Surface wrinkling of a channelized flow

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We consider an inclined rectangular duct of constant cross-section conveying viscous fluid and covered by an elastic plate. The fluid is described by the Stokes equations and the plate by the Föppl–von Kármán equations. The equations admit an equilibrium solution in which the plate is flat and fluid flows underneath due to gravity. This base flow induces a varying traction across the plate, which can lead to out-of-plane buckling due to the associated in-plane shear. Linear stability analysis demonstrates that buckling occurs for sufficiently thin plates on steep slopes and deep channels. The most unstable modes take the form of either symmetric or antisymmetric downslope-directed chevron patterns that travel downstream at a fraction of the average speed of the base flow. An analogous analysis shows that similar buckling also occurs if the elastic plate is replaced by a thin skin of very viscous fluid. Our description provides a simple model for the formation of ropy pahoehoe lava.

Keywords: elastic plate; buckling; lava flow; ropy pahoehoe

1. Introduction

The interaction of a fluid with an overlying elastic ‘skin’ (or, indeed, any rheologically distinct superficial layer) can generate a variety of buckling patterns. Common examples include the flow-induced wrinkling of the thin skins formed atop hot milk, wax or crème anglaise. Likewise, it has been suggested that motion of the molten interior of a lava flow can buckle the overlying solidified crust, creating a characteristic ‘ropy’ appearance (Fink & Fletcher 1978; figure 1). In these examples, the fluid flow plays an essential role in forming the wrinkles of the skin by exerting the stresses that promote buckling. However, the role played can be more passive, such as in semiconductor manufacturing.
in which an elastic plate floating on a viscous layer is compressed by external forces, with the fluid merely controlling the time scale of buckling (Huang & Suo 2002).

The goal of the current article is to explore a model problem in which fluid flow directly induces buckling. More specifically, we consider the gravity-driven flow of a viscous fluid through a rectangular duct overlain by a thin elastic plate. Because the flow is driven by gravity alone, without a pressure gradient, an equilibrium state can be established in which the surface skin remains flat. This equilibrium and its linear perturbations can be described with relative mathematical simplicity, in contrast to pressure-driven flows that generally form conduits that taper in the streamwise direction. The fluid traction acting on the base of the skin varies across the duct and generates shear within it, potentially inducing out-of-plane buckling when tractions are sufficiently strong. This configuration is one of the simplest idealizations of the particular fluid–structure interaction problem in question, and bears similarities to the buckling of an isolated elastic plate under gravity or externally imposed shear (Balmforth et al. 2008).

To model the configuration, we treat the viscous fluid with the Stokes approximation (valid for relatively slow flows) and represent the elastic plate as a thin Hookean isotropic elastic solid satisfying the Föppl–von Kármán equations (Love 1944). The latter are the simplest plate equations that incorporate both the bending and compressional terms necessary to account for out-of-plane buckling. The choice of this model for the skin is not essential, and we supplement our study of the elastically plated duct with an exploration of a related problem in which the skin is composed of a yet-more viscous, thin immiscible fluid (see also Teichman 2002).

Despite the common occurrence of flow-induced buckling of a superficial skin, relatively few preceding studies exist on the problem. Perhaps the closest earlier work to our current study is by Luo & Pozrikidis (2006, 2007). They considered the buckling of an arbitrarily shaped elastic section in an otherwise rigid plate.
suspended in flowing fluid. Compression and shear are induced in the elastic section by the flow, resulting in a complicated buckling instability. However, although the elastic portion of the plate deforms out of its plane on buckling, these authors do not incorporate the resulting feedback on the fluid dynamics. This simplifies the analysis, but alters the nature of the instability since fluid pressure variations cannot provide a restoring force.

The paper is structured as follows. In §2, we formulate the problem and note the assumptions required. In §3, we present the flat base state profile about which we perturb. We analyse the linear stability numerically in §4. A brief comparison of this theory with a qualitative experiment is presented in §5. In §6, we consider the particular geological application of ropy pahoehoe lava. Finally, in §7, we summarize our most significant findings. Appendix A furnishes the derivation of the Föppl–von Kármán equations in the present context and outlines their limits of validity. Appendix B briefly discusses the supplementary problem in which the elastic skin is replaced by a very viscous fluid one.

