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Stochastic reduction in nonlinear quantum mechanics

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Stochastic extensions of the Schrödinger equation have attracted attention recently as plausible models for state reduction in quantum mechanics. Here we formulate a general approach to stochastic Schrödinger dynamics in the case of a nonlinear state space of the type proposed by Kibble. We derive a number of new identities for observables in the nonlinear theory, and establish general criteria on the curvature of the state space sufficient to ensure collapse of the wave function.

Keywords: nonlinear quantum mechanics; quantum measurement; stochastic differential geometry; Kähler diffusion; holomorphic sectional curvature

1. Introduction

A generalization of quantum mechanics was considered by Mielnik (1974), who introduced the notion of nonlinear observables. Two alternative extensions of the standard quantum theory were then proposed by Kibble (1979). The first of these alternatives is based on the phase space formulation of quantum mechanics. In this case the quantum mechanical state manifold is taken to be the space of rays through the origin of Hilbert space. It can be shown then that the Schrödinger equation reduces to Hamilton's equation of classical mechanics (see, for example, Cantoni 1975, 1982; Page 1987; Anandan & Aharonov 1990; Gibbons 1992; Hughston 1995; Ashtekar & Schilling 1998; Field & Hughston 1999; Marsden & Ratiu 1999; Brody & Hughston 2001), except that the quantum Hamiltonian is of a special restricted form. Thus a natural generalization is to remove this constraint. When such trajectories are lifted from the space of rays to Hilbert space we obtain nonlinear wave equations.

The general properties of nonlinear observables were subsequently analysed in detail by Weinberg (1989*a*, *b*). Following this, it was pointed out by Gisin (1989) that the evolution of the density matrix is not autonomous in the nonlinear mechanics of Kibble and Weinberg, and that this property may be physically undesirable. However, it was also indicated by Gisin (1989) that there is another type of nonlinear quantum theory for which the evolution of the density matrix is, in fact, autonomous, and a number of other desirable features of linear evolution are extended in a natural way. This is the stochastic framework for quantum dynamics developed by Pearle (1986), Ghirardi *et al*. (1986, 1987, 1990), Diosi (1988), Gisin & Percival (1992), Percival (1994), and others. See Pearle (2000) for a recent review of the relevant literature.

The stochastic theories are of significance because they exhibit natural reductive properties: starting from a given initial state, the system evolves stochastically in such a way to ensure collapse to an eigenstate of one or more designated observables.

Kibble's second alternative for a nonlinear quantum theory is in essence to consider a general Kähler manifold as the phase space of quantum mechanics, instead of the space of rays. The idea is that in the presence of interactions the states accessible to a quantum system constitute a curved space $\mathfrak{M}$ which has the structure of a complex manifold endowed with a compatible symplectic structure. The dynamics of the state are then governed by a Hamiltonian flow which is also an isometry of the manifold.

In the present article we consider stochastic state reduction models within the framework of Kibble's second theory. The advantage of a stochastic approach in this context is that it leads to a probabilistic interpretation, a feature hitherto missing in the nonlinear theory. Remarkably, many of the key features of the basic stochastic reduction models carry through to a fully nonlinear state space. The main results of the paper are to determine general criteria sufficient to ensure state reduction in the nonlinear theory, and to express these criteria directly in terms of geometrical features of the state manifold. Thus it is the geometry of the quantum state manifold that determines whether reduction takes place, and if so, how rapidly.

The structure of the paper is as follows. After introducing the relevant state space geometry and elements of stochastic calculus on manifolds, a number of identities concerning the properties of quantum observables are established in lemmas 3.1–5.3. These results are then applied to formulate general theorems governing reduction processes on nonlinear state spaces.

2. Geometry of quantum mechanics

Let us first recall briefly the phase space formulation of quantum theory. We consider a finite dimensional complex Hilbert space $\mathcal{H}$ of which a typical element is denoted ψ^{α} ($\alpha = 0, 1, \dots, n$). Given the Hamiltonian operator H^{α}_{β}, the dynamics of the state is determined by the Schrödinger equation,

$$\mathrm{i}\hbar\partial_t\psi^{\alpha} = H^{\alpha}_{\beta}\psi^{\beta}. \tag{2.1}$$

The conventional quantum expectation $F^{\alpha}_{\beta}\bar{\psi}_{\alpha}\psi^{\beta}/\bar{\psi}_{\gamma}\psi^{\gamma}$ of an observable F^{α}_{β} in the state ψ^{α} is invariant under the scale transformations $\psi^{\alpha} \to \lambda\psi^{\alpha}$ ($\lambda \in \mathbb{C} - \{0\}$). Hence we can work with the space of equivalence classes of state vectors modulo such transformations, i.e. the complex projective space P^n.

