

# On the rigorous foundations of the Fokas method for linear elliptic partial differential equations

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In this paper, we address some of the rigorous foundations of the Fokas method, confining attention to boundary value problems for linear elliptic partial differential equations on bounded convex domains. The central object in the method is the global relation, which is an integral equation in the spectral Fourier space that couples the given boundary data with the unknown boundary values. Using techniques from complex analysis of several variables, we prove that a solution to the global relation provides a solution to the corresponding boundary value problem, and that the solution to the global relation is unique. The result holds for any number of spatial dimensions and for a variety of boundary value problems.

**Keywords:** Fokas; partial differential equations; convex

## 1. Introduction

Fokas (2008) has introduced a new, unified method for boundary value problems in which the analysis takes place entirely in spectral space. The central object of interest in this approach is global relation. The global relation is an integral equation in spectral space that couples all the boundary values. This integral equation has analytic dependence on a spectral parameter ( $\lambda \in \mathbf{C}^n$ ) that allows the use of powerful results from complex analysis. This method differs considerably from the classical approach that involves analysis of Fredholm operators and proves the existence of a solution to the Dirichlet problem through a related boundary integral equation.

Fokas' method has proved extremely powerful in the study of boundary value problems for linear (Fokas & Kapaev 2003; Crowdy & Fokas 2004; Dassios & Fokas 2005; Pelloni 2005; Ashton 2010) and integrable nonlinear (Fokas 2005; Fokas *et al.* 2005; Fokas & Lenells 2010; Pelloni & Pinotsis 2010) partial differential equations (PDEs). The new approach has also proved to be advantageous in non-integrable cases (Ablowitz *et al.* 2006; Ashton & Fokas 2011; Deconinck & Oliveras 2011). In these studies, the analysis is carried out on a formal level: one assumes that the global relation specifies the unknown boundary values, and then one constructs a solution. The legitimacy of this assumption

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can then be checked *a posteriori*. More recently, the philosophy of the Fokas method has been used in the study of rigorous problems (Ashton 2011), providing a new approach to the classical results of Hörmander (1985*a,b*) regarding the hypoellipticity of partial differential operators with constant coefficients. Here, we continue to address rigorous aspects of the new approach. We prove that it is possible to eliminate the *assumption* that the global relation specifies the unknown boundary values. In particular, our result establishes, *a priori*, that a solution to the global relation is unique and it provides the solution to the corresponding Dirichlet problem for linear elliptic PDEs on convex domains.

In this paper, we study the basic elliptic equations

$$-\Delta u + \beta^2 u = 0 \quad \text{in } \Omega, \quad (1.1a)$$

and

$$u = f \quad \text{on } \Gamma, \quad (1.1b)$$

where  $\Omega$  is a bounded, convex domain in  $\mathbf{R}^n$  with boundary,  $\Gamma$ . We will assume for simplicity and ease of presentation that  $\Gamma$  is smooth and  $f \in C(\Gamma)$ . These regularity conditions can be considerably relaxed (e.g.  $\Gamma$  Lipschitz and  $f \in L^p(\Gamma)$  for some  $p \geq 1$ ) and we will discuss such generalizations at the end of the paper. The constant  $\beta$  is assumed to be either: (i) zero, corresponding to Laplace; (ii) real, corresponding to modified-Helmholtz or; (iii) imaginary, corresponding to Helmholtz. In the latter case, we make the additional assumption that  $\beta^2$  is not an eigenvalue, so that the solution to the corresponding Dirichlet problem is unique. The method of proof holds for more general elliptic PDEs with constant coefficients, but to make the presentation clear, we confine attention to these well-known cases. Associated with each of these cases is the symbol of the corresponding differential operator  $P(D) = D^2 + \beta^2$ , where  $D = -i\partial$ , and is given by

$$P(\lambda) = \lambda^2 + \beta^2, \quad (1.2)$$

where  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n$ . The algebraic variety defined by the zero set of  $P$  is denoted by  $Z_P$ . The ellipticity of the PDEs is reflected in the fact that  $Z_P \cap \mathbf{R}^n$  is bounded, i.e.  $P(\lambda)$  has no arbitrarily large, real zeros. We will often find it convenient to use the notation  $\lambda = (\lambda_1, \lambda')$  with  $\lambda' \in \mathbf{C}^{n-1}$ .