2. Formulation

As shown in figure 2, we consider a duct of infinite length and rectangular cross section (having width $2y_0$ and depth $2\alpha y_0$, where $\alpha$ is the aspect ratio) inclined at angle $\theta$ to the horizontal. The duct is described by a Cartesian coordinate system with $\hat{x}$ directed downslope, $\hat{y}$ across the slope and $\hat{z}$ perpendicular to the slope; the origin is located on the centre line. The bottom, $\hat{z} = -\alpha y_0$, and sides, $\hat{y} = \pm y_0$, are rigid; the top, $\hat{z} = \alpha y_0 + \zeta(\hat{x}, \hat{y}, \hat{t})$, is an isotropic Hookean elastic plate of thickness $2d \ll 2y_0$, Young’s modulus $E$ and Poisson ratio $\nu$, where $\zeta$ is the out-of-plane displacement of the initially flat plate. An incompressible, inertialess Newtonian fluid of density $\rho$ and viscosity $\mu$ fills the duct and flows downslope under gravity.

The fluid motion is described by Stokes’ equations

$$\nabla \cdot \hat{\sigma} = \rho g (-\sin \theta, 0, \cos \theta), \quad \hat{\sigma} = -\hat{p} I + \mu (\nabla \hat{u} + (\nabla \hat{u})^\top), \quad \nabla \cdot \hat{u} = 0, \quad (2.1)$$

where $\nabla = (\partial / \partial \hat{x}, \partial / \partial \hat{y}, \partial / \partial \hat{z})$, $\hat{\sigma}$ is the stress tensor, $g$ is the acceleration due to gravity, $\hat{p}$ is the pressure, $\hat{u}$ is the velocity, $I$ is the identity tensor, and $^\top$ denotes the transpose. On the rigid lower and side surfaces of the duct, we impose no-slip,

$$\hat{u} = 0 \quad \text{on} \quad \hat{y} = \pm y_0 \quad \text{and on} \quad \hat{z} = -\alpha y_0. \quad (2.2)$$

Figure 2. An inclined rectangular duct with rigid base and sides and a thin elastic plate as the top surface. Fluid supplied far upslope fills the channel and flows downslope under the influence of gravity alone.
The strains in the elastic plate are assumed to be small so that the plate may
be described by the Föppl–von Kármán equations in Eulerian coordinates. In
appendix A, we give a discussion of the underlying assumptions that are required
for these equations to apply. We also assume that the weight of the plate is
negligible compared with fluid tractions acting upon it and that elastic waves are
much faster than the adjustment time scale of the coupled elastic-fluid system so
that the plate is in instantaneous equilibrium. Thus, the in-plane and out-
of-plane force balances are given by
\[
\hat{\mathbf{P}}_h \cdot \hat{\mathbf{N}} = \hat{\mathbf{r}}_h, \quad \frac{2d^3 E}{3(1-\nu^2)} \hat{\mathbf{P}}^4_h \hat{\mathbf{z}} = -\hat{\mathbf{r}}_z + \hat{\mathbf{P}}_h \cdot (\hat{\mathbf{N}} \cdot \hat{\mathbf{P}}_h \hat{\mathbf{z}}),
\]
respectively, where the subscript \( h \) denotes in-plane (\( \hat{x} \) and \( \hat{y} \)) components,
\( \hat{\mathbf{P}}_h = (\partial / \partial \hat{x}, \partial / \partial \hat{y}) \), and \( \hat{\mathbf{r}} = (\hat{r}_h, \hat{r}_z) \) is the fluid traction acting on the base of
the plate. Here, the in-plane stresses (integrated over the thickness) and strains are
\[
\hat{\mathbf{N}} = \frac{2dE}{1-\nu^2} [\nu \text{tr}(\hat{\mathbf{e}})] l_h + (1-\nu) \hat{\mathbf{e}}, \quad \hat{\mathbf{e}} = \frac{1}{2} (\hat{\mathbf{P}}_h \hat{\mathbf{\xi}}^i + \hat{\mathbf{P}}_h \hat{\mathbf{\xi}}^j + \hat{\mathbf{P}}_h \hat{\mathbf{\xi}}^k \hat{\mathbf{P}}_h \hat{\mathbf{\xi}}),
\]
respectively, where \( \hat{\mathbf{\xi}} = (\hat{\xi}, \hat{\eta}) \) is the in-plane displacement and \( \text{tr}(\cdot) \) is the trace.
The terms in the second equation of (2.3) represent a resistance to bending, out-
of-plane forcing by the fluid and a buckling term coupling in-plane stresses with
out-of-plane displacement. The plate is clamped along its lateral edges to the
sides of the duct,
\[
\hat{\mathbf{\xi}} = \hat{\eta} = \hat{\mathbf{\xi}} = \frac{\partial \hat{\mathbf{\xi}}}{\partial \hat{\eta}} = 0 \quad \text{on } \hat{\eta} = \pm y_0.
\]