We regard P^n as a real manifold $\varGamma$ of dimension $2n$. It is known that $\varGamma$ has a natural symplectic structure ω_{ab}, as well as a Riemannian structure given by the Fubini–Study metric g_{ab}. These two structures are compatible in the sense that there exists an integrable complex structure J^a_b on $\varGamma$, satisfying

$$J^a_c J^c_b = -\delta^a_b, \tag{2.2}$$

such that $\nabla_a J^b_c = 0$ and $g^{ac}\omega_{cb} = J^a_b$, where ∇_a is the covariant derivative associated with g_{ab} and $g_{ac}g^{cb} = \delta^b_a$. We use Roman indices $(a, b, \dots)$ for tensorial operations on $\varGamma$. The compatibility conditions make $\varGamma$ a Kähler manifold.

The special feature that identifies $\varGamma$ as the quantum phase space is that the Schrödinger equation can be expressed in the Hamiltonian form,

$$\hbar\,\mathrm{d}x^a = 2\omega^{ab}\nabla_b H(x)\,\mathrm{d}t. \tag{2.3}$$

Here $\omega^{ab} = g^{ac}g^{bd}\omega_{cd}$ is the inverse symplectic structure and $H(x)$ is the Hamiltonian function on Γ, given by the expectation of the operator H^{α}_{β} in the equivalence class of state vectors corresponding to the point $x \in \Gamma$. In particular, if $\psi^{\alpha}(x)$ denotes any representative vector in the fibre of $\mathcal{H} - \{0\}$ above the point x, then

$$H(x) = \frac{\bar{\psi}_{\alpha}(x)H^{\alpha}_{\beta}\psi^{\beta}(x)}{\bar{\psi}_{\gamma}(x)\psi^{\gamma}(x)}. \tag{2.4}$$

A Hamiltonian vector field $Z^a = \omega^{ab}\nabla_b H$ on Γ satisfies the Killing equation

$$\nabla_{(a}Z_{b)} = 0 \tag{2.5}$$

if and only if $H(x)$ is the expectation of a quantum observable in the state x, and is thus of the form (2.4) for some choice of H^{α}_{β}. Therefore, the Schrödinger evolution preserves the distance, and hence the transition probability, between any two given states. The energy eigenstates are the fixed points of the flow, at which $\nabla_a H = 0$. The value of $H(x)$ at a fixed point is the corresponding eigenvalue.

3. Nonlinear quantum state spaces

The first alternative of Kibble for a nonlinear extension of standard quantum theory is to replace the observable $H(x)$ by a general function on Γ. Then the resulting trajectories are Hamiltonian but no longer Killing, and the corresponding evolution on $\mathcal{H}$ is governed by a nonlinear wave equation. This is not the generalization we consider here.

For the analysis of Kibble's second alternative, it will be useful first to develop briefly a differential geometric framework for the standard operations of quantum mechanics (we set $\hbar = 1$). If $F(x)$ and $G(x)$ are observables, the expectation of their commutator is also an observable, given by the Poisson bracket $2\omega^{ab}\nabla_a F\nabla_b G$. Then if x_t is a Schrödinger trajectory and $F_t = F(x_t)$, it follows that

$$\mathrm{d}F_t = 2\omega^{ab}\nabla_a F\nabla_b H\,\mathrm{d}t, \tag{3.1}$$

where H is the Hamiltonian. This tells us how the expectation of F changes along the flow generated by H. For any observable $F(x)$, we define the associated dispersion by

$$V^F = g^{ab}\nabla_a F\nabla_b F. \tag{3.2}$$

In the linear theory $V^F(x)$ is the squared uncertainty of F in the state x. That is to say,

$$V^F(x) = \frac{\bar{\psi}_{\alpha}(x)F^{\alpha}_{\gamma}F^{\gamma}_{\beta}\psi^{\beta}(x)}{\bar{\psi}_{\delta}(x)\psi^{\delta}(x)} - \left(\frac{\bar{\psi}_{\alpha}(x)F^{\alpha}_{\beta}\psi^{\beta}(x)}{\bar{\psi}_{\gamma}(x)\psi^{\gamma}(x)}\right)^2. \tag{3.3}$$