As noted earlier, the object of central importance in the Fokas method is the global relation (Fokas 2008). In each of the cases (i)–(iii), the global relation takes the form

$$\int_{\Gamma} e^{-i\lambda \cdot x} [g(x) + i(\lambda \cdot \nu_x) f(x)] d\Gamma_x = 0, \quad \lambda \in Z_P. \quad (1.3)$$

Here,  $\nu_x$  denotes the outward normal to  $\Gamma$  at  $x$ ,  $f \in C(\Gamma)$  represents the known Dirichlet data and  $g$  denotes the unknown Neumann boundary value. The set  $Z_P$  is usually described by a local parametrization, e.g. for  $n = 2$  and  $\beta \neq 0$

$$\lambda_1 = \frac{\beta}{2} \left( k - \frac{1}{k} \right) \quad \text{and} \quad \lambda_2 = \frac{i\beta}{2} \left( k + \frac{1}{k} \right), \quad k \in \mathbf{C}.$$

If one *assumes* the existence of a solution to the boundary value problem, then the global relation can be seen as a consequence of Green's theorem

$$\int_{\Omega} (a\Delta b - b\Delta a) dx = \int_{\Gamma} \left( a \frac{\partial b}{\partial \nu} - b \frac{\partial a}{\partial \nu} \right) d\Gamma$$

with  $a = u$  and  $b = e^{-i\lambda \cdot x}$  with  $\lambda \in Z_P$ . However, from our point of view, the starting point is (1.3) and we assume that there exists some  $g \in C(\Gamma)$ , so that this equation is satisfied for the given  $f \in C(\Gamma)$ .

**Theorem 1.1.** *Let  $\Omega \subset \mathbf{R}^n$  be a bounded, convex domain with smooth boundary  $\Gamma$ . Suppose there exists a function  $g \in C(\Gamma)$  such that*

$$\int_{\Gamma} e^{-i\lambda \cdot x} [g(x) + i(\lambda \cdot \nu_x) f(x)] d\Gamma_x, \quad \lambda \in Z_P$$

for a given function  $f \in C(\Gamma)$ . Then, there exists a solution to (1.1) and  $g$  corresponds to the unknown Neumann boundary value.

**Corollary 1.2.** *The solution to the global relation is unique.*

The proof relies heavily on the Paley–Wiener–Schwartz theorem that states: if an entire function  $U(\lambda)$  satisfies the estimate

$$|U(\lambda)| \leq C(1 + |\lambda|)^N e^{H_K(\text{Im } \lambda)}, \quad \lambda \in \mathbf{C}^n \quad (1.4)$$

for some compact, convex set  $K \subset \mathbf{R}^n$  and some constants  $C, N \geq 0$ , then  $U(\lambda)$  is the Fourier transform of a distribution with support contained in  $K$ , i.e.  $U = \hat{u}$  for some  $u \in \mathcal{E}'(\mathbf{R}^n)$  and  $\text{supp}(u) \subset K$ . The function  $H_K$  denotes the supporting function of  $K$ ,

$$H_K(y) = \sup_{x \in K} x \cdot y,$$

which completely determines the compact, convex set  $K$ . A comprehensive account of this result is given in Hörmander (1985a, §7.3).

Our proof to theorem 1.1 will be constructive: under the assumption that there exists a function  $g$  satisfying the global relation (1.3), we will construct the unique solution to the corresponding Dirichlet problem. In this respect, we define  $U(\lambda)$ ,  $\lambda \in \mathbf{C}^n$ , by

$$\begin{aligned} U(\lambda) &= \frac{-1}{P(\lambda)} \int_{\Gamma} e^{-i\lambda \cdot x} [g(x) + i(\lambda \cdot \nu_x) f(x)] d\Gamma_x \\ &= \frac{Q(\lambda)}{P(\lambda)} \end{aligned}$$

with  $Q$  defined accordingly.

**Lemma 1.3.** *The function  $U(\lambda)$  is entire.*

*Proof.* It is clear that  $U$  is analytic away from  $Z_P$ , so we confine attention to neighbourhoods of  $Z_P$ . Following Hörmander (1985a, theorem 7.3.7), it is enough to show that the function  $U(\lambda)$  is well defined as  $\lambda$  approaches  $Z_P$  along a particular axis. From (1.2), we see that the differential

$$dP = \sum_{i=1}^n \frac{\partial P}{\partial \lambda_i} d\lambda_i = 2 \sum_{i=1}^n \lambda_i d\lambda_i$$

vanishes if and only if  $\lambda = 0$ . So if  $\beta \neq 0$  (i.e. Helmholtz or modified Helmholtz), then there are no critical points on  $Z_P$ , because  $dP(\zeta) \neq 0$  for  $\zeta \in Z_P$ , and in this

