) L_{-n}(\cos \vartheta) K_{n-\frac{1}{2}}(|\lambda| R) \right),
\]

(1.8)

where \(L_n(t) = nP_n(t) - iP_n^{(1)}(t)\) with \(P_n^{(m)}(t)\), \(t \in [-1,1]\), being the associated Legendre polynomial of the first kind of order \(n\) and rank \(m\) (for \(m = 0\), the superscript is omitted), \(K_{n+\frac{1}{2}}(\cdot)\) is a modified spherical Bessel function of the third kind (see Bateman & Erdelyi 1953, §7.2.6), and real-valued coefficients \(A_n\) are such that the series (1.8) converges for all \(\vartheta \in [0, \pi]\) and \(R\) greater than or equal to the radius of the sphere.

Proof. Proved similarly to proposition 2 in Zabarankin (2010).

The rest of this work is organized into three sections. Section 2 constructs representations for the velocity field, pressure and magnetic fields in the fluid and body in terms of four generalized analytic functions. Section 3 shows that in all three cases (i)–(iii), the corresponding boundary-value problems for the four involved functions have unique solutions and can be reduced to boundary integral equations based on the Cauchy integral formula for generalized analytic functions. Section 4 demonstrates the approach of the generalized analytic functions in solving the MHD problem for a sphere and in finding minimum-drag prolate spheroids subject to a volume constraint.

2. Magnetohydrodynamics

The steady flow of an electrically conducting viscous incompressible fluid is governed by the (stationary) MHD equations

\[
\begin{align*}
\rho (U \cdot \nabla) U & = - \nabla \varphi + \rho \nu \Delta U + \mu [J \times H], \\
\text{div } U & = 0, \\
\text{curl } E & = q/\epsilon, \\
\text{curl } H & = J \\
\text{div } H & = 0,
\end{align*}
\]

(2.1)

where \(\Delta U \equiv \nabla \cdot \nabla \nabla U - \nabla \text{curl} \nabla \nabla U, U\) is the fluid velocity, \(\varphi\) is the pressure in the fluid, \(E\) and \(H\) are the electric and magnetic fields, respectively, \(J\) is the current density, \(q/\epsilon\) is the excess charge density, \(\nu\) is the kinematic viscosity, \(\rho\) is the fluid density, \(\mu\) is the magnetic permeability, \(\sigma\) is the electric conductivity and \(\epsilon\) is the dielectric constant (see Cowling 1957; Cabannes 1970; Chester 1957). All constants are measured in metre, kilogram and second (MKS) units.

Suppose a non-magnetic solid body of revolution is immersed in a uniform flow in the presence of a uniform magnetic field. It is assumed that in the cylindrical coordinate system \((r, \varphi, z)\), the body’s axis of revolution is parallel to the \(z\)-axis, and that at infinity, the flow and magnetic fields are constant and aligned with the \(z\)-axis, i.e. \(U|_\infty = V_\infty k\) and \(H|_\infty = H_\infty k\), where \(V_\infty\) and \(H_\infty\) are constants. Under different approximations of the nonlinear equations (2.1), this MHD problem
was considered in Chester (1957, 1961, 1962), Gotoh (1960a,b), Imai (1960) and Stewartson (1956). Let \( \mathbf{u} \) be the disturbance of the velocity field: \( \mathbf{U} = \mathbf{V}_\infty (\mathbf{k} + \mathbf{u}) \), and let \( \mathbf{h}^\pm \) and \( \mathbf{h}^- \) be the disturbances of the magnetic field in the body and fluid, respectively: \( \mathbf{H} = \mathbf{H}_\infty (\mathbf{k} + \mathbf{h}^+) \) in the body and \( \mathbf{H} = \mathbf{H}_\infty (\mathbf{k} + \mathbf{h}^-) \) in the fluid. On the surface \( S \) of the body and at infinity, \( \mathbf{u} \) and \( \varphi \) satisfy the conditions

\[
\mathbf{u}|_S = -\mathbf{k}, \quad \mathbf{u}|_\infty = 0 \quad \text{and} \quad \varphi|_\infty = 0.
\]

(2.2)

In the body, the magnetic field satisfies \( \text{curl} \mathbf{H} = 0 \) and \( \text{div} \mathbf{H} = 0 \), or in terms of the disturbance \( \mathbf{h}^+ \),

\[
\text{curl} \mathbf{h}^+ = 0 \quad \text{and} \quad \text{div} \mathbf{h}^+ = 0.
\]

(2.3)

As in Chester (1957) and Gotoh (1960b), it is assumed that the fluid and body have the same magnetic permeability. In this case, the magnetic field is continuous across the surface of the body\(^4\) and also \( \mathbf{h}^- \) vanishes at infinity,

\[
\mathbf{h}^+|_S = \mathbf{h}^-|_S \quad \text{and} \quad \mathbf{h}^-|_\infty = 0.
\]

(2.4)

The problem (2.1)–(2.4) is axially symmetric. In this case, the velocity, pressure and electromagnetic field are independent of the angular coordinate \( \varphi \), i.e.

\[
\mathbf{u} = u_r(r, z)\mathbf{e}_r + u_z(r, z)\mathbf{k}, \quad \varphi = \varphi(r, z),
\]

\[
\mathbf{h}^\pm = h_r^\pm(r, z)\mathbf{e}_r + h_z^\pm(r, z)\mathbf{k}, \quad \text{and} \quad \mathbf{E} = E_\varphi(r, z)\mathbf{e}_\varphi.
\]

The equation \( \text{curl} \mathbf{E} = 0 \) implies \( rE_\varphi = c \), where \( c \) is a constant. Since in the fluid, \( [\mathbf{U} \times \mathbf{H}] \) and \( \text{curl} \mathbf{H} \) vanish at infinity, we have \( c = 0 \), and thus, \( \mathbf{E} \equiv 0 \) everywhere.

Rescaling the linear dimensions, velocity, pressure and magnetic field by \( a, V_\infty, V_\infty \rho v/a \) and \( H_\infty \), respectively, where \( a \) is the half of the diameter of the body, and assuming \( \mathbf{u} \) and \( \mathbf{h}^- \) to be small, we can rewrite equations (2.1) without \( \mathbf{E} \) and \( \mathbf{J} \) in the linearized dimensionless form

\[
\begin{align*}
R e (k \cdot \text{grad}) \mathbf{u} &= -\text{grad} \varphi + \Delta \mathbf{u} + M^2[[[(\mathbf{u} - \mathbf{h}^-) \times \mathbf{k}] \times \mathbf{k}], \\
\text{curl} \mathbf{h}^- &= R e_m[[(\mathbf{u} - \mathbf{h}^-) \times \mathbf{k}], \\
\text{div} \mathbf{u} &= 0 \quad \text{and} \quad \text{div} \mathbf{h}^- = 0,
\end{align*}
\]

(2.5)

where \( Re = V_\infty a/\nu \) is the Reynolds number, \( M = \mu H_\infty a/\sqrt{\sigma/(\rho v)} \) is the Hartmann number and \( R e_m = V_\infty a \mu \sigma \) is the magnetic Reynolds number.

For \( M = 0 \), the first equation in (2.5) reduces to the Oseen equations, and the problem for the velocity and pressure becomes uncoupled from the magnetic field. For solving the Oseen equations, see Zabarankin (2010). For \( M \neq 0 \), there are three cases to analyse:

(a) \( R e_m \neq 0 \) and \( R e_m Re \neq M^2 \) (\( S \neq 1 \)): a solution to (2.5) is represented by two \( H \)-analytic functions and one \( r \)-analytic function, whereas a solution to equations (2.3) is given by a single \( r \)-analytic function;

(b) \( R e_m = 0 \): the magnetic field is constant everywhere, i.e. \( \mathbf{h}^\pm = 0 \), and a solution to equations (2.5) is represented by two \( H \)-analytic functions (the case of \( Re = 0 \) yields no simplification); and

\(^4\)If the fluid and body have different magnetic permeabilities, then across the body’s surface, the tangential component of the magnetic field \( \mathbf{H} \) and the normal component of the magnetic induction \( \mathbf{B} \) should be continuous.

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(c) $Re_m Re = M^2$ ($S = 1$): a solution to equations (2.5) is represented by one $H$-analytic function and two $r$-analytic functions, and a solution to equations (2.3) is given by a single $r$-analytic function.

Let $\mathbb{D}^+$ and $\mathbb{D}^-$ be the regions occupied by the body and fluid, respectively, with $S$ being their common boundary (the surface of the body), and let $\mathbb{D}^\pm$ be the open region corresponding to the interior of the cross section of $\mathbb{D}^\pm$ in the right-half $rz$-plane ($r \geq 0$). In this case, $\ell$ denotes the positively oriented common boundary of $\mathbb{D}^+$ and $\mathbb{D}^-$, i.e. $\ell$ is the cross section of $S$ in the right-half $rz$-plane and is either a closed curve or has the endpoints on the $z$-axis, and $\ell'$ denotes the reflection of $\ell$ over the $z$-axis.

**Theorem 2.1 (solution representation, case (a)).** In the axially symmetric case with $M \neq 0$ and $Re_m Re \neq M^2$, a solution to equations (2.5) and (2.3) is given by

$$u_z + iu_r = \frac{1 - 2\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1 z} G_1 - \frac{1 - 2\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 z} G_2 - \frac{1}{Re_m Re - M^2} G_3, \quad (2.6)$$

$$h_z^+ + ih_r^+ = G_3^+, \quad (2.8)$$

and

$$h_z^+ - ih_r^+ = -\frac{2\lambda_1}{\lambda_1 - \lambda_2} (e^{\lambda_1 z} G_1 - e^{\lambda_2 z} G_2) - \frac{1}{Re_m Re - M^2} G_3^- \quad (2.7)$$

where $G_1$ and $G_2$ are $H$-analytic functions in $\mathbb{D}^-$ satisfying system (1.3) with $\lambda = \lambda_{1,2} = (Re + Re_m \pm \sqrt{(Re - Re_m)^2 + 4M^2})/4$, respectively, and vanishing at infinity; $G_3^+$ and $G_3^-$ are $r$-analytic functions in $\mathbb{D}^+$ and $\mathbb{D}^-$, respectively, with $G^-$ vanishing at infinity; and $\lambda = Re_m/(Re_m Re - M^2)$.