The plate displacement and fluid velocities are connected through the
kinematic boundary condition,
\[
\hat{u} = \hat{v} = 0, \quad \hat{w} = \frac{\partial \hat{\mathbf{\xi}}}{\partial \tau} \quad \text{on } \hat{z} = \alpha y_0 + \hat{\mathbf{\xi}},
\]
where \( \hat{\tau} \) is time. We have neglected the \( \hat{u} \cdot \hat{\mathbf{v}} (\hat{\xi}, \hat{\eta}, \hat{\mathbf{\xi}}) \) and time-derivative terms in
the in-plane equations because the in-plane displacements must be small relative
to the out-of-plane displacements (essential scalings for deriving the Föppl–von
Kármán equations, as given in (A 4) in appendix A) whereas the three velocity
components are all of similar order. The traction on the plate \( \hat{\mathbf{r}} \) is exerted by fluid
shear stresses as follows:
\[
\hat{\mathbf{r}} = \hat{\sigma} \cdot \hat{\mathbf{n}} \quad \text{on } \hat{z} = \alpha y_0 + \hat{\mathbf{\xi}},
\]
where \( \hat{n} \) is the normal to the plate.

\( (a) \) Non-dimensionalization

We non-dimensionalize the governing equations based on the half-width of the
channel and the gravitational driving force as follows:
\[
\begin{align*}
\hat{u} &= u \rho g y_0^2 \sin \theta / \mu, \quad \hat{p} = p \rho g y_0 \sin \theta, \quad \hat{\sigma} = \sigma \rho g y_0 \sin \theta, \quad \hat{\mathbf{r}} = \tau \rho g y_0 \sin \theta, \\
\hat{\mathbf{\xi}} &= \xi y_0, \quad \hat{\mathbf{\xi}} = \zeta y_0, \quad \hat{\mathbf{N}} = N \rho g y_0^2 \sin \theta, \quad \hat{\mathbf{e}} = \mathbf{e}, \quad \hat{x} = x y_0, \quad \hat{\tau} = t \mu / \rho g y_0 \sin \theta.
\end{align*}
\]
Subject to

\( v \)

with

\( \theta \)

The dimensionless parameters are the inclination of the duct \( \theta \), the aspect ratio \( a \), the Poisson ratio of the plate \( \nu \),

\[
G = \frac{3(1-\nu^2)p_g y_0^4 \sin \theta}{2d^3 E},
\]

measuring the relative importance of gravity-induced fluid forcing to bending stiffness and \( G = G d^2/3y_0^3 \). \( G \) is large when the plate is relatively flimsy, and it is small when the plate is relatively rigid. \( G \) measures how much strain is introduced in the base state by the in-plane stresses. Because linear instability is controlled by these stresses, rather than the strains, \( G \) will not feature in the stability analysis.

