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Exact solution of some hyperbolic systems with source terms

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In this paper we present a methodology for constructing exact solutions to hyperbolic systems with source or forcing terms. We illustrate the approach as applied to some inhomogeneous systems arising in environmental problems, namely the inhomogeneous linearized shallow-water equations. We solve exactly a generalized Riemann problem that models dam-break problems in channels with variable bottom elevation and variable width. The exact solutions obtained are very valuable in the process of assessing the performance of numerical methods intended for solving more-complex problems; they can also be useful in providing new ideas to be used in the design of numerical methods for more-general inhomogeneous systems.

Keywords: hyperbolic equations; source terms; exact solutions; Riemann problem; shock-capturing methods; Godunov methods

1. Introduction

In the last few decades important contributions to the design, analysis and application of numerical methods for solving hyperbolic systems of conservation laws have been made. Godunov-type methods (Godunov 1959) in particular, have, over the years, been shown to be very satisfactory for solving such problems, particularly the higher-order versions developed in the last two to three decades (see the textbooks on the subject by LeVeque (1992), Godlewski & Raviart (1996) and Toro (1999, 2001)). The analysis of properties of these methods for inhomogeneous systems, such as stability, monotonicity and conservation, is difficult; as a result, most researchers assess their methods via comparisons with known exact solutions. Traditional test examples are shock-tube problems in gas dynamics and dam-break problems in free-surface shallow-water flows. Conventionally, these tests relate to homogeneous hyperbolic systems, that is, no source terms are present. When source terms are included, the situation is far from clear and work in this area is relatively recent. These works have made clear the importance, for example, of solving the steady case conservatively, using appropriate techniques for dealing with source terms.

In order to study the design and performance of numerical methods applicable to the time-dependent and inhomogeneous cases, we have carried out the study of model inhomogeneous hyperbolic systems in which the source terms depend explicitly on space and time and in which the evolutionary character of the solution is determined not just by the advection terms, but also by the source terms. In a previous work (Toro & Vázquez-Cendón 2001), we formulated source terms in which the time dependency is explicit in such a way that they produced a marked increase or decrease of the norm of the solution with time; we also analysed the behaviour of schemes developed by the authors of this paper for that kind of problem. There are many situations of practical interest in which the source terms are of a geometric nature and these are found to be particularly challenging for numerical methods. This is the reason why, in this paper, we centre our attention on equations with source terms that depend on the *spatial variable* and that allow us to find and interpret the solution of initial-value problems (IVPs) for a family of hyperbolic conservation laws with source terms. In addition, we analyse the exact solutions for the linearized shallow-water equations with source terms and, more specifically, we study dam-break problems with variable bed elevation and channel width. In order to assess our solutions, we also apply a different method, proposed by Watson *et al.* (1992). Their method is applicable to the nonlinear shallow-water equations with a single source term due to a linearly varying bed elevation. In this case the equations may be transformed to a homogeneous system; we use this transformation in the linear case to verify an alternative method based on characteristic theory to solve the corresponding generalized Riemann problem. We also compare our exact solutions with numerical solutions; details about the numerical methods are omitted here.

Section 2 of this paper introduces general inhomogeneous scalar and non-scalar hyperbolic problems and a method of solution. In § 3 we apply the solution methodology to the shallow-water equations and solve a generalized Riemann problem using two analytical methods and one numerical method. Conclusions are given in § 4.

2. Hyperbolic system with source terms

This section is devoted to the presentation of model linear hyperbolic systems whose source terms are such that the form of the sought exact solution is predetermined. Specifically, we seek solutions that allow the translation of the initial conditions of an associated homogenous IVP, and whose solutions contain information related to a function that depends only on distance. In § 3 we show that a generalized Riemann problem associated with our model system corresponds to the linearized shallow-water equations with variable bed elevation and width variation.