In this case, the pressure and scalar vortex function $\omega$ ($\omega = curl u = \omega e_\theta$ is the vorticity) are determined by

$$\varphi = \text{Re} \left[ \frac{2(Re - 1)}{\lambda_1 - \lambda_2} (\lambda_2 e^{\lambda_1 z} G_1 - \lambda_1 e^{\lambda_2 z} G_2) + \frac{Re}{Re_m Re - M^2} G_3 \right] \quad (2.9)$$

and

$$\omega = \frac{1}{\lambda_1 - \lambda_2} \text{Im}[(Re - 2\lambda_2) e^{\lambda_1 z} G_1 - (Re - 2\lambda_1) e^{\lambda_2 z} G_2]. \quad (2.10)$$

**Proof.** With $\omega = curl u$ and the identity $(k \cdot \text{grad})u = -\text{curl}[k \times u]$, the first equation in (2.5) can be rewritten in the form

$$\text{curl}(\omega - Re[k \times u]) - M^2 [k \times (k \times (u - h^-))] + \text{grad} \varphi = 0. \quad (2.11)$$

The continuity equation $\text{div} u = 0$ implies that $u$ and $\omega$ satisfy the identity

$$\text{curl}[k \times u] + \text{grad}(k \cdot u) - [k \times \omega] = 0, \quad (2.12)$$

whereas $\text{div} h^- = 0$ implies a similar identity for $h^-$,

$$[k \times \text{curl} h^-] = \text{curl}[k \times h^-] + \text{grad}(k \cdot h^-),$$

with which the second equation in (2.5) can be represented in the form

$$\text{curl}[k \times h^-] + \text{grad}(k \cdot h^-) + Re_m [k \times (k \times (u - h^-))] = 0. \quad (2.13)$$
Forming a linear combination of (2.11), (2.12) and (2.13) with constant weights $\beta_1$, $\beta_2$ and $\beta_3$, respectively, we have

$$\text{curl}(\beta_1 \omega + [k \times ((\beta_2 - \beta_1 \text{Re})u + \beta_3 h^-)])$$

$$- [k \times (\beta_2 \omega + (\beta_1 M^2 - \beta_3 \text{Re}_m)(k \times (u - h^-)))]$$

$$+ \text{grad}(\beta_1 \varphi + \beta_2 (k \cdot u) + \beta_3 (k \cdot h^-)) = 0. \quad (2.14)$$

Let $\beta_3 = -(\beta_2 - \beta_1 \text{Re})$ and $(\beta_2 - \beta_1 \text{Re})/\beta_1 = (\beta_1 M^2 - \beta_3 \text{Re}_m)/\beta_2$, then $(\beta_2/\beta_1)^2 - (\text{Re} + \text{Re}_m)\beta_2/\beta_1 + \text{Re}_m \text{Re} - M^2 = 0$, hence $\beta_2/\beta_1 = 2\lambda_k$, $k = 1, 2$, and equation (2.14) becomes a particular case of system (1.2),

$$\text{curl} \ A_k - 2\lambda_k [k \times A_k] + \text{grad} \ \Psi_k = 0 \text{ and div} \ A_k = 0, \ k = 1, 2, \quad (2.15)$$

where

$$A_k = \omega + (2\lambda_k - \text{Re})[k \times (u - h^-)]$$

and

$$\Psi_k = \varphi + 2\lambda_k (k \cdot u) + (\text{Re} - 2\lambda_k)(k \cdot h^-). \quad (2.16)$$

In the axially symmetric case, $A_k = A_k e_\varphi$, $k = 1, 2$, and system (2.15) reduces to system (1.3) determining $H$-analytic functions $G_k$,

$$\Psi_k + iA_k = 2e^{\lambda_k z} G_k, \ k = 1, 2, \quad (2.17)$$

with $\lambda_k$ defined in the theorem.

On the other hand, if $\beta_2 = 0$ and $\beta_1 M^2 - \beta_3 \text{Re}_m = 0$, then equation (2.14) becomes the relationship for related potentials,

$$\text{curl} \ A_3 + \text{grad} \ \Psi_3 = 0 \text{ and div} \ A_3 = 0, \quad (2.18)$$

where

$$A_3 = \text{Re}_m \omega + [k \times (M^2 h^- - \text{Re}_m \text{Re} u)] \quad \text{and} \quad \Psi_3 = \text{Re}_m \varphi + M^2 (k \cdot h^-). \quad (2.19)$$

In the axially symmetric case, $A_3 = A_3 e_\varphi$ and system (2.18) determines an $r$-analytic function $G_3^-$ in $D^-$,

$$\Psi_3 + iA_3 = G_3^- \quad (2.20)$$

Equations (2.15)–(2.20) can be reformulated in the complex form

$$2e^{\lambda_k z} G_k = \varphi + i\omega + 2\lambda_k (u_z + iu_r) - i\text{Re} u_r + (\text{Re} - 2\lambda_k)(h^-_z + ih^-_r), \ k = 1, 2,$$

and

$$G_3^- = \text{Re}_m (\varphi + i\omega) - i\text{Re} \text{Re}_m u_r + M^2 (h^-_z + ih^-_r),$$

from which the representations (2.6), (2.7), (2.9) and (2.10) follow. The condition $M \neq 0$ guarantees that $\lambda_1 \neq \lambda_2$.

Finally, in the axially symmetric case, $h^+$, being an irrotational solenoidal field (see equations (2.3)), can be represented in the form (2.8). ■

**Corollary 2.2 (solution representation, case (b)).** In the axially symmetric case with $M \neq 0$ and $\text{Re}_m = 0$, the magnetic disturbance is zero everywhere: $h^+ \equiv 0$, and
the velocity, pressure and vorticity that solve equations (2.5) can be represented by

\[ u_z + i u_r = \frac{1}{\lambda_1 - \lambda_2} (e^{\lambda_1 z} G_1 - e^{\lambda_2 z} G_2) \]

and \[ \varphi + i \omega = e^{\lambda_1 z} \left( G_1 - \frac{Re}{2(\lambda_1 - \lambda_2)} G_1 \right) + e^{\lambda_2 z} \left( G_2 + \frac{Re}{2(\lambda_1 - \lambda_2)} G_2 \right). \]

where \( G_1 \) and \( G_2 \) are \( H \)-analytic functions satisfying system (1.3) with \( \lambda = \lambda_{1,2} = (Re \pm \sqrt{Re^2 + 4M^2})/4 \), respectively, and vanishing at infinity.

**Detail.** For \( Re_m = 0 \), the representation (2.7) reduces to \( h^{-}_z + i h^{-}_r = G^{-}_3 / M^2 \), which with the representation (2.8) and the boundary condition (2.4) implies \( G^{-}_3 / M^2 = G^+_3 \) on \( \ell \). Consequently, as \( G^{-}_3 \) vanishes at infinity, by the Sokhotski–Plemelj formula (1.6), \( G^+_3 \equiv 0 \) in \( \mathcal{D}^\pm \), respectively, and the representation (2.21) follows from the representations (2.6), (2.9) and (2.10).

**Corollary 2.3.** For \( M \neq 0 \) and \( Re_m = Re = 0 \), representation (2.21) simplifies to

\[ u_z + i u_r = \frac{1}{M} (e^{Mz/2} G_1 - e^{-Mz/2} G_2) \]

and \[ \varphi + i \omega = e^{Mz/2} G_1 + e^{-Mz/2} G_2, \]

where \( G_1 \) and \( G_2 \) are \( H \)-analytic functions satisfying system (1.3) with \( \lambda = \lambda_{1,2} = \pm M/2 \), respectively. This representation is similar to the solution form suggested by Chester (1957) in the case of \( Re = Re_m = 0 \).

**Corollary 2.4.** For \( M = 0 \) and \( Re \neq 0 \), representation (2.21) reduces to representations (22) and (23) in Zabarankin (2010) for the velocity, pressure and vorticity for the axially symmetric Oseen flow of a non-conducting fluid.

**Theorem 2.5 (solution representation, case (c)).** In the axially symmetric case with \( M \neq 0 \) and \( Re_m Re = M^2 \), a solution to equations (2.5) and (2.3) is given by

\[ u_z + i u_r = \frac{1}{Re + Re_m} \left( \frac{2Re e^{\lambda z} G_1}{Re + Re_m} + \left( \frac{Re_m (z - \frac{i}{2} r)}{Re + Re_m} - \frac{Re}{Re + Re_m} \right) G_2 + Re_m G_3^- \right), \]

\[ h^{-}_z + i h^{-}_r = \frac{Re_m}{Re + Re_m} \left( - \frac{2e^{\lambda z} G_1}{Re + Re_m} + \left( z - \frac{i}{2} r + \frac{1}{Re + Re_m} \right) G_2 + G_3^- \right) \]

and \( h^+_z + i h^+_r = G^+_3 \),

where \( \lambda = (Re + Re_m)/2 \); \( G_1 \) is an \( H \)-analytic function in \( \mathcal{D}^- \) that satisfies system (1.3) with \( \lambda \) and vanishes at infinity; \( G_2 \) and \( G_3^- \) are \( r \)-analytic functions in \( \mathcal{D}^- \) that vanish at infinity; and \( G^+_3 \) is an \( r \)-analytic function in \( \mathcal{D}^+ \).