3. Base state

Our base state (denoted by subscript 0) is driven by gravity alone; the plate is flat and the flow is steady and unidirectional. The flow profile is (Rosenhead 1963)

\[
2u_0(y, z) = \alpha^2 - z^2 - 4\alpha^2 \left(\frac{2}{\pi}\right)^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^3} \frac{\cosh\left(\frac{(2n+1)\pi y}{2a}\right) \cos \left(\frac{(2n + 1)\pi z}{2a}\right)}{\cosh\left(\frac{(2n+1)\pi}{2a}\right)},
\]

with \( v_0 = w_0 = 0 \) and \( p_0 = (\alpha - z) \cot \theta \).

The fluid flow imposes a traction on the plate, causing downstream displacement according to

\[
\frac{(1-\nu)}{G} \xi_0(y) = \alpha(1-y^2) - 2 \frac{3}{\alpha} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{4\alpha^3}{(2n + 1)^4} \frac{\cosh\left(\frac{(2n+1)\pi y}{2a}\right)}{\cosh\left(\frac{(2n+1)\pi}{2a}\right)},
\]

with \( \eta_0 = \zeta_0 = 0 \). For convenience, we define the scaled shear stress,

\[
S(y, \alpha) = \frac{(1-\nu)}{G} \frac{\partial \xi_0}{\partial y} = -2\alpha y + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{4\alpha^2}{(2n + 1)^3} \frac{\sinh\left(\frac{(2n+1)\pi y}{2a}\right)}{\cosh\left(\frac{(2n+1)\pi}{2a}\right)}.
\]
For shallow ducts (small $\alpha$), the flow becomes independent of $y$ away from the edges; the corresponding shear profile is $S(y, \alpha) \sim -2\alpha y$. For deep ducts (large $\alpha$), the flow close to the plate becomes independent of the depth of the duct and the corresponding shear profile becomes independent of $\alpha$. Sample velocity profiles and displacements for the base state are shown in Figure 3.

4. Linear stability

To examine the stability of the base state, we perturb each of the variables about that equilibrium, $f = f_0 + f_1$, where $f$ denotes any of $u$, $v$, $w$, $p$ and $\zeta$, and the subscript 1 refers to the perturbation. We then linearize in the perturbation amplitudes $f_1$ and calculate the growth rate, $\omega = \omega_r + i\omega_i$, of infinitesimal normal mode perturbations with downstream wavenumber $k$: $f_1 = f_1(y, z)\exp(ikx - i\omega t)$. This furnishes the eigenvalue problem

$$
-k^2u_1 + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} = ikp_1,
$$

$$
-k^2v_1 + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} = \frac{\partial p_1}{\partial y},
$$

$$
-k^2w_1 + \frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2} = \frac{\partial p_1}{\partial z},
$$

$$
i ku_1 + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0,
$$

with boundary conditions

$$
\begin{align*}
    u_1 &= v_1 = w_1 = 0 & \text{on } y = \pm 1 & \text{and on } z = -\alpha, \\
    u_1 + \frac{1}{2} \frac{\partial S}{\partial y} \zeta_1 &= v_1 = w_1 + i\omega \zeta_1 = 0 & \text{on } z = \alpha,
\end{align*}
$$

Figure 3. The base state (a) downslope velocity profile $2u_0$ for $\alpha = 1$ (contours at intervals of 0.1 from 0.1) and (b) downslope plate displacement profile $(1 - r)\zeta_0/G$ for $\alpha = 0.1, 0.5, 1, 2$ and 10.
where
\[
\frac{1}{\mathcal{G}} \left( \partial_y \zeta_1 - 2k \frac{\partial^2 \zeta_1}{\partial y^2} + k^4 \zeta_1 \right) = p_1 - \zeta_1 \cot \theta + ikS \frac{\partial \zeta_1}{\partial y}
\]
(4.7)
and
\[
\zeta_1 = \frac{\partial \zeta_1}{\partial y} = 0 \quad \text{on} \quad y = \pm 1.
\]
(4.8)

Note that the in-plane displacement equations are no longer needed, and that we have used the in-plane force balance (the first equation of (2.12)) to replace \( \partial u_0/\partial z \big|_{z=\alpha} \) with \((1/2)\partial S/\partial y\) in the kinematic condition (the first equation of (4.6)).

The out-of-plane plate equation (4.7) regulates instability: the first three terms describe the stabilizing effects of bending stiffness, fluid pressure and hydrostatic restoring force, respectively, whereas the final term describes the destabilizing effect of differential fluid traction on the base of the plate.