case the analysis is simpler. We treat this case first and consider  $U(\zeta_1 + \tau, \zeta')$  as  $\tau \rightarrow 0$  for  $\zeta \in Z_P$ . Taylor's theorem gives

$$Q(\zeta_1 + \tau, \zeta') = \tau \frac{\partial Q}{\partial \lambda_1}(\zeta) + \mathcal{O}(\tau^2),$$

because  $Q(\zeta) = 0$ . Recall that for  $\beta \neq 0$ ,  $\partial P / \partial \lambda_1(\zeta) \neq 0$  for  $\zeta \in Z_P$ , so we have

$$\lim_{\tau \rightarrow 0} U(\zeta_1 + \tau, \zeta') = \lim_{\tau \rightarrow 0} \frac{Q(\zeta_1 + \tau, \zeta')}{P(\zeta_1 + \tau, \zeta')} = \frac{\partial Q / \partial \lambda_1(\zeta)}{\partial P / \partial \lambda_1(\zeta)}$$

which is well-defined. For the case  $\beta = 0$ , one has to work slightly harder because  $dP(\zeta)$  vanishes at  $\zeta = 0 \in Z_P$ . This is reflected in the fact  $P(\tau, 0, \dots, 0)$  has a second-order zero at  $\tau = 0$ . It suffices to show  $\partial Q / \partial \lambda_1(0) = 0$ , so that the limit

$$\lim_{\tau \rightarrow 0} \frac{Q(\tau, 0, \dots, 0)}{P(\tau, 0, \dots, 0)}$$

will be well-defined. We consider a surface in  $Z_P$  that passes through  $\zeta = 0$  defined by  $\zeta(\sigma) = (\sigma, i\sigma, 0, \dots, 0)$  with  $\sigma \in \mathbf{C}$ . Then,  $Q(\zeta(\sigma)) = 0$  for all  $\sigma \in \mathbf{C}$  and we have

$$0 = \frac{d}{d\sigma} Q(\zeta(\sigma)) = \frac{\partial Q}{\partial \lambda_1}(\zeta(\sigma)) + i \frac{\partial Q}{\partial \lambda_2}(\zeta(\sigma)).$$

Taking  $\sigma = 0$  and comparing real and imaginary parts yields the desired result. ■

We also have the following simple estimates on  $Q$

$$\begin{aligned} |Q(\lambda)| &\leq \max\{\|f\|_\infty, \|g\|_\infty\} \int_\Gamma e^{x \cdot \text{Im} \lambda} (1 + |\nu_x||\lambda|) d\Gamma_x \\ &\leq C(1 + |\lambda|) \sup_{x \in \Gamma} [e^{x \cdot \text{Im} \lambda}] \\ &\leq C(1 + |\lambda|) e^{H_{\text{cl}(\omega)}(\text{Im} \lambda)}, \end{aligned}$$

where here and throughout this paper,  $C$  denotes some positive constant independent of  $\lambda$ . Similar estimates hold if  $f$  and  $g$  are in less restrictive function spaces. At this point, we recall the following lemma due to Hörmander (1985a, theorem 7.3.3).

**Lemma 1.4.** *If  $h(z)$  is an analytic function of  $z \in \mathbf{C}$  and  $p(z)$  is a polynomial with leading coefficient  $\alpha$ , then*

$$|\alpha h(0)| \leq \sup_{|z|=1} |h(z)p(z)|.$$

Employing this lemma with  $h(z) = U(z + \lambda_1, \lambda')$  and  $p(z) = P(z + \lambda_1, \lambda')$ , we find

$$\begin{aligned} |U(\lambda)| &\leq \sup_{|z|=1} |U(z + \lambda_1, \lambda')P(z + \lambda_1, \lambda')| \\ &= \sup_{|z|=1} |Q(z + \lambda_1, \lambda')| \\ &\leq C(1 + |\lambda|) e^{H_{\text{cl}(\omega)}(\text{Im} \lambda)} \end{aligned} \tag{1.5}$$

by our previous estimate. So  $U$  satisfies an estimate of the form (1.4), so it is the Fourier transform of a distribution  $u \in \mathcal{E}'(\mathbf{R}^n)$  whose support is contained in  $\text{cl}(\mathcal{Q})$ . We set  $U = \hat{u}$ , so that

$$\hat{u}(\lambda) = \frac{-1}{P(\lambda)} \int_{\Gamma} e^{-i\lambda \cdot x} [g(x) + i(\lambda \cdot \nu_x)f(x)] d\Gamma_x.$$