In this case, the pressure and vortex functions are determined by

\[ \varphi = Re \left[ \frac{Re_m Re}{Re + Re_m} \left( \frac{2e^{\lambda z} G_1}{Re + Re_m} - \left( z - \frac{i}{2} r + \frac{1}{Re + Re_m} \right) G_2 - G_3^- \right) + G_2 \right] \]

and

\[ \omega = \frac{1}{Re + Re_m} \text{Im}[2Re e^{\lambda z} G_1 + Re_m G_2]. \]
Proof. The proof is partially based on the proof of theorem 2.1. When \( R_e m \, R e = M^2 \), relationships (2.15)–(2.17) hold for \( \lambda_1 = (R e + R e_m)/2 \) and \( \lambda_2 = 0 \). For \( \lambda_1 = (R e + R e_m)/2 \), equation (2.17) simplifies to

\[
2e^{(R e + R e_m)z/2} G_1 = \varphi + i \omega + (R e + R e_m)(u_z + i u_r) - i R e \, u_r - R e_m(h_z^- + i h_r^-),
\]

where \( G_1 \) is the \( H \)-analytic function defined in this theorem. The case of \( \lambda_2 = 0 \) means that equation (2.15) reduces to relationship (2.18) for the related potentials. Consequently, a different approach is required. As \( R e m \, R e = M^2 \), multiplying equation (2.13) by \( R e \) and adding to equation (2.11), we obtain

\[
curl(\omega - R e \, [k \times (u - h^-)]) + \text{grad}(\varphi + R e \, (k \cdot h^-)) = 0,
\]

which with the second equation in equations (2.5) and conditions \( \text{div} \, u = 0 \) and \( \text{div} \, h^- = 0 \) can be rewritten as

\[
\Delta(u + (R e/R e_m)h^-) = \text{grad}(\varphi + R e(k \cdot h^-)) \quad \text{and} \quad \text{div}(u + (R e/R e_m)h^-) = 0.
\]

System (2.28) is similar to the Stokes equations for a viscous incompressible fluid. In the axially symmetric case, its solution is given by proposition 7 in Zabarankin (2008):

\[
u_z + i u_r + \frac{R e}{R e_m}(h_z^- + i h_r^-) = \left( z - \frac{i}{2} r \right) G_2 + G_3^- \]

and

\[
\varphi + i \omega - i R e \, u_r + R e(h_z^- + i h_r^-) = G_2,
\]

where \( G_2 \) and \( G_3^- \) are the \( r \)-analytic functions defined in this theorem. Observe that the conditions \( M \neq 0 \) and \( R e_m \, R e = M^2 \) guarantee that \( R e_m \neq 0 \). The representations (2.22), (2.23), (2.25) and (2.26) follow from equations (2.27) and (2.29), whereas the representation (2.24) remains the same as in theorem 2.1.

The advantage of the representations (2.6)–(2.10) and (2.21)–(2.26) compared with the existing solution forms, e.g. in Chester (1957); Gotoh (1960a,b) and Yosinobu (1960), is that these representations, being linear combinations of the generalized analytic functions, involve no derivates of those functions and simultaneously represent the velocity, vorticity, pressure and magnetic fields in the fluid and body. This fact considerably simplifies reducing the MHD problem to boundary integral equations based on the generalized Cauchy integral formula.

3. Boundary-value problems and integral equations

(a) Case (a): \( M \neq 0, R e_m \neq 0 \) and \( R e_m \, R e \neq M^2 \)

For \( M \neq 0, R e_m \neq 0 \) and \( R e_m \, R e \neq M^2 \), the MHD problem (2.2)–(2.5) reduces to the boundary-value problem for the four generalized analytic functions \( G_1, G_2 \)
and $G_3^\pm$ defined in theorem 2.1,

\[
(1 - 2\lambda_2 x)e^{\lambda_1 z}G_1 - (1 - 2\lambda_1 x)e^{\lambda_2 z}G_2 - \frac{\lambda_1 - \lambda_2}{Re_m Re - M^2} G_3^- = \lambda_2 - \lambda_1, \quad \zeta \in \ell, \tag{3.1}
\]

and

\[
e^{\lambda_1 z}G_1 - e^{\lambda_2 z}G_2 = (\lambda_2 - \lambda_1)(G_3^+ + 1), \quad \zeta \in \ell.
\]

**Proposition 3.1.** The boundary-value problem (3.1) has a unique solution.

**Proof.** The proposition is equivalent to the fact that the homogenous problem (3.1) has only zero solution, which with the representations (2.6)–(2.10) corresponds to (2.5) with the zero boundary conditions, i.e. $u|_S = 0$.

The first equation in (2.5) can be recast in the form

\[
\text{curl} \, \omega + \text{grad} \varphi + Re \frac{\partial u}{\partial z} - M^2[k \times [k \times (u - h^-)]] = 0. \tag{3.2}
\]

With the identities

\[
u \cdot \text{curl} \, \omega = \text{div}(\omega \times u) + |\omega|^2
\]

and

\[
u \cdot \text{grad} \, \varphi = \text{div}(\varphi \, u) - \varphi \text{ div} \, u = \text{div}(\varphi \, u),
\]

the scalar product of equation (3.2) and $u$ takes the form

\[
\text{div} \left( [\omega \times u] + \varphi u + Re \frac{1}{2} |u|^2 k \right) + |\omega|^2 + M^2 u_r(u_r - h^-_r) = 0. \tag{3.3}
\]

On the other hand, with the identity

\[
[k \times h^-] \cdot \text{curl} \, h^- = \text{div} \left( \frac{1}{2} |h^-|^2 - h^- (k \cdot h^-) \right), \tag{3.4}
\]

the scalar product of the second equation in equations (2.5) and $[k \times h^-]$ results in

\[
h^-_r (h^-_r - u_r) - \frac{1}{Re_m} \text{div} \left( \frac{1}{2} |h^-|^2 - h^- (k \cdot h^-) \right) = 0, \tag{3.5}
\]

provided that $Re_m \neq 0$. The linear combination (3.3) + $M^2 (3.5)$ reduces to

\[
\text{div} \left( [\omega \times u] + \varphi u + Re \frac{1}{2} |u|^2 k - \frac{M^2}{Re_m} \left( \frac{1}{2} |h^-|^2 - h^- (k \cdot h^-) \right) \right) + |\omega|^2 + M^2 (u_r - h^-_r)^2 = 0. \tag{3.6}
\]

Since curl $h^+ = 0$ in $\mathbb{D}^+$, we can also write

\[
[k \times h^+] \cdot \text{curl} \, h^+ = \text{div} \left( \frac{1}{2} |h^+|^2 - h^+ (k \cdot h^+) \right) = 0. \tag{3.7}
\]

Let $\mathbb{D}_R$ be the region bounded by the body’s surface $S$ and by a sphere $S_R$ with large radius $R$ and centre at the origin, so that $\mathbb{D}_R \to \mathbb{D}^-$ as $R \to \infty$. Applying the divergence theorem to the linear combination of the volume integrals,
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\[
\iint_{\mathbb{D}_R} (3.6) \, dV - M^2 / Re_m \iint_{\mathbb{D}^+} (3.7) \, dV, \text{ and using the boundary conditions } u = 0 \text{ and } h^+ = h^- \text{ on } S, \text{ we obtain}
\]
\[
\mathcal{I}_R + \iint_{\mathbb{D}_R} \left( |\omega|^2 + M^2(u_r - h_r^-)^2 \right) \, dV = 0,
\]

where
\[
\mathcal{I}_R = \int_{S_R} \mathbf{n} \cdot \left( (\omega \times u) + \varphi u + \frac{Re}{2} |u|^2 k - \frac{M^2}{Re_m} \left( \frac{1}{2} |h^-|^2 - h^- (k \cdot h^-) \right) \right) \, dS,
\]

with \( \mathbf{n} \) being the outward normal. Note that equation (3.8) holds for multiply connected \( \mathbb{D}_R \), since \( u, \omega, \varphi \) and \( h^\pm \) are all single-valued functions. \(^5\)

Next we show that \( \mathcal{I}_R \to 0 \) as \( R \to \infty \). Let \( C_R \) be the positively oriented cross section of \( S_R \) in the right-half \( rz \)-plane, i.e. \( C_R \) is a semicircle, and let \( \partial / \partial s \) and \( \partial / \partial n \) be the tangential and normal derivatives for \( C_R \), respectively. With the identities
\[
\frac{\partial r}{\partial s} = \frac{\partial z}{\partial n} \quad \text{and} \quad \frac{\partial r}{\partial n} = -\frac{\partial z}{\partial n},
\]
and the surface element \( dS = r \, ds \, d\varphi \), where \( ds \) is the differential of the length of \( C_R \), we have
\[
\mathcal{I}_R = 2\pi Re \left[ \int_{C_R} \left( (u_z + iu_r) \left( \varphi + i\omega + \frac{Re}{2} (u_z - iu_r) \right) + \frac{M^2}{2Re_m} (h_z^- + ih_r^-)^2 \right) r \, d\zeta \right].
\]

Now with the representations (2.6)–(2.7) and (2.9)–(2.10), the integral (3.10) reduces to
\[
\mathcal{I}_R = 2\pi Re \left[ \int_{C_R} \left( k_1 e^{2\lambda_1 z} G_1^2 + k_2 e^{2\lambda_2 z} G_2^2 - \frac{(G_3^-)^2}{2Re_m(Re_m Re - M^2)} \right) r \, d\zeta \right],
\]

where \( k_1 = (Re/2 - 2\lambda_2 + 2\lambda_2^2) / (\lambda_1 - \lambda_2)^2 \) \( \text{ and } k_2 = (Re/2 - 2\lambda_1 + 2\lambda_1^2) / (\lambda_1 - \lambda_2)^2 \). Observe that \( \lambda_1 \neq \lambda_2 \) as \( M \neq 0 \).