We solved the systems (4.1)–(4.8) numerically using a Chebyshev collocation scheme (Trefethen 2000) with the boundary conditions imposed explicitly (Weideman & Reddy 2000) and \( u_1 \) and \( p_1 \) eliminated. The resulting generalized eigenvalue problem was solved using the QZ algorithm. We found that employing an equal number of collocation points (between 24 and 32) in each direction yielded reasonable convergence rates for most values of \( \alpha \).

The dispersion relations for the four least-stable modes for a square duct on a 45° slope (\( \alpha = \cot \theta = 1 \)) and \( \mathcal{G} = 1000/\sqrt{2} \) are plotted in figure 4; associated contour plots of out-of-plane displacements for a selection of wavenumbers are provided in figure 5. For \( k \to 0 \), the dominant mode \( E_2 \) corresponds to over- or under-filling the channel uniformly in \( x \) and travels faster than the base state flow. At intermediate and large wavenumbers, an even \( E_1 \) and odd \( O_1 \) pair of modes dominate, and for a range of intermediate wavenumbers both are unstable. These modes are destabilized by the traction exerted by the fluid on the base of the plate. They are stabilized at large wavenumber by bending stiffness and at small wavenumber by a combination of cross-stream bending stiffness and the restoring pressure force of the fluid. The out-of-plane displacements for these modes take the form of patterns of downslope-directed chevrons with crests having an appreciable angle to the axis of the channel, despite the base displacement being small (cf. the shear-induced buckles of an isolated elastic plate; Mansfield 1964; Wong & Pellegrino 2006; Balmforth et al. 2008). At smaller \( k \), the odd mode (figure 5b) with fewer cross-slope oscillations is favoured, while at larger \( k \) the effect of shear is increased and the even mode (figure 5a), with more oscillations in \( y \) (augmenting the destabilizing shear), is favoured. At large wavenumber, the eigenfunctions become concentrated near the sidewalls where there is maximum shear. Little deflection occurs near the centre-line, with the result that the flows to either side of the channel do not interact and the growth rates of the dominant odd and even pair become indistinguishable. These unstable waves propagate downslope at a fraction of the average base state velocity. A multitude of other modes exist with decreasing growth rates, as illustrated by the additional odd mode \( O_2 \) in figures 4 and 5d. These modes are similar to the dominant modes, but with an increasing number of oscillations in \( y \).

The growth rates and phase speeds of the most unstable mode are shown on the \((k, \mathcal{G})\)-plane for the same square duct (with \( \alpha = \cot \theta = 1 \)) in figure 6. As \( \mathcal{G} \) increases, the resistance of the plate to bending decreases and the flat base state
becomes unstable above a critical value $G \geq G_c$ over an expanding window of wavenumbers. The fourth-order form of the bending stiffness ensures that this term is large even for moderate wavenumbers and cross-stream scales of variation, so the plate must be very flimsy ($G$ must be very large) to ensure instability. Beyond $G_c$, the most unstable modes are found at increasingly shorter wavelengths and have slowly decreasing wavespeeds. A selection of neutral stability curves for different slope angles and aspect ratios is shown in figure 7. For steeper slopes, onset occurs for smaller driving forces and longer waves, because the component of gravity driving the underlying shear flow is increased and, to a lesser extent, because the stabilizing hydrostatic pressure is reduced. As the channel becomes deeper, larger velocities can be generated with associated larger shears on the base of the plate. Onset thus occurs at smaller driving force and for longer waves. However, the behaviour becomes independent of the aspect ratio once $\alpha$ exceeds two or so.

The behaviour of the critical value $G_c$ as a function of slope angle and aspect ratio is shown in figure 8. Figure 8a also displays contours of growth rate and wavenumber for the most unstable mode above onset with $\alpha = 1$.