To begin with, we pose the general IVP for a scalar inhomogeneous equation, namely a partial differential equation (PDE) and an initial condition (IC),

$$\text{PDE:}\quad \partial_t q(x,t) + \lambda \partial_x q(x,t) = s(x), \tag{2.1}$$

$$\text{IC:}\quad q(x,0) = \tilde{q}^{(0)}(x) + r(x), \tag{2.2}$$

where the wave-propagation speed λ is constant, $\tilde{q}^{(0)}$ is the initial condition of the corresponding homogeneous IVP, $s \equiv 0$, and the function r is associated with the source term when seeking a solution of the form

$$q(x,t) = \tilde{q}^{(0)}(x - \lambda t) + r(x). \tag{2.3}$$

Substituting (2.3) in (2.1) gives

$$\partial_t q(x,t) + \lambda \partial_x q(x,t) = -\lambda(\tilde{q}^{(0)})'(x - \lambda t) + \lambda((\tilde{q}^{(0)})'(x - \lambda t) + r'(x)) = \lambda r'(x). \tag{2.4}$$

Hence, the sought function s is

$$s(x) = \lambda r'(x). \tag{2.5}$$

Next we extend these ideas to hyperbolic systems with source terms of the form

$$\text{PDEs:} \quad \partial_t W(x,t) + \bar{A}\partial_x W(x,t) = S(x), \tag{2.6}$$

$$\text{ICs:} \quad W(x,0) = \tilde{W}^{(0)}(x) + R(x) = \begin{pmatrix} \tilde{w}_1^{(0)}(x) \\ \tilde{w}_2^{(0)}(x) \end{pmatrix} + \begin{pmatrix} r_1(x) \\ r_2(x) \end{pmatrix}, \tag{2.7}$$

where W is the vector of unknowns, S is the vector of source terms and $\bar{A}$ is a diagonalizable matrix whose eigenvalues λ_1 and λ_2 define the diagonal matrix Λ, and the corresponding eigenvectors V_1 and V_2 are the columns of the base-transformation matrix X. The source terms are of the form

$$S(x) = X \begin{pmatrix} \lambda_1 r_1'(x) \\ \lambda_2 r_2'(x) \end{pmatrix}. \tag{2.8}$$

The characteristic variables defined by the vector Q satisfy the IVP

$$\text{PDEs:} \quad \partial_t Q(x,t) + \Lambda\partial_x Q(x,t) = X^{-1}S(x), \tag{2.9}$$

$$\text{ICs:} \quad Q(x,0) = \tilde{Q}^{(0)}(x) + R(x). \tag{2.10}$$

Here, $\tilde{Q}^{(0)}(x)$ is the initial condition of the corresponding IVP for $S \equiv 0$. Then, in view of (2.3), each of the characteristic variables is

$$q_i(x,t) = \tilde{q}_i^{(0)}(x - \lambda_i t) + r_i(x) \quad \Rightarrow \quad \tilde{q}_i^{(0)}(x - \lambda_i t) = q_i(x,t) - r_i(x), \tag{2.11}$$

and the initial conditions satisfy

$$\tilde{q}_i^{(0)}(x) = q_i(x,0) - r_i(x) = q_i^{(0)}(x) - r_i(x). \tag{2.12}$$

Finally, sought solutions are of the form

$$\begin{aligned} W(x,t) &= XQ(x,t) \\ &= X \begin{pmatrix} \tilde{q}_1^{(0)}(x - \lambda_1 t) + r_1(x) \\ \tilde{q}_2^{(0)}(x - \lambda_2 t) + r_2(x) \end{pmatrix} \\ &= X \begin{pmatrix} q_1^{(0)}(x - \lambda_1 t) - r_1(x - \lambda_1 t) + r_2(x) \\ q_2^{(0)}(x - \lambda_2 t) - r_1(x - \lambda_2 t) + r_2(x) \end{pmatrix}. \end{aligned} \tag{2.13}$$

The initial conditions of the characteristic variables can be expressed in terms of the initial conditions of the conserved variables using the base-transformation matrix, namely, $W^{(0)}(x) = XQ^{(0)}(x)$. Therefore, the solution of the system is determined by the initial conditions of the problem, naturally, and by functions r_1 and r_2, which are arbitrary.

3. Application to the shallow-water equations

An interesting example of a hyperbolic system, and one that is also a motivation for the present work, is furnished by the linearized shallow-water equations in channels in which the source terms govern the variable bed elevation and the variable width. The unknowns w_1 and w_2 are the water depth h and the particle velocity u, respectively. The expressions for the matrix $\bar{A}$ and for the source terms are given by

$$\bar{A} = \begin{pmatrix} \bar{u} & \bar{h} \\ g & \bar{u} \end{pmatrix}, \qquad S(x) = \begin{pmatrix} -\bar{h}\bar{u}\dfrac{B'(x)}{B(x)} \\ 2\bar{u}^2\dfrac{B'(x)}{B(x)} - gb'(x) \end{pmatrix}, \tag{3.1}$$

where $\bar{h}$ and $\bar{u}$ denote averages for water depth and velocity, as distinct from h and u. The function defining the bed elevation is denoted by $b(x)$ and $B(x)$ represents the width of the channel at each position x. The eigenvalues and eigenvectors of $\bar{A}$ are

$$\lambda_1 = \bar{u} - c, \qquad \lambda_2 = \bar{u} + c, \qquad c = \sqrt{g\bar{h}}, \qquad X = \begin{pmatrix} \bar{h} & \bar{h} \\ -c & c \end{pmatrix}. \tag{3.2}$$