As a consequence of the inequality

$$(g_{ab}X^aX^b)(g_{ab}Y^aY^b) \geqslant (g_{ab}X^aY^b)^2 + (\omega_{ab}X^aY^b)^2, \tag{3.4}$$

which holds for all X^a and Y^a, we obtain the Heisenberg relation

$$V^FV^G \geqslant (\omega^{ab}\nabla_a F\nabla_b G)^2 \tag{3.5}$$

if we set $X^a = \omega^{ab}\nabla_b F$ and $Y^a = \omega^{ab}\nabla_b G$.

Kibble's second alternative for a nonlinear quantum theory is to let the state space be a general Kähler manifold $\mathfrak{M}$, with metric g_{ab}, symplectic structure ω_{ab}, and complex structure J^a_b. In the nonlinear theory, we say that a real function $F(x)$ on $\mathfrak{M}$ is an observable if and only if the corresponding Hamiltonian vector field $X^a = \omega^{ab}\nabla_b F$ is an isometry. This definition fully agrees with the usual characterization of observables when $\mathfrak{M} = \Gamma$. We note that if $\mathfrak{M}$ is compact and has vanishing first Betti number, then any Killing field on $\mathfrak{M}$ is Hamiltonian, with at least two distinct eigenstates (Frankel 1959). As in the linear theory, we interpret $F(x)$ as the expectation of the result of a measurement of the given observable in the state $x \in \mathfrak{M}$.

With this interpretation in mind, we now proceed to deduce a number of general properties satisfied by observables in the nonlinear theory.

Lemma 3.1. *If $F(x)$ and $G(x)$ are observables, so is their commutator.*

The proof is as follows. If X^a and Y^a are Hamiltonian flows, then their Lie bracket is

$$X^b\nabla_b Y^a - Y^b\nabla_b X^a = \omega^{ab}\nabla_b(\omega_{cd}X^cY^d). \tag{3.6}$$

If $F(x)$ and $G(x)$ are generators of X^a and Y^a, then

$$\omega_{cd}X^cY^d = \omega^{cd}\nabla_c F\nabla_d G. \tag{3.7}$$

Furthermore, if X^a and Y^a are Killing, so is their Lie bracket. Therefore, the Hamiltonian flow generated by the commutator of $F(x)$ and $G(x)$ is Killing. As a consequence we also obtain the following nonlinear generalization of an identity due to Adler & Horwitz (2000).

Lemma 3.2. *If $F(x)$ and $G(x)$ are observables, then*

$$\nabla_b F\nabla^b\nabla^a G - \nabla_b G\nabla^b\nabla^a F = \omega^{ab}\nabla_b(\omega^{cd}\nabla_c F\nabla_d G). \tag{3.8}$$

If F and G commute, then any non-degenerate critical point of F is a critical point of G.

As in the linear theory, the eigenstates of an observable $F(x)$ are the points of $\mathfrak{M}$ at which $\nabla_a F = 0$. The value of $F(x)$ at a critical point is the corresponding eigenvalue. In the nonlinear theory $V^F(x)$ does not in general have an interpretation as a moment, but nevertheless remains a measure of the dispersion of $F(x)$ in the given state. In particular, the Heisenberg relation (3.5) holds.

The Schrödinger trajectories in the nonlinear theory are generated by a Hamiltonian $H(x)$, which we assume to be an observable. The following result shows that for any observable commuting with the Hamiltonian, its dispersion is constant along the Schrödinger trajectory. This fact will be used later to derive the stochastic dynamics of the energy dispersion.