The representation (1.8) implies that in the spherical coordinates \( (R, \vartheta, \varphi) \) related to the cylindrical coordinates in the ordinary way, \( G_i(R, \vartheta) = f_i(\vartheta) R^{-1} e^{-|\lambda_i| R} + \mathcal{O}(R^{-2} e^{-|\lambda_i| R}) \) as \( R \to \infty \) for \( i = 1, 2 \), where \( f_i(\vartheta) \) is a bounded complex-valued function with \( |f_i(\vartheta)| < K \) for some constant \( K \) and \( \vartheta \in [0, \pi] \), whereas example 5 in Zabarankin (2010) implies that \( G_3^- = \mathcal{O}(R^{-2}) \) as \( R \to \infty \). Thus, with \( \zeta = Re^{i\vartheta}, \vartheta \in [0, \pi] \), we can evaluate
\[
|\mathcal{I}_R| \leq 2\pi K^2 \int_0^\pi \left( \sum_{j=1}^2 |k_j| e^{-2R (|\lambda_j| - |\lambda_j| \cos \vartheta)} \right) \sin \vartheta \, d\vartheta + o(1)
\]
\[
= 2\pi K^2 \sum_{j=1}^2 \frac{|k_j|}{2|\lambda_j| R} (1 - e^{-4|\lambda_j| R}) + o(1) \to 0 \quad \text{as} \quad R \to \infty,
\]

If \( \mathbb{D}_R \) is multiply connected, we can make crosscuts in \( \mathbb{D}_R \) to make \( \mathbb{D}_R \) simply connected, and since \( u, \omega, \varphi \) and \( h^\pm \) are single valued, they have the same values on the banks of a crosscut.
where \( \lambda_i \neq 0 \) provided that \( \Re_m \Re \neq M^2 \). Consequently, passing \( R \) to infinity in equation (3.8), we obtain \( \iint_{D^+} (|\omega|^2 + M^2(u_r - h_r^+)^2) \, dV = 0 \) so that \( \omega = 0 \) and \( u_r = h_r^- \) in \( D^- \), which along with the representations (2.6), (2.7) and (2.10) imply that \( \Im G_j = 0 \), \( j = 1, 2 \), in \( D^- \). It follows from the system (1.3) that \( \Re G_j = c_i e^{-\lambda_iz} \), \( i = 1, 2 \), where \( c_i \) are real-valued constants. But since \( G_1 \) and \( G_2 \) vanish at infinity, \( c_j = 0 \), \( j = 1, 2 \), and thus, \( G_1 = 0 \) and \( G_2 = 0 \) in \( D^- \). Then, the first and second equations of the homogeneous problem (3.1) imply \( G_3^+ = 0 \) and \( G_3^- = 0 \) on \( \ell \), and it follows from the Cauchy integral formula (1.5) for \( r \)-analytic functions that \( G_3^+ = 0 \) in \( D^\pm \), respectively.

The next proposition is considered from the mathematical point of view only. It will be used in determining homogeneous solutions to boundary integral equations that follow from the boundary-value problem (3.1).

**Proposition 3.2 (homogeneous conjugate boundary-value problem, case (a)).** Let \( G_1 \) and \( G_2 \) be \( H \)-analytic functions in \( D^+ \), and let \( G_3^\pm \) be \( r \)-analytic functions in \( D^\pm \), respectively, with \( G_3^- \) vanishing at infinity. Under the assumptions \( M \neq 0 \), \( \Re_m \neq 0 \) and \( \Re_m \Re \neq M^2 \), the homogeneous conjugate boundary-value problem

\[
(1 - 2\lambda_2x)e^{\lambda_1 z} G_1 - (1 - 2\lambda_1 x)e^{\lambda_2 z} G_2 - \frac{\lambda_1 - \lambda_2}{\Re_m \Re - M^2} G_3^+ = 0, \quad \zeta \in \ell, \tag{3.11}
\]

and

\[
e^{\lambda_1 z} G_1 - e^{\lambda_2 z} G_2 - (\lambda_2 - \lambda_1) G_3^- = 0, \quad \zeta \in \ell,
\]

has the solution

\[
G_1 = c e^{-\lambda_1 z}, \quad G_2 = c e^{-\lambda_2 z}, \quad G_3^+ = 2\Re_m c \quad \text{and} \quad G_3^- = 0, \tag{3.12}
\]

where \( c \) is an arbitrary real-valued constant, and \( \lambda_1, \lambda_2 \) are defined in theorem 2.1.

**Proof.** The formulae (2.6)–(2.10), in which \( G_3^+ \) and \( G_3^- \) are interchanged and \( G_1, G_2 \) and \( G_3^\pm \) are defined in this proposition, represent \( u, \varphi, \omega \) and \( h^+ \) that satisfy equations (2.5) in \( D^+ \) and represent \( h^- \) that satisfy equations (2.3) in \( D^- \). In this case, equations (2.5) in \( D^+ \) and (2.3) in \( D^- \) will be called conjugate equations for equations (2.5) in \( D^- \) and (2.3) in \( D^+ \). The homogeneous boundary-value problem (3.11) corresponds to the conjugate equations with the boundary conditions \( u = 0 \) and \( h^+ = h^- \) on \( S \), where \( h^- \) vanishes at infinity. To find a solution to this problem, we repeat the proof of proposition 3.1 so that

\[
\mathcal{I}_R + \iint_{D^+} (|\omega|^2 + M^2(u_r - h_r^+)^2) \, dV = 0, \tag{3.13}
\]

where

\[
\mathcal{I}_R = -\frac{M^2}{\Re_m} \int_{S_R} n \cdot \left( \frac{1}{2} |h^-|^2 - h^- (k \cdot h^-) \right) \, dS.
\]

As in the proof of proposition 3.1, it can be shown that \( |\mathcal{I}_R| \rightarrow as R \rightarrow \infty \), and passing \( R \) to infinity in equation (3.13), we have \( \omega = 0 \) and \( u_r = h_r^+ \) in \( D^+ \). Then, the representations (2.6)–(2.10) with interchanged \( G_3^+ \) and \( G_3^- \) imply that \( \Im G_1 = 0 \) and \( \Im G_2 = 0 \) in \( D^+ \), and it follows from system (1.3) that \( \Re G_j = c_i e^{-\lambda_i z}, \; i = 1, 2 \), where \( c_i \) are real-valued constants. The second equation in the problem (3.11) implies \( G_3^+ = (c_1 - c_2)/(\lambda_2 - \lambda_1) \) on \( \ell \), and by the Cauchy
integral formula (1.5) for r-analytic functions, we obtain $G^+_3 = (c_1 - c_2)/(\lambda_2 - \lambda_1)$ in $D^-$. In this case, $G^+_3$ vanishes at infinity only if $c_1 = c_2$, and consequently, $G^+_3 = 0$ in $D^-$. Similarly, the first equation in (3.11) implies $G^+_3 = 2Re_m c$ on $\ell$, where $c = c_1 = c_2$, and by the generalized Cauchy integral formula (1.5), we have $G^+_3 = 2Re_m c$ in $D^+$.

In general, as in the classic theory of boundary-value problems for ordinary analytic functions (Muskheilishvili 1992), the number of unknown generalized analytic functions should be equal to the number of real-valued boundary conditions, and the problem (3.1) can be reduced to integral equations for the boundary values of $G_1$ and $G_2$ based on the generalized Cauchy integral formula.

**Theorem 3.3 (boundary integral equations, case (a)).** Let $M \neq 0$, $Re_m \neq 0$ and $Re_m Re \neq M^2$. The boundary-value problem (3.1) yields two integral equations for the boundary values of $F_k(\zeta) = e^{\lambda_k z} G_k(\zeta)$, $k = 1, 2$,

$$
\frac{1}{2\pi i} \oint_{\ell \cup \ell'} (W_r(\zeta, \tau) - e^{\lambda_k(z-z_1)}W_{r'}(\zeta, \tau, \lambda_k)) F_k(\tau) d\tau
+ \frac{1 - 2x\lambda_k}{2x(\lambda_1 - \lambda_2)} (F_1(\zeta) - F_2(\zeta)) = -\frac{1}{2x}, \quad \zeta \in \ell, \quad k = 1, 2, \quad (3.14)
$$

where $\lambda_1, \lambda_2$ are defined in theorem 2.1. A solution to equations (3.14) is determined up to a real-valued constant $c$, i.e. $F_k(\zeta) = c$, $k = 1, 2$, is a homogeneous solution to equations (3.14). Let $\hat{F}_k(\zeta)$, $k = 1, 2$, solve equations (3.14), then $G_1(\zeta)$ and $G_2(\zeta)$ on $\ell$ are determined by

$$
G_k(\zeta) = (\hat{F}_k(\zeta) - c)e^{-\lambda_k z}, \quad k = 1, 2, \quad \zeta \in \ell, \quad (3.15)
$$

where

$$
c = \frac{1}{2} \hat{F}_k(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \hat{F}_k(\tau)e^{\lambda_k(z-z_1)}W_{r'}(\zeta, \tau, \lambda_k) d\tau, \quad \zeta \in \ell. \quad (3.16)
$$

**Proof.** The necessary and sufficient conditions for $G_1(\zeta), G_2(\zeta)$ and $G^+_3(\zeta), \zeta \in \ell$, to be boundary values for the corresponding generalized analytic functions are given by the generalized Sokhotski–Plemelj formulae (1.6)

$$
\frac{1}{2} G_k(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} G_k(\tau)W_{r'}(\zeta, \tau, \lambda_k) d\tau = 0, \quad k = 1, 2, \quad \zeta \in \ell, \quad (3.17a)
$$

and

$$
\frac{1}{2} G^+_3(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} G^+_3(\tau)W_r(\zeta, \tau) d\tau = 0, \quad \zeta \in \ell. \quad (3.17b)
$$

Expressing $G^+_3(\zeta)$ and $G^-_3(\zeta)$ on $\ell$ from the boundary-value problem (3.1) in terms of $G_1(\zeta)$ and $G_2(\zeta)$, then substituting them into corresponding equations in (3.17b) and solving the latter for the integral terms, we have

$$
\frac{1 - 2x\lambda_k}{2x(\lambda_1 - \lambda_2)} (F_1(\zeta) - F_2(\zeta)) + \frac{1}{2} F_k(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} F_k(\tau)W_r(\zeta, \tau) d\tau = -\frac{1}{2x},
$$

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for \( k = 1, 2 \) and \( \zeta \in \ell \). Then, equation (3.17a) is multiplied by \( e^{\lambda k z} \) and is recast in terms of \( F_k(\zeta) \),

\[
\frac{1}{2} F_k(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} F_k(\tau) e^{\lambda k (z - \tau)} W_H(\zeta, \tau, \lambda_k) \, d\tau = 0, \quad k = 1, 2, \quad \zeta \in \ell.
\]

Subtracting this equation from the previous one for corresponding \( k \), we obtain equations (3.14).