Simple calculations give the functions r_1 and r_2 that satisfy (2.8) for the source terms of this model, namely,

$$r_1(x) = \frac{1}{2c\lambda_1}\left\{-\bar{u}(2\bar{u}+c)\log\left(\frac{B(x)}{B(x_\mathrm{I})}\right) + g(b(x) - b(x_\mathrm{I}))\right\}, \tag{3.3}$$

$$r_2(x) = \frac{1}{2c\lambda_2}\left\{+\bar{u}(2\bar{u}-c)\log\left(\frac{B(x)}{B(x_\mathrm{I})}\right) + g(b(x) - b(x_\mathrm{I}))\right\}. \tag{3.4}$$

Here, the point x_I is the lower integration limit. The solutions for the characteristic variables of the corresponding homogeneous problem are

$$\tilde{q}_1(x) = \tfrac{1}{2}\left\{\frac{h^{(0)}(x)}{\bar{h}} - \frac{u^{(0)}}{c}(x)\right\} - r_1(x), \tag{3.5}$$

$$\tilde{q}_2(x) = \tfrac{1}{2}\left\{\frac{h^{(0)}(x)}{\bar{h}} + \frac{u^{(0)}}{c}(x)\right\} - r_2(x). \tag{3.6}$$

Therefore, the solution of the shallow-water equations with variable bed elevation and channel width is given by

$$\begin{aligned} h(x,t) &= \bar{h}[\tilde{q}_1^{(0)}(x-\lambda_1 t) + \tilde{q}_2^{(0)}(x-\lambda_2 t)] + \bar{h}[r_1(x) + r_2(x)] \\ &= \tfrac{1}{2}[h^{(0)}(x-\lambda_1 t) + h^{(0)}(x-\lambda_2 t)] \\ &\quad + \frac{\bar{h}}{2c}[-u^{(0)}(x-\lambda_1 t) + u^{(0)}(x-\lambda_2 t)] \\ &\quad - \bar{h}[r_1(x-\lambda_1 t) + r_2(x-\lambda_2 t) + r_1(x) + r_2(x)], \end{aligned} \tag{3.7}$$

$$\begin{aligned} u(x,t) &= c[-\tilde{q}_1(x-\lambda_1 t) + \tilde{q}_2(x-\lambda_2 t)] + c[-r_1(x) + r_2(x)] \\ &= \frac{c}{2\bar{h}}[-h^{(0)}(x-\lambda_1 t) + h^{(0)}(x-\lambda_2 t)] \\ &\quad + \frac{\bar{h}}{2c}[u^{(0)}(x-\lambda_1 t) + u^{(0)}(x-\lambda_1 t)] \\ &\quad + c[-r_1(x-\lambda_1 t) + r_2(x-\lambda_2 t) - r_1(x) + r_2(x)]. \end{aligned} \tag{3.8}$$

A simpler version of this problem was introduced in Toro (2001, § 4.6), but we note that the solution given there contains an error.

(*a*) *A generalized Riemann problem*

Here we consider a generalized Riemann problem for the linearized shallow-water equations, generalized in the sense that (i) the equations include source terms and (ii) the initial conditions have a linear distribution. We note that in the conventional Riemann problem the initial data are piece-wise constant. We assume that bed elevation is a linear function of distance and thus $b'(x) = -\alpha$: const.; the function $B(x)$ is assumed to be constant. We make these assumptions in order to compare our solution with that obtained from applying the alternative solution method proposed by Watson *et al.* (1992). Thus the initial conditions are

$$h^{(0)}(x) = \tilde{h}^{(0)}(x) - b(x), \qquad \tilde{h}^{(0)}(x) = \begin{cases} h_{\mathrm{l}} & \text{if } x < 0, \\ h_{\mathrm{r}} & \text{if } x \geqslant 0, \end{cases} \tag{3.9}$$

$$u^{(0)}(x) = 0, \qquad \tilde{u}^{(0)}(x) = 0. \tag{3.10}$$

In this case, the expressions for the functions r_1 and r_2 are

$$r_1(x) = -\frac{1}{2c\lambda_1} g\alpha x, \qquad r_2(x) = \frac{1}{2c\lambda_2} g\alpha x. \tag{3.11}$$