Lemma 3.3. *If F is an observable that commutes with the Hamiltonian H, then*

$$\omega^{ab}\nabla_a H\nabla_b V^F = 0. \tag{3.9}$$

The proof is as follows. Equation (3.2) implies that

$$\omega^{ab}\nabla_a H \nabla_b V^F = 2\omega^{ab}\nabla_a H \nabla_b \nabla_c F \nabla^c F, \tag{3.10}$$

and thus

$$\omega^{ab}\nabla_a H \nabla_b V^F = 2\nabla_c(\omega^{ab}\nabla_a H \nabla_b F)\nabla^c F - 2\nabla_c(\omega^{ab}\nabla_a H)\nabla_b F \nabla^c F. \tag{3.11}$$

The first term on the right vanishes because H and F commute, whereas the second term vanishes on account of the Killing equation (2.5) satisfied by the Hamiltonian vector field $Z^a = \omega^{ab}\nabla_b H$.

4. Stochastic differential geometry

To proceed further we now introduce the elements of stochastic differential geometry (see, for example, Emery 1989; Ikeda & Watanabe 1989; Norris 1992). The basic process we consider is the Wiener process W_t defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ-field on Ω, and $\mathbb{P}$ is the probability measure. The filtration of $\mathcal{F}$, which determines the causal structure of the probability space, is given by a parametrized family $\{\mathcal{F}_t\}$ $(0 \leqslant t < \infty)$ of nested σ-subfields satisfying $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for any $s \leqslant t < \infty$. We say that W_t is a Wiener process if it is continuous, $W_0 = 0$, $W_t - W_s$ $(0 \leqslant s < t)$ is independent of the information $\mathcal{F}_s$ up to time s, and $W_t - W_s$ is normally distributed with mean zero and variance $t - s$.

A process σ_t is said to be adapted to the filtration $\mathcal{F}_t$ generated by W_t if its random value at time t is determined by the history of W_t up to that time. If σ_t is $\mathcal{F}_t$-adapted, then the stochastic integral

$$M_t = \int_0^t \sigma_s \, \mathrm{d}W_s \tag{4.1}$$

exists, provided σ_t is almost surely square-integrable. If the variance of M_t exists, then M_t satisfies the martingale conditions $\mathbb{E}[|M_t|] < \infty$ and $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$, where $\mathbb{E}[-]$ denotes expectation with respect to the measure $\mathbb{P}$. The second condition implies that, given the history of the Wiener process up to time s, the expectation of M_t for $t \geqslant s$ is given by its value at s. The variance of M_t is determined by the Itô isometry

$$\mathbb{E}[M_t^2] = \mathbb{E}\left[\int_0^t \sigma_s^2 \, \mathrm{d}s\right]. \tag{4.2}$$

A general Itô process is defined by an integral of the form,

$$x_t = x_0 + \int_0^t \mu_s \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}W_s, \tag{4.3}$$

where the adapted processes μ_t and σ_t are called the drift and the volatility of x_t. A convenient way to express the content of (4.3) is to write

$$\mathrm{d}x_t = \mu_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}W_t, \tag{4.4}$$

and to regard the initial condition x_0 as implicit. In the special case $\mu_t = \mu(x_t)$ and $\sigma_t = \sigma(x_t)$, where $\mu(x)$ and $\sigma(x)$ are prescribed functions, the process x_t is said to be a diffusion.

This analysis can be generalized to the case of a diffusion x_t taking values on a manifold $\mathfrak{M}$, driven by a standard m-dimensional Wiener process W_t^i $(i = 1, 2, \ldots, m)$. Let ∇_a be a torsion-free connection on $\mathfrak{M}$, and suppose $\mu^a(x)$ and $\sigma_i^a(x)$ are $m+1$ vector fields on $\mathfrak{M}$. The general diffusion process on the manifold $\mathfrak{M}$ is governed by a stochastic differential equation of the form,

$$\mathrm{d}x^a = \mu^a\,\mathrm{d}t + \sigma_i^a\,\mathrm{d}W_t^i, \tag{4.5}$$

where $\mathrm{d}x^a$ is the covariant Itô differential associated with the given connection (cf. Hughston 1996), and a summation is understood to be taken over the index i. For the characterization of the general diffusion process it suffices to specify a connection on $\mathfrak{M}$, and a metric is not required. The quadratic relation

$$\mathrm{d}x^a\,\mathrm{d}x^b = h^{ab}\,\mathrm{d}t, \tag{4.6}$$

where $h^{ab} = \sigma_i^a\sigma^{bi}$, follows from the Itô identities

$$\mathrm{d}t^2 = 0, \quad \mathrm{d}t\,\mathrm{d}W_t^i = 0 \quad \text{and} \quad \mathrm{d}W_t^i\,\mathrm{d}W_t^j = \delta^{ij}\,\mathrm{d}t.$$