Now we will show that the solution to the boundary-value problem (3.1) is determined by the formulae (3.15) and (3.16) and that a homogenous solution to equations (3.14) is \( F_k(\zeta) = c, \ k = 1, 2 \). It is known that if a boundary-value problem for ordinary analytic functions is reduced to boundary integral equations, then a homogeneous solution to those equations is a solution to the homogeneous conjugate boundary-value problem (Muskhelishvili 1992). Thus, the approach is to reduce equations (3.14) to the homogeneous conjugate boundary-value problem (3.11), whose solution is given by the formulae (3.12). This explains the role of the problem (3.11).

Let \( \hat{F}_k(\zeta), \ k = 1, 2 \), and solve equations (3.14). Then, let \( \Theta^+_k(\zeta), \ k = 1, 2 \), be \( H \)-analytic functions in \( D^+ \) determined by the generalized Cauchy-type integral

\[
\Theta^+_k(\zeta) = \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \hat{F}_k(\tau) e^{-\lambda_k \tau} W_H(\zeta, \tau, \lambda_k) \, d\tau, \quad k = 1, 2, \quad \zeta \in D^+, \tag{3.18}
\]

and let \( \Phi^+(\zeta) \) be \( r \)-analytic functions in \( D^\pm \), respectively, also determined by the generalized Cauchy-type integrals

\[
\Phi^+(\zeta) = \frac{1}{2\pi i} \oint_{\ell \cup \ell'} ((1 - 2\lambda_1 x) \hat{F}_2(\tau) - (1 - 2\lambda_2 x) \hat{F}_1(\tau) + \lambda_2 - \lambda_1) W_r(\zeta, \tau) \, d\tau
\]

and

\[
\Phi^-(\zeta) = \frac{1}{2\pi i} \oint_{\ell \cup \ell'} (\hat{F}_2(\tau) - \hat{F}_1(\tau)) W_r(\zeta, \tau) \, d\tau
\]

where \( \zeta \in D^\pm \) for \( \Phi^\pm \), respectively. With the introduced functions \( \Theta^+_k, \Theta^+_1 \) and \( \Phi^\pm \) and the corresponding generalized Sokhotski–Plemelj formulae (1.6) for these functions, the difference of equation (3.14) for \( k = 1 \) and \( k = 2 \) takes the form

\[
e^{\lambda z} \Theta^+_2(\zeta) - e^{\lambda_1 z} \Theta^+_1(\zeta) = \Phi^-(\zeta), \quad \zeta \in \ell.
\]

Similarly, the linear combination \((1 - 2\lambda_2 x) \cdot (3.14)|_{k=1} - (1 - 2\lambda_1 x) \cdot (3.14)|_{k=2}\) reduces to

\[
(1 - 2\lambda_1 x) e^{\lambda_2 z} \Theta^+_2(\zeta) - (1 - 2\lambda_2 x) e^{\lambda_1 z} \Theta^+_1(\zeta) = \Phi^+(\zeta), \quad \zeta \in \ell.
\]

The last two equations form a system equivalent to the homogeneous conjugate boundary-value problem (3.11) with \( \Theta^+_k = G_k, \ k = 1, 2, \ \Phi^+ = -(\lambda_1 - \lambda_2)/ (Re_m Re - M^2) G^+_3 \) and \( \Phi^- = (\lambda_1 - \lambda_2) G^-_3 \). Consequently, by proposition 3.2, the
only solution this system has is \( \Theta_k^+ = c e^{-\lambda_k z}, \quad k = 1, 2, \quad \Phi^+ = 2(\lambda_2 - \lambda_1) \chi c \) and \( \Phi^- = 0 \), where \( c \) is a real-valued constant. In this case, the representation (3.18) can be rearranged in the form

\[
\frac{1}{2\pi i} \oint_{\ell \cup \ell'} (\hat{F}_k(\tau) - c) e^{-\lambda_k z} W_H(\zeta, \tau, \lambda_k) \, d\tau = 0, \quad k = 1, 2, \quad \zeta \in D^+.
\]

For \( \zeta \) approaching \( \ell \) within \( D^+ \), the above equation reduces to the generalized Sokhotski–Plemelj formula

\[
\frac{1}{2}(\hat{F}_k(\zeta) - c) e^{-\lambda_k z} + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} (\hat{F}_k(\tau) - c) e^{-\lambda_k z} W_H(\zeta, \tau, \lambda_k) \, d\tau = 0, \quad (3.19)
\]

for \( k = 1, 2 \) and \( \zeta \in \ell \), which is the necessary and sufficient condition for \( (\hat{F}_k(\zeta) - c) e^{-\lambda_k z}, \quad \zeta \in \ell \), to be the boundary value of an \( H \)-analytic function in \( D^- \) that vanishes at infinity. Thus, the solution (3.15) and (3.16) follows from equation (3.19).

Finally, let \( \tilde{F}_k(\zeta), \quad k = 1, 2, \quad \zeta \in \ell \) be another solution to equations (3.14). Similarly, we can show that \( G_k(\zeta) = (\hat{F}_k(\zeta) - \tilde{c}) e^{-\lambda_k z}, \quad k = 1, 2, \quad \zeta \in \ell \), solve the boundary-value problem (3.1), where \( \tilde{c} \) is a real-valued constant. However, by proposition 3.1, the boundary-value problem (3.1) has a unique solution, and consequently, \( \hat{F}_k(\zeta) - \tilde{F}_k(\zeta) = c - \tilde{c}, \quad k = 1, 2, \quad \zeta \in \ell \), which means that a solution to the integral equations (3.14) is determined up to a real-valued constant, and the proof is finished.

Several remarks are in order.

**Remark 3.4 (logarithmic singularity).** The kernels in equations (3.14) have logarithmic singularity.

**Remark 3.5 (alternative form).** For efficient solving, the integral equation (3.14) needs to be rewritten in the form similar to eqns (67) and (68) in Zabarankin (2010).

**Remark 3.6 (conic endpoint).** If \( \ell \) is piece-wise smooth (has salient points), in particular has conic endpoints, the integral equation (3.14) holds for all \( \zeta \in \ell \), except for salient points and conic endpoints, i.e. almost everywhere.

**Remark 3.7 (multiply connected body).** The integral equation (3.14) holds for multiply connected regions, e.g. in the MHD problem for a torus.

The boundary integral equations (3.14) can be solved by the quadratic error minimization method. Let \( \ell \) be parametrized by \( \zeta = \zeta(t) \in C^1[-1, 1] \), and let \( F_j = F_j(t), \quad t \in [-1, 1], \quad j = 1, 2, \) be unknown boundary values. For brevity, equations (3.14) are recast as

\[
A_j(F_1, F_2) = f_j, \quad t \in [-1, 1], \quad j = 1, 2,
\]

where \( A_j, j = 1, 2, \) are corresponding integral operators, and \( f_1(t) = f_2(t) = -1/(2\chi) \). The functions \( F_j, j = 1, 2, \) can be approximated by finite function series

\[
F_1(t) = \sum_{k=1}^{n} (a_k + ib_k) T_{k-1}(t) \quad \text{and} \quad F_2(t) = \sum_{k=1}^{n} c_k T_k(t) + id_k T_{k-1}(t), \quad (3.20)
\]

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where \( T_k(t) \) is the Chebyshev polynomial of the first kind, and \( a_k, b_k, c_k \) and \( d_k \), \( k = 1, \ldots, n \), are real-valued coefficients. Observe that the real part in the series approximating \( F_2 \) contains no constant term, whereas the series for \( F_1 \) does. This is because the same real-valued constant in place of \( F_1 \) and \( F_2 \) is a homogeneous solution to equations (3.14).

Unknown coefficients \( a_k, b_k, c_k \) and \( d_k \), \( k = 1, \ldots, n \), can be found by minimizing the quadratic error in satisfying equations (3.14), i.e.

\[
\min_{a_k, b_k, c_k, d_k} \sum_{j=1}^{2} \| A_j(F_1, F_2) - f_j \|^2, \tag{3.21}
\]

with the inner product and norm for complex-valued functions \( f(t) \) and \( g(t) \) introduced by

\[
\langle f, g \rangle = \text{Re} \left[ \int_{-1}^{1} f(t) \overline{g(t)} \, dt \right], \quad \| f \| = \sqrt{\langle f, f \rangle}.
\]

As \( A_j, j = 1, 2 \), are linear operators, the problem (3.21) is unconstrained quadratic optimization and reduces to a system of linear algebraic equations for \( a_k, b_k, c_k \) and \( d_k \), \( k = 1, \ldots, n \).