Therefore, the solutions are of the form

$$h(x,t) = \begin{cases} h_{\mathrm{l}} + \alpha x - \alpha t\bar{u} & \text{if } x/t < \lambda_1, \\ \frac{1}{2}(h_{\mathrm{l}} + h_{\mathrm{r}}) + \alpha x - \alpha t\bar{u} & \text{if } \lambda_1 < x/t < \lambda_2, \\ h_{\mathrm{r}} + \alpha x - \alpha t\bar{u} & \text{if } x/t > \lambda_2, \end{cases} \tag{3.12}$$

$$u(x,t) = \begin{cases} 0 & \text{if } x/t < \lambda_1, \\ -(c/2\bar{h})(h_{\mathrm{r}} - h_{\mathrm{l}}) & \text{if } \lambda_1 < x/t < \lambda_2, \\ 0 & \text{if } x/t > \lambda_2. \end{cases} \tag{3.13}$$

We can also write these solutions in terms of the solution $\tilde{W} = (\tilde{h}(x,t), \tilde{u}(x,t))^{\mathrm{T}}$ of the associated homogeneous Riemann problem,

$$h(x,t) = \tilde{h}(x,t) - b(x) - \alpha t\bar{u}, \tag{3.14}$$

$$u(x,t) = \tilde{u}(x,t). \tag{3.15}$$

From these relations one can conclude that, in the case in which $\bar{u}$ is zero, the water depth of the homogeneous problem $\tilde{h}$ coincides with the value for the free-surface position $h + b$ everywhere.

In what follows we illustrate the way to obtain the solution by the method of Watson *et al.* (1992). The change of variables allows the transformation of the nonlinear shallow-water equations with a source term due to a bed elevation given by a linear distribution ($b'(x) = -\alpha$) into a homogeneous system. The transformation is as follows:

$$\chi = x - \tfrac{1}{2} g\alpha t^2, \qquad \tau = t. \tag{3.16}$$

The original variables h and u expressed in terms of the new spatial and temporal variables are given by

$$\hat{h}(\chi, \tau) = h(x,t), \qquad \hat{u}(\chi, \tau) = u(x,t) - \alpha g t. \tag{3.17}$$

Therefore, the entries of the matrix $\bar{A}$ in terms of the new variables are

$$\hat{\bar{h}} = \bar{h}, \qquad \hat{\bar{u}}(\tau) = \bar{u} - \alpha g t \quad \Rightarrow \quad \bar{u} = \hat{\bar{u}}(\tau) + \alpha g t. \tag{3.18}$$

In view of the following relations being satisfied,

$$\partial_t h(x,t) = \partial_\tau \hat{h}(\chi,\tau) - \alpha g t \partial_\chi \hat{h}(\chi,\tau), \qquad \partial_x h(x,t) = \partial_\chi \hat{h}(\chi,\tau), \tag{3.19}$$

$$\partial_t u(x,t) = \partial_\tau \hat{u}(\chi,\tau) - \alpha g t \partial_\chi \hat{u}(\chi,\tau) + g\alpha, \qquad \partial_x u(x,t) = \partial_\chi \hat{u}(\chi,\tau), \tag{3.20}$$

one arrives at the following IVP for the new variables,

$$\text{PDEs:} \quad \partial_\tau \hat{W}(\chi,\tau) + \hat{\bar{A}}(\tau)\partial_\chi \hat{W}(\chi,\tau) = 0, \tag{3.21}$$

$$\text{ICs:} \quad \tilde{W}(\chi,0) = W^{(0)}(x), \tag{3.22}$$

where

$$\hat{W}(x,t) = \begin{pmatrix} \hat{h}(\chi,\tau) \\ \hat{u}(\chi,\tau) \end{pmatrix}, \qquad \hat{\bar{A}}(\tau) = \begin{pmatrix} \hat{\bar{u}} & \hat{\bar{h}} \\ g & \hat{\bar{u}} \end{pmatrix}. \tag{3.23}$$