Employing the inverse Fourier transform on  $\mathcal{E}'(\mathbf{R}^n)$ , we deduce

$$u(x) = \int_{\Gamma} \left[ f(y) \frac{\partial E_P}{\partial \nu_y}(x, y) - g(y) E_P(x, y) \right] d\Gamma_y, \quad (1.6)$$

where  $E_P$  denotes the fundamental solution associated with the operator  $P(D)$ , defined by  $P(D)E_P = \delta$ . In particular, we know from our previous discussion that the right-hand side of (1.6) vanishes outside  $\text{cl}(\mathcal{Q})$ . This means that  $u \rightarrow 0$  as  $x \rightarrow \Gamma$  non-tangentially from outside  $\mathcal{Q}$ . Standard results from the theory of boundary integral equations, e.g. Hsiao & Wendland (2008, §1) give rise to the relations

$$u_- = \text{DL}_{\Gamma, P}(f) + \frac{1}{2}f - \text{SL}_{\Gamma, P}(g) \quad (1.7a)$$

and

$$u_+ = \text{DL}_{\Gamma, P}(f) - \frac{1}{2}f - \text{SL}_{\Gamma, P}(g), \quad (1.7b)$$

where  $\text{SL}_{\Gamma, P}$  and  $\text{DL}_{\Gamma, P}$  are the operators associated with the single- and double-layer potentials for  $(\Gamma, P)$ . The notation  $u_{\pm}$  denotes limits of  $u$  being taken onto  $\Gamma$  from below and above, respectively. We know that  $u_+ = 0$ , so on subtracting these equations, we find  $u_- = f$ . It is also clear that  $u(x)$  satisfies the relevant PDEs in  $\mathcal{Q}$ , because  $E_P(x, y)$  does for  $x \neq y$ . So,  $u$  is the unique solution to the corresponding Dirichlet problem.

It is now straightforward to deduce the uniqueness of the solution to the global relation. Indeed, if there were two solutions  $g_1, g_2$  to the global relation (1.3), then equation (1.6) would provide *two* solutions to the corresponding Dirichlet problem. This would contradict the classical results pertaining to the uniqueness of these solutions. This concludes our proof. ■

**Remark 1.5.** The convexity condition cannot be dropped. Indeed, if  $\mathcal{Q}$  is not convex then the estimate in (1.5) only tells us that the support of  $u$  is contained in the convex hull of  $\mathcal{Q}$ . In this case, the argument that  $u_+ = 0$  breaks down.

**Remark 1.6.** Equation (1.6) establishes the relationship between the classical approach and the integral representations arising in the Fokas method. See also Ashton (2008, theorem 3) and Fokas & Zyskin (2002).

## 2. Generalizations

The arguments presented can easily be modified to deal with more general problems, at the expense of introducing more sophisticated machinery. To this end

- One can assume the boundary is merely Lipschitz continuous. In this case, the normal vector  $\nu_x$  is defined on all of  $\Gamma$  except a set of measure zero (with respect to the surface measure  $d\Gamma$ ) and the layer potential equations (1.7) must be understood in the trace sense.

- The regularity of the boundary data can be weakened so that it only belongs to  $L^p(\Gamma)$  with  $p \geq 1$ , the standard spaces of Lebesgue integrable functions on  $\Gamma$ . The estimates for  $Q$  now require an application of Hölder's inequality and again the layer potential equations (1.7) must be understood in the trace sense.
- In changing the elliptic operator  $P(D)$ , one must first construct the appropriate global relation (Fokas 2008) and repeat the proof of lemma 1.3 with  $Q(\lambda)$  defined accordingly. This is particularly straightforward if there are no critical points on  $Z_P$ , i.e.  $dP(\zeta) \neq 0$  for  $\zeta \in Z_P$ . Finally, one must invoke the appropriate results from the theory of boundary integral equations to use the analogue of (1.7).
- Different classes of boundary data can be given, so the proof requires little or no adjustment if, for instance, the Neumann data are given and the Dirichlet data are unknown. However, in this case, one cannot infer the uniqueness to the solution to the global relation, because the solution to the Neumann problem is not unique.

The first of these generalizations makes the main result directly applicable to problems on convex polygons and more generally, convex polyhedra. As an immediate application, our result provides a step towards justification of the numerous numerical studies in this area, e.g. Fornberg & Flyer (2011), Fulton *et al.* (2004), Sifalakis *et al.* (2008) and Smitherman *et al.* (2010), where the global relation is solved numerically for the unknown Neumann boundary values.

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