Then for any smooth function $\phi(x)$ on $\mathfrak{M}$ we define the associated process $\phi_t = \phi(x_t)$, and Itô's formula takes the form,

$$\mathrm{d}\phi_t = (\nabla_a\phi)\,\mathrm{d}x^a + \tfrac{1}{2}(\nabla_a\nabla_b\phi)\,\mathrm{d}x^a\,\mathrm{d}x^b, \tag{4.7}$$

or, more explicitly,

$$\mathrm{d}\phi = (\mu^a\nabla_a\phi + \tfrac{1}{2}h^{ab}\nabla_a\nabla_b\phi)\,\mathrm{d}t + \sigma_i^a\nabla_a\phi\,\mathrm{d}W_t^i. \tag{4.8}$$

The probability law for x_t is characterized by a density function $\rho(x,t)$ on $\mathfrak{M}$ that satisfies the Fokker–Planck equation,

$$\frac{\partial\rho}{\partial t} = -\nabla_a(\mu^a\rho) + \tfrac{1}{2}\nabla_a\nabla_b(h^{ab}\rho). \tag{4.9}$$

The diffusion is said to be non-degenerate if h^{ab} is of maximal rank. In particular, if g_{ab} is a Riemannian metric on $\mathfrak{M}$ and ∇_a is the associated Levi-Civita connection, then if $h^{ab} = \sigma^2 g^{ab}$, the process x_t is a Brownian motion with drift on $\mathfrak{M}$, with volatility parameter σ. Apart from Brownian motion with drift, there are also other types of diffusion processes that can arise on a manifold. In particular, if the tensor h^{ab} is not of maximal rank, then the diffusion is degenerate, which is the case of relevance to the present consideration.

5. Stochastic reduction

Our intention in this article is to generalize the Schrödinger evolution of standard quantum mechanics to a stochastic process on a nonlinear quantum state manifold $\mathfrak{M}$. Specifically, we consider a stochastic reduction model of the type introduced by Hughston (1996), for which the dynamical trajectories are governed by the following stochastic differential equation:

$$\mathrm{d}x_t^a = (2\omega^{ab}\nabla_b H - \tfrac{1}{4}\sigma^2\nabla^a V^H)\,\mathrm{d}t + \sigma\nabla^a H\,\mathrm{d}W_t. \tag{5.1}$$

When $\mathfrak{M}$ is the state space $\mathit{\Gamma}$ of linear quantum mechanics, then (5.1) has the following interpretation. The first term in the drift generates the unitary part of the evolution, while the second term creates a tendency for the system to evolve to a state of lower energy variance. The volatility term is given by the gradient of the Hamiltonian, and generates fluctuations that die down as the system approaches an eigenstate. The parameter σ controls the magnitude of the fluctuations. Starting from any initial state, the state vector collapses to an energy eigenstate, with collapse probability given by the Dirac transition probability. The rate of collapse is determined by σ, for which a plausible choice is given by

$$\sigma^2 = \sqrt{G\hbar^{-3}c^{-5}} \sim (8\,\mathrm{MeV}^2\,\mathrm{s})^{-1}, \tag{5.2}$$

as discussed in Hughston (1996) and Adler & Horwitz (2000). If the evolution of the density function $\rho(x,t)$ associated with the process (5.1) is lifted to $\mathcal{H}$, then by a projection argument we recover the Lindblad form of the density matrix dynamics appropriate to an open system (Gisin 1989; Adler 2000; Wiseman & Diosi 2001).

In the nonlinear theory, there is no analogue of the density matrix, and a general mixed quantum state is characterized by a density function $\rho(x,t)$ on $\mathfrak{M}$. The dynamical evolution of a mixed state generated by the process (5.1) is governed by the Fokker–Planck equation (4.9), which in the present context takes the form,

$$\frac{\partial\rho}{\partial t} = -2\omega^{ab}\nabla_a\rho\nabla_b H + \tfrac{1}{4}\sigma^2\rho\nabla^2 V + \tfrac{1}{4}\sigma^2\nabla_a\rho\nabla^a V + \tfrac{1}{2}\sigma^2\nabla_a\nabla_b(\rho\nabla^a H\nabla^b H), \tag{5.3}$$

where ∇^2 is the Laplace–Beltrami operator on $\mathfrak{M}$.