The initial conditions are defined by (3.9) and (3.10). The eigenvalues of the matrix $\hat{\bar{A}}(\tau)$ depend on time and can be expressed in terms of the matrix $\bar{A}$,

$$\hat{\lambda}_1 = \hat{\bar{u}} - c = \lambda_1 - g\alpha t, \qquad \hat{\lambda}_2 = \hat{\bar{u}} + c = \lambda_2 - g\alpha t. \tag{3.24}$$

The solution of the IVP (3.21), (3.22) can be obtained from standard theory of linear hyperbolic systems, noting that the characteristics curves are not straight lines in this case,

$$\frac{\mathrm{d}\chi}{\mathrm{d}\tau}(\tau) = \lambda_k - g\alpha\tau, \tag{3.25}$$

$$\chi(0) = \chi_0, \tag{3.26}$$

$$\chi(\tau) - \chi_0 = \lambda_k \tau - \tfrac{1}{2} g\alpha\tau^2 \to \chi(0) = \chi(\tau) - \lambda_k\tau - \tfrac{1}{2} g\alpha\tau^2, \quad k = 1,2. \tag{3.27}$$

Therefore, the characteristic variables are

$$\hat{q}_k(\chi,\tau) = \hat{q}^{(0)}(\chi_0) = \hat{q}^{(0)}(\chi - \lambda_k\tau + \tfrac{1}{2} g\alpha\tau^2) = \hat{q}^{(0)}(x - \lambda_k t), \quad k = 1,2. \tag{3.28}$$

If we assume that the initial velocity is identically zero, the initial conditions of the characteristic variables in terms of the initial conditions (3.22) are given by

$$\begin{aligned} \hat{q}_1(\chi,\tau) &= \hat{q}_2(\chi,\tau) \\ &= \frac{1}{2\bar{h}} \hat{h}(\chi,0) \\ &= \frac{1}{2\bar{h}} \hat{h}^{(0)}(x) \\ &= \frac{1}{2\bar{h}} (\tilde{h}^{(0)}(x) - b(x)) \\ &= \frac{1}{2\bar{h}} (\tilde{h}^{(0)}(x) + \alpha x). \end{aligned} \tag{3.29}$$

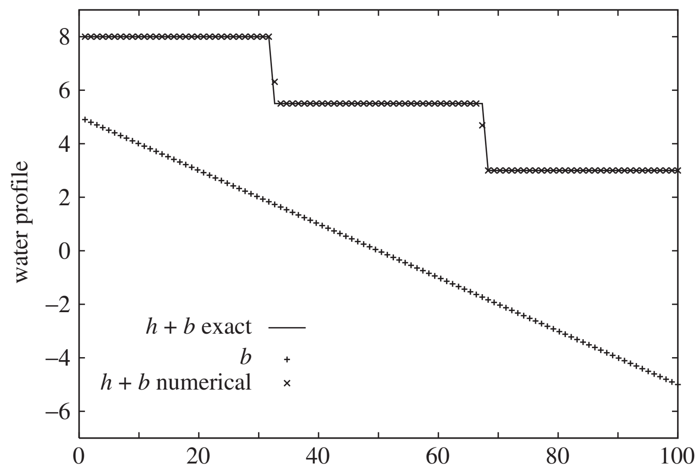

Figure 1. Computed (symbols) and exact (line) free-surface elevation $h(x,t)+b(x)$ at time $t=2.5$. The channel bottom elevation $b(x)$ is also displayed.

Then the solution of the IVP (3.21), (3.22) is

$$\hat{h}(\chi,\tau)=\tfrac{1}{2}\{\tilde{h}^{(0)}(\chi-\lambda_1\tau+\tfrac{1}{2}g\alpha\tau^2)+\tilde{h}^{(0)}(\chi-\lambda_2\tau+\tfrac{1}{2}g\alpha\tau^2)\}$$
$$+\alpha\chi-\alpha\tau\bar{u}+\tfrac{1}{2}g\alpha^2\tau^2, \tag{3.30}$$
$$\hat{u}(\chi,\tau)=-\frac{c}{2\bar{h}}\{\tilde{h}^{(0)}(\chi-\lambda_1\tau+\tfrac{1}{2}g\alpha\tau^2)-\tilde{h}^{(0)}(\chi-\lambda_2\tau+\tfrac{1}{2}g\alpha\tau^2)\}-g\alpha\tau. \tag{3.31}$$

Transforming to the original variables, we obtain

$$h(x,t)=\tfrac{1}{2}\{\tilde{h}^{(0)}(x-\lambda_1 t)+\tilde{h}^{(0)}(x-\lambda_2 t)\}+\alpha x-\alpha t\bar{u}, \tag{3.32}$$
$$\begin{aligned}u(x,t)&=\hat{u}(\chi,\tau)+g\alpha t\\&=-\frac{c}{2\bar{h}}\{\tilde{h}^{(0)}(\chi-\lambda_1 t)-\tilde{h}^{(0)}(x-\lambda_2 t)\},\end{aligned} \tag{3.33}$$

which coincides with the solution obtained by our method.