(b) Case (b): \( M \neq 0 \) and \( \text{Re}_m = 0 \)

For \( M \neq 0 \) and \( \text{Re}_m = 0 \), the MHD problem (2.2)–(2.5) reduces to the boundary-value problem for the \( H \)-analytic functions \( G_1 \) and \( G_2 \) defined in corollary 2.2,

\[
e^{\lambda_1 z} G_1 - e^{\lambda_2 z} G_2 = \lambda_2 - \lambda_1, \quad z \in \ell. \tag{3.22}
\]

**Proposition 3.8.** The boundary-value problem (3.22) has a unique solution.

**Proof.** The proof is similar to the proof of proposition 3.1. In this case, corollary 2.2 shows that the disturbance of the magnetic field in the fluid and body is zero, i.e. \( h^\pm = 0 \). With \( h^- = 0 \), equation (3.8) and the surface integral (3.10) reduce to

\[
\mathcal{I}_R + \iint_{D_R} (|\omega|^2 + M^2 u_r^2) \, dV = 0 \tag{3.23}
\]

and

\[
\mathcal{I}_R = 2\pi \text{Re} \left[ \int_{C_R} \left( (u_z + i u_r) \left( \varphi + i \omega + \frac{\text{Re}}{2} (u_z - i u_r) \right) \right) r \, d\zeta \right],
\]

respectively. Using the representation (2.21), we obtain

\[
\mathcal{I}_R = \frac{2\pi}{\lambda_1 - \lambda_2} \text{Re} \left[ \int_{C_R} \left( e^{2\lambda_1 z} G_1^2 - e^{2\lambda_2 z} G_2^2 \right) r \, d\zeta \right],
\]

and the rest of the proof is analogous to the proof of proposition 3.1. \( \blacksquare \)

\(^6f(t)\) and \( g(t) \) are viewed as two-dimensional vector functions from the direct sum \( L^2([-1, 1]) \oplus L^2([-1, 1]) \).

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Proposition 3.9 (homogeneous conjugate boundary-value problem, case (b)). Let $G_1$ and $G_2$ be $H$-analytic functions in $D^+$. Under the assumption $M \neq 0$, the homogeneous conjugate boundary-value problem

$$e^{\lambda_1 z} G_1 - e^{\lambda_2 z} G_2 = 0, \quad \zeta \in \ell,$$

has the solution

$$G_1 = c e^{-\lambda_1 z} \quad \text{and} \quad G_2 = c e^{-\lambda_2 z},$$

where $c$ is an arbitrary real-valued constant, and $\lambda_1, \lambda_2$ are defined in corollary 2.2.

Proof. Proved similar to proposition 3.2. ■

With the generalized Cauchy integral formula for $H$-analytic functions, the boundary-value problem (3.22) readily reduces to a boundary integral equation.

Theorem 3.10 (boundary integral equation, case (b)). Let $M \neq 0$ and $\text{Re} m = 0$ and let $F_1(\zeta) = e^{\lambda_1 z} G_1(\zeta)$. The boundary-value problem (3.22) reduces to the integral equation for the boundary value of $F_1$,

$$\frac{1}{2\pi i} \oint_{\ell \cup \ell'} (e^{\lambda_2(z-z_1)} W_H(\zeta, \tau, \lambda_2) - e^{\lambda_1(z-z_1)} W_H(\zeta, \tau, \lambda_1)) F_1(\tau) d\tau = \lambda_2 - \lambda_1,$$

for $\zeta \in \ell$. A solution to equation (3.26) is determined up to a real-valued constant $c$. Let $\hat{F}_1(\zeta)$ solve equation (3.26), then $G_1(\zeta)$ in the problem (3.22) is given by

$$G_1(\zeta) = (\hat{F}_1(\zeta) - c)e^{-\lambda_1 z}, \quad \zeta \in \ell,$$

where

$$c = \frac{1}{2} \hat{F}_1(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \hat{F}_1(\tau)e^{\lambda_1(z-z_1)} W_H(\zeta, \tau, \lambda_1) d\tau, \quad \zeta \in \ell.$$

Proof. The integral equation (3.26) is derived similar to equations (3.14). In the second part of the proof that determines a homogeneous solution to equation (3.26), $H$-analytic functions $\Theta^+_k, k = 1, 2$, in $D^+$ are introduced by the generalized Cauchy-type integrals

$$\Theta^+_1(\zeta) = \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \hat{F}_1(\tau)e^{-\lambda_1 z_1} W_H(\zeta, \tau, \lambda_1) d\tau, \quad \zeta \in D^+,$$

and

$$\Theta^+_2(\zeta) = \frac{1}{2\pi i} \oint_{\ell \cup \ell'} (\hat{F}_1(\tau) + \lambda_1 - \lambda_2)e^{-\lambda_2 z_1} W_H(\zeta, \tau, \lambda_2) d\tau, \quad \zeta \in D^+,$$

where $\hat{F}_1$ is a solution to equation (3.26). With these functions and corresponding generalized Sokhotski–Plemelj formulae (1.6), the integral equation (3.26) reduces to the homogenous conjugate boundary-value problem (3.24), whose solution is given by the formulae (3.25). The rest of the proof is analogous to that of theorem 3.3. ■

The integral equation (3.26) can be solved by the quadratic error minimization method outlined at the end of §3a. Remarks 3.4–3.7 also hold for equation (3.26).
(c) Case (c): $M \neq 0$ and $Re_m Re = M^2$

For $M \neq 0$ and $Re_m Re = M^2$, the MHD problem (2.2)–(2.5) reduces to the boundary-value problem for the four generalized analytic functions $G_1, G_2$ and $G_3^\pm$ defined in theorem 2.5,

$$2e^{i\zeta} G_1 - G_2 + (Re + Re_m)(G_3^+ + 1) = 0, \quad \zeta \in \ell,$$

and

$$Re_m \left( \left( z - \frac{i}{2} r \right) G_2 + G_3^- + 1 \right) - Re G_3^+ = 0, \quad \zeta \in \ell.$$

(3.29)

**Proposition 3.11.** The boundary-value problem (3.29) has a unique solution.

**Proof.** The assumptions $M \neq 0$ and $Re_m Re = M^2$ imply $Re_m \neq 0$, and consequently, as in the proof of proposition 3.1, we obtain equation (3.8) with the surface integral (3.10). With the representations (2.22)–(2.26), the integral (3.10) reduces to

$$I_R = 2\pi Re \left[ \int_{C_R} \left( (u_z + i u_r) G_2 + \frac{Re}{2(Re + Re_m)} (2e^{i\zeta} G_1 - G_2)^2 \right) r \, d\zeta \right].$$

Since in this case, $G_1$ and $G_2$ are $H$-analytic and $r$-analytic functions, respectively, we have $G_1(R, \varphi) = f(\varphi) R^{-1} e^{-|\varphi|R} + O(R^{-2} e^{-|\varphi|R})$ as $R \to \infty$, where $f(\varphi)$, $\varphi \in [0, \pi]$, is a bounded complex-valued function, and $G_2 = O(R^{-2})$ as $R \to \infty$. The rest of the proof is completely analogous to the proof of proposition 3.1. ■

**Proposition 3.12** (homogeneous conjugate boundary-value problem, case (c)). Let $G_1$ and $G_2$ be $H$-analytic and $r$-analytic functions in $D^+$, respectively, and let $G_3^\pm$ be $r$-analytic functions in $D^\pm$, respectively, with $G_3^-$ vanishing at infinity. Under the assumptions $M \neq 0$ and $Re_m Re = M^2$, the homogeneous conjugate boundary-value problem

$$2e^{i\zeta} G_1 - G_2 + (Re + Re_m) G_3^- = 0, \quad \zeta \in \ell,$$

and

$$Re_m \left( \left( z - \frac{i}{2} r \right) G_2 + G_3^+ \right) - Re G_3^- = 0, \quad \zeta \in \ell.$$

(3.30)

has the solution

$$G_1 = c e^{-\lambda z}, \quad G_2 = 2c, \quad G_3^+ = -c(2z - ir) \quad \text{and} \quad G_3^- = 0,$$

(3.31)

where $\lambda = (Re + Re_m)/2$, and $c$ is an arbitrary real-valued constant.

**Proof.** Proved similarly to proposition 3.2. ■
Theorem 3.13 (boundary integral equations, case (c)). Let $M \neq 0$ and $Re_{in} Re = M^2$, and let $\lambda = (Re + Re_{in})/2$, then the boundary-value problem (3.29) reduces to two integral equations for $F_1(\zeta) = e^{i\zeta} G_1(\zeta)$ and $F_2(\zeta) = G_2(\zeta), \zeta \in \ell$,

$$\frac{1}{2\pi i} \oint_{\ell \cup \ell'} (e^{i(z-z_1)} W_H(\zeta, \tau, \lambda) - W_r(\zeta, \tau)) F_1(\tau) d\tau + F_1(\zeta) - \frac{1}{2} F_2(\zeta) = 0$$

and

$$\frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left( z - z_1 + \frac{i}{2}(r_1 - r) \right) W_r(\zeta, \tau) F_2(\tau) d\tau - \frac{Re(2F_1(\zeta) - F_2(\zeta))}{Re_{in}(Re + Re_{in})} = \frac{Re + Re_{in}}{Re_{in}}.$$

Equations (3.32) determine $F_1$ and $F_2$ up to $c$ and $2c$, respectively, where $c$ is a real-valued constant. Let $\hat{F}_k, k = 1, 2$, solve equations (3.32), then the solution to the boundary-value problem (3.29) is given by

$$G_1(\zeta) = (\hat{F}_1(\zeta) - c)e^{-2z} \quad \text{and} \quad G_2(\zeta) = \hat{F}_2(\zeta) - 2c, \quad \zeta \in \ell,$$

where

$$2c = \frac{1}{2} \hat{F}_2(\zeta) + \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \hat{F}_2(\tau) W_r(\zeta, \tau) d\tau, \quad \zeta \in \ell.$$