In the case of a general nonlinear quantum phase space $\mathfrak{M}$, we obtain the following characterization of the dynamics of the energy.

Theorem 5.1. *The Hamiltonian process $H_t = H(x_t)$ is a martingale, given by the stochastic integral,*

$$H_t = H_0 + \sigma\int_0^t V_s\,\mathrm{d}W_s, \tag{5.4}$$

where $V_t = V^H(x_t)$.

Therefore, under the stochastic dynamics (5.1), the energy of the system is weakly conserved in the sense that the expectation of the value of the Hamiltonian at any future time is given by the initial value H_0. In particular, if reduction occurs, the martingale property ensures that the expectation of the terminal value of the energy, which is given by the sum of the energy eigenvalues weighted by the associated transition probabilities, is H_0. This in turn justifies the interpretation of $H(x)$ as the expectation of the energy in the given state $x \in \mathfrak{M}$. The proof of theorem 5.1 follows from an application of Itô's formula (4.8). Alternatively, the conservation of the energy expectation

$$\mathbb{E}[H(x_t)] = \int \rho(x,t)H(x)\,\mathrm{d}x \tag{5.5}$$

can be shown by a direct use of the Fokker–Planck equation (5.3).

The fact that H_t is a martingale does not in itself imply that reduction occurs. For a reduction to energy eigenstates, we require $\lim_{t\to\infty} V_t = 0$. To determine the circumstances under which this occurs, we consider the dynamics of V_t.

Lemma 5.2. *The process* $V_t = \nabla_a H \nabla^a H$ *satisfies*

$$\mathrm{d}V_t = \sigma^2 (\nabla^a H \nabla^b H \nabla^c H \nabla_a \nabla_b \nabla_c H)\,\mathrm{d}t + \sigma(\nabla^a H \nabla_a V)\,\mathrm{d}W_t. \tag{5.6}$$

The proof is as follows. We note that according to Itô's formula (4.8) we have

$$\mathrm{d}V_t = (2\omega^{ab}\nabla_a V \nabla_b H - \tfrac{1}{4}\sigma^2 \nabla_a V \nabla^a V + \tfrac{1}{2}\sigma^2 \nabla^a H \nabla^b H \nabla_a \nabla_b V)\,\mathrm{d}t \\ + \sigma \nabla^a H \nabla_a V\,\mathrm{d}W_t. \tag{5.7}$$

The first term in the drift vanishes by lemma 3.3. The remaining two terms in the drift combine to yield (5.6), because

$$\nabla^a H \nabla^b H \nabla_a \nabla_b V = 2\nabla^a H \nabla^b H \nabla^c H \nabla_a \nabla_b \nabla_c H + \tfrac{1}{2}\nabla_a V \nabla^a V. \tag{5.8}$$

For state reduction, we need to show that the drift of V_t is negative, and hence that V_t is a supermartingale. To obtain a suitable criterion for this we proceed as follows. If X_a is a Killing field, the cyclic identity $\nabla_{[a}\nabla_b X_{c]} = 0$ implies that

$$\nabla_c \nabla_a X_b = R_{abc}{}^d X_d, \tag{5.9}$$

where the Riemann tensor is defined by

$$\nabla_a \nabla_b A_c - \nabla_b \nabla_a A_c = -R_{abc}{}^d A_d \tag{5.10}$$

for any vector field A_a. As shown in Cirelli *et al.* (1990) we therefore have the following lemma.