Some remarks regarding the comparison of the two methods for solving the generalized Riemann problem are in order. First we note that, in the method of Watson *et al.* (1992), the advection part of the equations may be nonlinear, but the geometry of the channel is restricted to the simple case in which the bed elevation varies linearly and the channel width is constant. On the other hand, the method proposed in this paper allows more general channel geometries, but the advection part of the equation must be linear. In addition, our method provides solutions for problems with source terms that allow the assessment of various discretization schemes to deal with conservation laws with source terms, especially geometric source terms. This area is the subject of current investigations by the authors.

(*b*) *Numerical solutions*

As a way of illustrating one of the uses of our exact solutions, we consider a test problem and compare the exact solution with the numerical solution obtained from the method proposed in Vázquez-Cendón (1999). The test is a dam-break problem

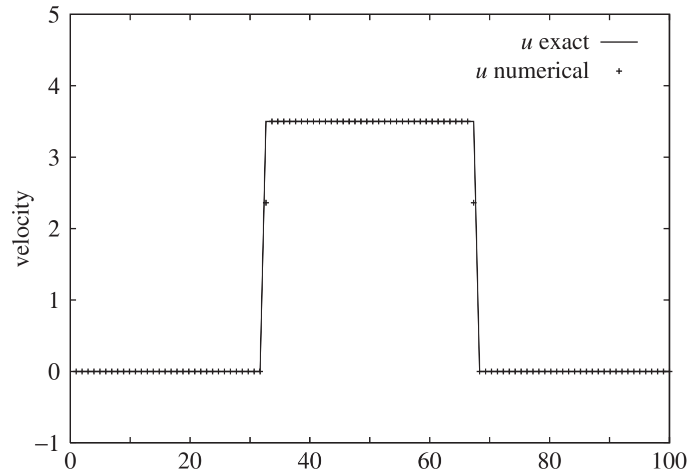

Figure 2. Computed (symbols) and exact (line) particle velocity $u(x,t)$ at time $t = 2.5$.

in a channel 100 m in length and initial conditions

$$h(x,0) = \begin{cases} h_{\mathrm{L}} = 8\ \mathrm{m} & \text{if } x \leqslant 50, \\ h_{\mathrm{R}} = 3\ \mathrm{m} & \text{if } x > 50, \end{cases} \tag{3.34}$$

with $u(x,0) = u_{\mathrm{L}} = u_{\mathrm{R}} = 0\ \forall x$. As parameters of the problem we take $\bar{h} = 5$, $\bar{u} = 0$, $\alpha = 0.1$. In the computations, we take 100 cells and a Courant–Friedrichs–Levy (CFL) number of unity. Results are displayed at time $t = 2.5$ in figures 1 and 2, where symbols show the numerical solution and the full line shows the exact solution. Figure 1 shows results for the free-surface position; the channel bed profile is also shown. Figure 2 shows results for the particle velocity. By comparison with the exact solution, it is seen that the numerical solution is very accurate for this test problem.

4. Conclusions

Exact solutions to model hyperbolic systems with source terms have been presented. The source terms are assumed to depend explicitly on the spatial variable and serve as analogues for geometric source terms, which are renowned for causing difficulties to numerical methods. The methodology for deriving exact solutions has been applied to the linearized shallow-water equations with source terms due to bed elevation and width variation. In particular, a generalized Riemann problem for the linearized shallow-water equations with a source term due to bed elevation has been solved, and the solution has been compared with that obtained by an alternative method proposed by Watson *et al.* (1992). Finally, as a way of illustrating one of the potential uses of the exact solutions presented here, we solve a test problem numerically and compare exact and numerical solutions. Current investigations in which the findings of this paper are useful include the design of new numerical methods for solving nonlinear systems of hyperbolic conservation laws with source terms.

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