Proof. The derivation of equations (3.32) is similar to that of equations (3.14) (see the proof of theorem 3.3). In the second part of the proof that determines a homogeneous solution to equations (3.32) and also follows closely the proof of theorem 3.3, an $H$-analytic function $\Theta^+$ in $D^+$, $r$-analytic functions $\Phi_1^+$ and $\Phi_2^+$ in $D^+$, and an $r$-analytic function $\Phi_2^-$ in $D^-$ are introduced by the generalized Cauchy-type integrals

$$\Theta^+(\zeta) = \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \hat{F}_1(\tau) e^{-2z} W_H(\zeta, \tau, \lambda) d\tau, \quad \zeta \in D^+,$$

$$\Phi_1^+(\zeta) = \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \hat{F}_2(\tau) W_r(\zeta, \tau) d\tau, \quad \zeta \in D^+,$$

$$\Phi_2^+(\zeta) = \frac{1}{2\pi i} \oint_{\ell \cup \ell'} \left( \left( z_1 - \frac{i}{2} r_1 \right) \hat{F}_2(\tau) + \frac{Re(2\hat{F}_1(\tau) - \hat{F}_2(\tau))}{Re_{in}(Re + Re_{in})} \right)$$

$$\times W_r(\zeta, \tau) d\tau, \quad \zeta \in D^+,$$

and

$$\Phi_2^-(\zeta) = \frac{1}{2\pi i} \oint_{\ell \cup \ell'} (2\hat{F}_1(\tau) - \hat{F}_2(\tau)) W_r(\zeta, \tau) d\tau, \quad \zeta \in D^-,$$

where $\hat{F}_k, k = 1, 2$, are a solution to equations (3.32). With these functions and corresponding generalized Sokhotski–Plemelj formulae (1.6), the integral equations (3.32) reduce to the homogeneous conjugate boundary-value problem (3.30), whose solution is given by the formulae (3.31). The rest of the proof continues as in the proof of theorem 3.3.

The integral equations (3.32) can be solved by the quadratic error minimization method outlined at the end of §3a. Remarks 3.4–3.7 hold for equations (3.32) as well.

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The drag (force) exerted on the body by the electrically conducting viscous incompressible fluid in the presence of the magnetic field has the mechanic and magnetic components corresponding to the viscous stress and Lorentz force, respectively,

$$F = \int_S P_n \, dS - \mu \oint_{\Omega^-} [J \times H] \, dV,$$

where $P_n = \rho v (2 \partial U/\partial n + [n \times \text{curl} \, U]) - \varphi \, n$, and $n$ is the outward normal to $S$ (Cabannes 1970).

In the axially symmetric case, $\text{div} \, U = 0$ and the boundary conditions (2.2) imply $\partial U/\partial n = -[n \times \text{curl} \, U]$ on $S$; see e.g. the proof of proposition 8 in Zabarankin (2008). Also, the next proposition shows that the second term in $F$ vanishes.

**Proposition 3.14.** If at infinity, the flow and the magnetic field are uniform and parallel to the body’s axis of revolution, and if also the fluid and body have the same magnetic permeability, the Lorentz force has no direct contribution to the drag.

**Proof.** In the axially symmetric case, the resulting mechanic and Lorentz forces are parallel to the body’s axis of revolution, and consequently, it is sufficient to show that the projection of the resulting Lorentz force onto $k$ vanishes. Let $\mathbb{D}_R$ be the region bounded by the body’s surface $S$ and a sphere $S_R$ with large radius $R$ and centre at the origin. Using the identity (3.4) and the divergence theorem, we obtain

$$\int_{\mathbb{D}_R} k \cdot [J \times H] \, dV = -H_\infty^2 \int_{\mathbb{D}_R} [k \times h^-] \cdot \text{curl} \, h^- \, dV = -H_\infty^2 (I_R - I),$$

where

$$I = \int_S n \cdot \left( (k \cdot h^-) h^- - \frac{1}{2} |h^-|^2 k \right) \, dS,$$

and $I_R$ is defined as the integral $I$ in which $S$ is replaced by $S_R$. As in the proof of proposition 3.1, we can show that in all three cases (i)–(iii), $I_R \to 0$ as $R \to \infty$. On the other hand, as $h^+ = h^-$ on $S$ and $\text{curl} \, h^+ = 0$ in $\mathbb{D}^+$, using the identity similar to (3.4) for $h^+$, we have

$$I = \int_S n \cdot \left( (k \cdot h^+) h^+ - \frac{1}{2} |h^+|^2 k \right) \, dS = -\int_{\mathbb{D}^+} [k \times h^+] \cdot \text{curl} \, h^+ \, dV = 0.$$

Consequently, $\int_{\mathbb{D}_R} k \cdot [J \times H] \, dV \to 0$ as $R \to \infty$. 

Thus, in terms of dimensionless $u$ and $\varphi$, the only non-zero drag component $F_z = k \cdot F$ takes the form

$$F_z = -V_\infty \rho v a \int_S k \cdot ([n \times \text{curl} \, u] + \varphi n) \, dS,$$

which, as in the proof of proposition 11 in Zabarankin (2008), reduces to

$$F_z = -2\pi V_\infty \rho v a \text{Re} \left[ \int_\ell r(\varphi + i\omega) \, d\zeta \right]. \quad (3.35)$$
Let $C_D = -F_z/(6\pi V_\infty \rho a)$ be a dimensionless drag coefficient, where $6\pi V_\infty \rho a$ is the drag of the sphere with radius $a$ in the Stokes flow of a non-conducting fluid. Theorem 2.1, corollary 2.2, theorem 2.5 and the corresponding boundary-value problems (3.1), (3.22) and (3.29) imply that

$$\wp + i\omega = \begin{cases} 
\frac{Re - 2\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_2 z} G_1 - \frac{Re - 2\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_1 z} G_2 + Re, & \zeta \in \ell, \text{ in case (a)} , \\
2(e^{\lambda_1 z} G_1 + \lambda_1), & \zeta \in \ell, \text{ in case (b)} , \\
\frac{1}{Re + Re_m} (2Re e^{\lambda_2 z} G_1 + Re_m G_2) + Re, & \zeta \in \ell, \text{ in case (c)} .
\end{cases}$$

Now with this representation for $\wp + i\omega$ and the fact that $\text{Re}[\int _\ell r d\zeta] = 0$, the formula (3.35) yields the following result.

\textbf{Proposition 3.15 (drag).} Let $M \neq 0$, then the drag coefficient $C_D$ is determined as follows.

(i) If $Re_m \neq 0$ and $Re_m Re \neq M^2$, then

$$C_D = \text{Re} \left[ \int _\ell r \left( \frac{Re - 2\lambda_2}{3(\lambda_1 - \lambda_2)} e^{\lambda_2 z} G_1 - \frac{Re - 2\lambda_1}{3(\lambda_1 - \lambda_2)} e^{\lambda_1 z} G_2 \right) d\zeta \right], \quad (3.36)$$

where $G_1$, $G_2$, $\lambda_1$ and $\lambda_2$ are defined in theorem 2.1.

(ii) If $Re_m = 0$, then

$$C_D = \frac{2}{3} \text{Re} \left[ \int _\ell r e^{\lambda_1 z} G_1 d\zeta \right], \quad (3.37)$$

where $G_1$ and $\lambda_1$ are defined in corollary 2.2.

(iii) If $Re_m Re = M^2$, then

$$C_D = \frac{1}{3(Re + Re_m)} \text{Re} \left[ \int _\ell r(2Re e^{\lambda_2 z} G_1 + Re_m G_2) d\zeta \right], \quad (3.38)$$

where $G_1$, $G_2$ and $\lambda$ are defined in theorem 2.5.

4. Axially symmetric magnetohydrodynamic problems

\textit{(a) Sphere}

The purpose of solving the MHD problem for a sphere is threefold: (i) verifying the presented approach of generalized analytic functions based on the existing results for a sphere, (ii) testing accuracy of solutions to the derived boundary integral equations, and (iii) obtaining drag for various $Re$, $Re_m$ and $M$ to compare it with the drag of minimum-drag shapes in Zabarankin (2011).

Let $(R, \theta, \phi)$ be the spherical coordinates related to the cylindrical coordinates in the ordinary way, and let $S_a$ be a sphere with radius $a$ and centre at the origin.
(i) Case (a): $M \neq 0$, $Re_m \neq 0$ and $Re_m \Re \neq M^2$ ($S \neq 1$)

In the region exterior to $S_a$, the functions $G_1$ and $G_2$ in theorem 2.1 can be represented by a series similar to the series (1.8),

$$G_k(R, \vartheta) = \sqrt{\frac{\alpha}{R}} \sum_{n=1}^{\infty} A_{k,n} N_n(\cos \vartheta, R, \lambda_k), \quad k = 1, 2, \quad (4.1)$$

where $A_{k,n}$ are unknown real-valued coefficients and

$$N_n(t, R, \lambda) = L_n(t) \frac{K_{n+\frac{1}{2}}(|\lambda|R)}{K_{n+\frac{1}{2}}(|\lambda|a)} - (\text{sign } \lambda) L_{-n}(t) \frac{K_{n-\frac{1}{2}}(|\lambda|R)}{K_{n+\frac{1}{2}}(|\lambda|a)},$$

whereas $G_3^\pm$ can be represented by (see example 5 in Zabarankin (2010))

$$G_3^\pm(R, \vartheta) = \sum_{n=1}^{\infty} B_n^\pm(R/a)^{\pm n-1} L_{\mp n}(\cos \vartheta), \quad (4.2)$$

where $B_n^\pm$ are unknown real-valued coefficients.