Lemma 5.3. *If* $F(x)$ *is an observable on* $\mathfrak{M}$, *then*

$$\nabla_a \nabla_b \nabla_c F = R_{apbq} J^p_r \nabla^r F J^q_c. \tag{5.11}$$

Next, we define the holomorphic sectional curvature $\mathcal{K}_H$ of the Kähler manifold $\mathfrak{M}$ with respect to the J-invariant plane $\nabla^{[a} H J^{b]}_c \nabla^c H$ by the formula,

$$\mathcal{K}_H = -\frac{R_{apbq} J^p_c J^q_d \nabla^a H \nabla^b H \nabla^c H \nabla^d H}{(\nabla_a H \nabla^a H)^2}. \tag{5.12}$$

The meaning of $\mathcal{K}_H$ is as follows. At each point $x \in \mathfrak{M}$ we consider the tangent plane spanned by vectors $\nabla^a H$ and $J^b_c \nabla^c H$. Then the totality of the geodesic curves tangent to this plane at x forms a two-dimensional surface in $\mathfrak{M}$, and the Gauss curvature of this surface at x is $\mathcal{K}_H$. The minus sign appearing in the definition (5.12) arises as a consequence of our choice of conventions for the curvature tensor.

Theorem 5.4. *If* $\mathcal{K}_H > 0$, *then* V_t *is a supermartingale and (5.1) is a reduction process.*

The proof is by virtue of lemmas 5.2 and 5.3. Writing $\mathcal{K}_t = \mathcal{K}_H(x_t)$, we deduce that

$$V_t = V_0 - \sigma^2 \int_0^t \mathcal{K}_s V_s^2\,\mathrm{d}s + \sigma \int_0^t \nabla_a H \nabla^a V\,\mathrm{d}W_s, \tag{5.13}$$

and thus

$$\mathbb{E}[V_t|\mathcal{F}_s] \leqslant V_s, \tag{5.14}$$

the supermartingale condition. In particular, if we write $\bar{V}_t = \mathbb{E}[V_t]$, then it follows from (5.13) that

$$\frac{\mathrm{d}\bar{V}_t}{\mathrm{d}t} = -\kappa\sigma^2\bar{V}_t^2(1+\eta_t), \tag{5.15}$$

where

$$\bar{V}_t^2\eta_t = \mathbb{E}[(V_t-\bar{V}_t)^2] + \kappa^{-1}\mathbb{E}[(\mathcal{K}_t-\kappa)V_t^2], \tag{5.16}$$

and $\kappa = \inf_{\mathfrak{M}} \mathcal{K}_H$. Integrating, we obtain

$$\bar{V}_t = \frac{V_0}{1+\kappa\sigma^2 V_0(t+\xi_t)}, \tag{5.17}$$

where $\xi_t = \int_0^t \eta_s\,\mathrm{d}s$. The Hamiltonian sectional curvature is positive if and only if $\kappa > 0$, in which case $\xi_t \geqslant 0$. It follows that

$$\bar{V}_t \leqslant \frac{V_0}{1+\kappa\sigma^2 V_0 t}, \tag{5.18}$$

and thus $\lim_{t\to\infty} V_t = 0$ almost surely. Therefore, wave function collapse on the nonlinear state space is guaranteed if the holomorphic sectional curvature is positive. The characteristic reduction time-scale is

$$\tau = \frac{1}{\kappa\sigma^2 V_0}, \tag{5.19}$$

and for $t \gg \tau$ the uncertainty is reduced to a small fraction of its initial value.

6. Curvature and the energy uncertainty

Now we are in a position to determine the relationship between the initial energy uncertainty $V_0 = V^H(x_0)$ and the terminal variance of the energy as a result of a reduction. It follows from theorem 5.1, together with the Itô isometry, that

$$\mathbb{E}[(H_t-H_0)^2] = \sigma^2\bar{Q}_t, \tag{6.1}$$

where $H_0 = \mathbb{E}[H_t]$, $\bar{Q}_t = \mathbb{E}[Q_t]$, and $Q_t = \int_0^t V_s^2\,\mathrm{d}s$. On the other hand, if $\kappa > 0$, then from (5.13) we obtain

$$\bar{V}_t \leqslant V_0 - \kappa\sigma^2\bar{Q}_t. \tag{6.2}$$

Furthermore, if $\lambda = \sup_{\mathfrak{M}} \mathcal{K}_H$ and $\lambda > 0$, then (5.13) implies that

$$\bar{V}_t \geqslant V_0 - \lambda\sigma^2\bar{Q}_t. \tag{6.3}$$

Therefore, by theorem 5.4, if $\mathcal{K}_H > 0$, we obtain the following bounds for the terminal energy variance:

$$\frac{V_0}{\kappa} \geqslant \lim_{t\to\infty} \mathbb{E}[(H_t-H_0)^2] \geqslant \frac{V_0}{\lambda}. \tag{6.4}$$

In particular, if $\mathfrak{M} = \Gamma$, it follows from the relation (cf. Kobayashi & Nomizu 1969)

$$R_{abcd} = -\tfrac{1}{4}(g_{ac}g_{bd} - g_{bc}g_{ad} + \omega_{ac}\omega_{bd} - \omega_{bc}\omega_{ad} + 2\omega_{ab}\omega_{cd}) \tag{6.5}$$

that $\mathcal{K}_H = 1$ and V_0 is the terminal energy dispersion.

7. Commuting observables

An important issue in the consideration of state reduction processes is whether an energy-based dynamics suffices. To address this issue we examine the processes induced by (5.1) for observables other than H.

Theorem 7.1. *If an observable F commutes with the Hamiltonian, then under the stochastic Schrödinger dynamics (5.1) the process $F_t = F(x_t)$ is a martingale.*

The proof follows as a consequence of Itô's lemma with an application of lemmas 3.2 and 3.3. Theorem 7.1 generalizes a result of Adler & Horwitz (2000) obtained in the case $\mathfrak{M} = \Gamma$. To determine whether the system collapses to an eigenstate of F under (5.1) we require the concept of holomorphic bisectional curvature (Goldberg & Kobayashi 1967), which enters in a fundamental way into the following result.

Theorem 7.2. *If the observable F commutes with the Hamiltonian, then the stochastic equation for V_t^F is*

$$\mathrm{d}V_t^F = -\sigma^2 \mathcal{K}_{FH} V^F V^H \,\mathrm{d}t + \sigma \nabla_a H \nabla^a V^F \,\mathrm{d}W_t, \tag{7.1}$$

where the holomorphic bisectional curvature of $\mathfrak{M}$ with respect to the J-invariant planes determined by F and H is defined by

$$\mathcal{K}_{FH} = -\frac{R_{apbq} J_c^p J_d^q \nabla^a F \nabla^b H \nabla^c F \nabla^d H}{\nabla_a F \nabla^a F \nabla_b H \nabla^b H}. \tag{7.2}$$

To prove this result, we use Itô's formula (4.8) to obtain

$$\mathrm{d}V_t^F = \tfrac{1}{2}\sigma^2 (\nabla^a H \nabla^b H \nabla_a \nabla_b V^F - \nabla^a \nabla^b H \nabla_a H \nabla_b V^F)\,\mathrm{d}t + \sigma \nabla^a H \nabla_a V^F \,\mathrm{d}W_t. \tag{7.3}$$

Then, by use of lemmas 3.2, 3.3 and 5.3, a calculation shows that the two terms in the drift combine to give (7.1). In particular, if we substitute $V^F = \nabla_a F \nabla^a F$ into the drift terms and use lemma 3.2, we find that the drift in (7.3) takes the form

$$\sigma^2 (2\nabla^a H \nabla^b F \nabla^c H \nabla_a \nabla_b \nabla_c F - \nabla^a H \nabla^b F \nabla^c F \nabla_a \nabla_b \nabla_c H).$$

By use of lemma 5.3 and property (2.2) of the complex structure, we are then led to the desired expression involving the curvature tensor.

Theorem 7.3. *If the holomorphic bisectional curvature is positive, then for any observable F commuting with the Hamiltonian, the associated dispersion V_t^F is a supermartingale. If the critical points of H are non-degenerate, then (5.1) is a reduction process for F.*

In the linear theory, this result is intuitively expected. In particular, in the case of a non-degenerate Hamiltonian, the eigenstates of H also diagonalize any commuting observable F. We note that when $\mathfrak{M} = \Gamma$ we have

$$\mathcal{K}_{FH} = \tfrac{1}{2}(1 + \cos^2\theta), \tag{7.4}$$

where θ is the angle between the vectors $\nabla^a H$ and $\nabla^a F$. It follows that

$$\tfrac{1}{2} < \mathcal{K}_{FH} \leqslant 1 \tag{7.5}$$

for the linear theory, and thus the supermartingale condition is guaranteed.

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