Let the inner product $\langle \cdot, \cdot \rangle$ be defined as at the end of §3a. With the orthogonality property

$$\langle L_n(t), L_m(t) \rangle = 2m \delta_{mn}, \quad (4.3)$$

for all integer $n$ and $m$, where $\delta_{mn}$ is the Kronecker delta, and the representations (4.1) and (4.2), the inner product of the first equation in (3.1) with $L_{-m-1}(t)$ for $m \geq 0$ and the inner product of the second equation in (3.1) with $L_m(t)$ for $m \geq 1$ yield a real-valued infinite linear system for $A_{1,n}$ and $A_{2,n}$,

$$\sum_{n=1}^{\infty} \left( \frac{1 - 2\lambda_2 \xi_{mn}(\lambda_1) A_{1,n}}{\lambda_1 - \lambda_2} - \frac{1 - 2\lambda_1 \xi_{mn}(\lambda_2) A_{2,n}}{\lambda_1 - \lambda_2} \right) = 2\delta_{m0}, \quad m \geq 0, \quad \left\{ \begin{array}{l} \\
\end{array} \right. \quad (4.4)$$

and

$$\sum_{n=1}^{\infty} (\eta_{mn}(\lambda_1) A_{1,n} - \eta_{mn}(\lambda_2) A_{2,n}) = 0, \quad m \geq 1,$$

where $\xi_{mn}(\lambda) = \langle e^{\lambda a t} N_n(t, a, \lambda), L_{-m-1}(t) \rangle$ and $\eta_{mn}(\lambda) = \langle e^{\lambda a t} N_n(t, a, \lambda), L_m(t) \rangle$. System (4.4) can be truncated at some large $m$ and solved numerically.

With the second equation in (4.4), the drag coefficient (3.36) reduces to

$$C_D = \frac{2}{3} \sum_{n=1}^{\infty} A_{1,n} \eta_{1n}(\lambda_1).$$
Figure 1 shows $C_D$ as a function of $S$ in five cases: (i) $Re = Re_m = 1$; (ii) $Re = 1$, $Re_m = 3$; (iii) $Re = Re_m = 2$; (iv) $Re = 3$, $Re_m = 1$; and (v) $Re = Re_m = 3$.

(ii) Case (b): $M \neq 0$ and $Re_m = 0$

As in the previous case, the functions $G_1$ and $G_2$ in corollary 2.2 can be represented by the series (4.1). With the orthogonality property (4.3) and representations (4.1), the inner products of (3.22) with $e^{-\lambda_2 a t} L_m(t)$ for $m \geq 1$ and with $e^{-\lambda_1 a t} L_m(t)$ for $m \geq 1$ yield a real-valued infinite linear system for $A_{1,n}$ and $A_{2,n}$,

$$
\sum_{n=1}^{\infty} \tilde{\xi}_{mn} A_{1,n} - 2m A_{2,m} = b_{2,m} \quad \text{and} \quad 2m A_{1,m} - \sum_{n=1}^{\infty} \tilde{\eta}_{mn} A_{2,n} = b_{1,m}, \quad m \geq 1,
$$

(4.5)

where $\tilde{\xi}_{mn} = (e^{\lambda_1 a t} N_n(t, a, \lambda_1), e^{-\lambda_2 a t} L_m(t))$, $\tilde{\eta}_{mn} = (e^{\lambda_2 a t} N_n(t, a, \lambda_2), e^{-\lambda_1 a t} L_m(t))$ and $b_{k,m} = (\lambda_2 - \lambda_1, e^{-\lambda_1 a t} L_m(t))$, $k = 1, 2$. System (4.5) can be truncated at some large $m$ and solved numerically.

The drag coefficient (3.37) takes the form

$$
C_D = \frac{2}{3} \sum_{n=1}^{\infty} A_{1,n} (e^{\lambda_1 a t} N_n(t, a, \lambda_1), L_1(t)),
$$

which is in good agreement with Chester (1957): $C_D = 1 + 3M/8 + 7M^2/960 - 43M^3/7680 + O(M^4)$ for small $M$ up to 0.25.

(iii) Case (c): $M \neq 0$ and $Re_m Re = M^2$ ($S = 1$)

In this case, the $H$-analytic function $G_1$ and the $r$-analytic function $G_2$ in theorem 2.5 are represented by the series

$$
G_1(R, \vartheta) = \sqrt{\frac{a}{R}} \sum_{n=1}^{\infty} A_{1,n} N_n(\cos \vartheta, R, \lambda) \quad \text{and} \quad G_2(R, \vartheta) = \sum_{n=1}^{\infty} A_{2,n} \left( \frac{a}{R} \right)^{n+1} L_n(\cos \vartheta),
$$

(4.6)

where $\lambda = (Re + Re_m)/2$, and $N_n(\cos \vartheta, R, \lambda)$ was introduced in §4a(i), whereas the $r$-analytic functions $G_3^\pm$ are represented by the series (4.2).

With the orthogonality property (4.3) and representations (4.6) and (4.2), the inner products of the first equation in (3.29) with $L_m(t)$ for $m \geq 1$ and with $L_{-m-1}(t)$ for $m \geq 0$ yield a real-valued infinite linear system

$$
\sum_{n=1}^{\infty} \eta_{mn}(\lambda) A_{1,n} - m A_{2,m} = 0, \quad m \geq 1,
$$

and

$$
\sum_{n=1}^{\infty} \xi_{mn}(\lambda) A_{1,n} - (Re + Re_m)(m + 1) B_{m+1}^+ = (Re + Re_m) \delta_{m0}, \quad m \geq 0,
$$

(4.7)

where $\eta_{mn}(\lambda)$ and $\xi_{mn}(\lambda)$ are those in §4a(i).
Similarly, with equations (4.6), (4.2) and (4.3), the inner products of the second equation in (3.29) with $L_m(t)$ for $m \geq 1$ and with $L_{-m-1}(t)$ for $m \geq 0$ yield

$$a \left( \frac{m - 1}{2m - 1} A_{2,m-1} + \frac{3(m + 1)}{2m + 1} A_{2,m+1} \right) + 2B^*_m = 0, \ m \geq 1,$$

and

$$Re_m a \left( \frac{m + 1)(m+2)}{(2m+1)(2m+3)} A_{2,m+1} - Re(m+1)B^*_{m+1} = -Re_m \delta_{m0}, \ m \geq 0. \quad (4.8)$$

It follows from system (4.7) and the second equation in (4.8) that

$$\sum_{n=1}^{\infty} \left( \xi_{mn}(\lambda) - \frac{a Re_m(Re + Re_m)(m+2)}{Re(2m+1)(2m+3)} \eta_{(m+1)n}(\lambda) \right) A_{1,n} = \frac{(Re + Re_m)^2}{Re} \delta_{m0},$$

for $m \geq 0$. The latter can be truncated at some large $m$ and solved numerically.

The drag coefficient (3.38) simplifies to the formula

$$C_D = \frac{2}{3} \sum_{n=1}^{\infty} A_{1,n} \eta_{1n}(\lambda),$$

which is used to evaluate $C_D$ at $S=1$ in figure 1 and which agrees with Gotoh (1960b) for various small $Re$ and $Re_m$: $C_D \approx 1 + 3Re/8 - (19Re^2/320 + 2Re Re_m/15) + 1/7680(213 Re^3 + 256 Re^2 Re_m - 704 Re Re_m^2) + 1Re Re_m/30(Re + 4Re_m)$.

(b) Minimum-drag spheroids

This section solves the axially symmetric MHD problem for solid non-magnetic spheroids having the volume of a unit sphere, i.e. $4\pi/3$, and among those finds the spheroids that have the smallest drag for given $Re$, $Re_m$ and $M$. In the right-half $rz$-plane ($r \geq 0$), the cross section $\ell$ of spheroids is parametrized by $r(t) = a \cos(\pi t/2), \ z(t) = a^{-2} \sin(\pi t/2), \ t \in [-1,1]$, with $a \in (0,1]$. Then, for fixed $Re$, $Re_m$ and $M$, the spheroid drag becomes a function of $a$, and the minimum of this function along with the value of $a$ at which the minimum is attained is found by a modified bisection method. This is an iterative procedure that requires the involved boundary integral equations to be solved fast and accurately.

Let $C_D$ and $C_D^*$ be the drag coefficients for the minimum-drag spheroid and unit sphere, respectively, for the same $Re$, $Re_m$ and $S$, and let $\kappa = a^{-3}$ be the axes ratio of the corresponding minimum-drag spheroid. Figures 2 and 3 show the drag ratio $C_D/C_D^*$ and $\kappa$ as functions of $S \in [0,2]$ in three cases: (i) $Re = Re_m = 1$, (ii) $Re = Re_m = 2$, and (iii) $Re = Re_m = 3$. As a function of $S$, $C_D/C_D^*$ has a maximum at $S = 1$ (and is non-smooth at $S = 1$), and consequently, drag reduction is smallest for $S \gg 1$. Also, figure 2 indicates that drag reduction becomes more significant for $S \gg 1$, whereas figure 3 suggests that the shortest minimum-drag spheroids do not correspond to $S = 1$. The minimum-drag spheroids found are used as initial approximations for minimum-drag shapes in Zabarankin (2011).

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Figure 2. The drag ratio $C_D/C_D^*$ for the minimum-drag spheroid as a function of $S$ in three cases: (i) $Re = Re_m = 1$; (ii) $Re = Re_m = 2$; and (iii) $Re = Re_m = 3$.

Figure 3. The axes ratio $\kappa$ of the minimum-drag spheroid as a function of $S$ in three cases: (i) $Re = Re_m = 1$; (ii) $Re = Re_m = 2$; and (iii) $Re = Re_m = 3$.

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