Finally, figure 9 presents mode profiles for the most unstable mode (here $E_1$) and the most unstable wavenumbers for the $G$-values marked in figure 6. The characteristics of the out-of-plane displacement are identical to those described in figure 7. The figure also highlights the increasing concentration of the eigenfunctions to the regions near the sidewalls with increasing $G$. At onset, the perturbation to the velocity field is appreciable throughout the depth of the channel, but when the growth rates are larger only the uppermost fluid layer responds to the plate motion.

(a) Local analysis

A crude short-wavelength-style analysis can be used to complement the numerical stability theory. We assume that the perturbations vary more rapidly in $y$ than the background state and decompose the perturbations as $f_1 = f_1(z) \exp(ikx + my - \omega t)$. We then integrate the fluid equations through the depth of the flow and impose the boundary conditions at $z = \pm \alpha$ to obtain a
Figure 5. The out-of-plane displacements for the four modes (a) $E_1$, (b) $O_1$, (c) $E_2$ and (d) $O_2$ for $\alpha=\cot \theta = 1$ and $G = 1000/\sqrt{2}$ and (i) $k=0.5$, (ii) $k=1$, (iii) $k=2$ and (iv) $k=8$ as indicated by solid dots in figure 4a. Contours are at intervals of 0.2 of the maximum. Lighter (darker) curves indicate positive (negative) displacements. A shaded panel indicates the least stable mode for a given wavenumber.

\[
\omega = \frac{-k \sinh^2(2\alpha \kappa) \frac{\partial S}{\partial y} + i[4\alpha^2 \kappa^2 - \sinh^2(2\alpha \kappa)]\chi}{4\alpha \kappa + \sinh(4\alpha \kappa)}, \tag{4.9}
\]

Figure 6. Growth rates and wave speeds on the \((k, \mathcal{G})\)-plane for \(\alpha = \cot \theta = 1\). (a) Properties of the growth rate: thin solid curves are contours of \(\omega_i\) drawn at \(-500, -50, -5, -0.5, -0.05, 0.05, 0.5\) and 5; the bold curve indicates neutral stability. The dashed bold curve locates the most unstable wavenumber for given \(\mathcal{G}\). Dotted lines are local analysis approximations for the long- and short-wave cut-offs and the dash-dotted line is the most unstable wavenumber. (b) Properties of the wave speed: thin curves are contours of \(c = \omega_i / k\) drawn at intervals of 0.02 from 0.01. The dash-dotted curves give the approximate locations of a change in the dominant mode, where the wavespeed is discontinuous (this jump is not indicated where it is relatively small).

Figure 7. Neutral stability curves (solid) for (a) \(\cot \theta = 1\) and \(\alpha = 0.1, 0.5, 1\) and 2; (b) \(\alpha = 1\) and \(\cot \theta = 0.1, 0.5, 1, 2\) and 10. Dotted curves are the corresponding local analysis approximations.
where $k^2 = k_2^2 + m^2$ and $\chi = \kappa^4/G + \cot \theta + kmS$. Because $S$ is a function of $y$, this relation can only be viewed as a short-wavelength approximation, suitable when the length scale $m^{-1}$ is much shorter than the channel width.

To estimate the neutral stability curves, we look for the smallest value of $G$ that gives $u_i = 0$ over all possible $m$ and $y$. This minimization requires the value of $S_m(\alpha) = \max_y |S(y, \alpha)| = |S(\pm 1, \alpha)|$, which is plotted in figure 10. Taking the value at the sidewalls (although the short-wavelength form does not satisfy the boundary conditions there) provides a crude lower bound on the neutral stability curves: solutions of this algebraic problem are included in figures 6 and 7 for comparison with

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Figure 8. (a) Properties of the least stable mode on the $(\theta, G)$-plane for $\alpha = 1$. Solid curves are contours of growth rate, drawn at $0.1 \times 2^j$, $j = 1, \ldots, 4$; the bold curve is the neutral stability curve. Short-dashed curves are contours of wavenumber, drawn at intervals of 2 from $k = 6$. (b) Neutral stability curves in $(\theta, G)$ space for $\alpha = 0.1, 0.5, 1$ and 5.

Figure 9. Plots of the most unstable mode $E_1$ at the discs marked in figure 6 ((a) $G = 238$, (b) $G = 1000/\sqrt{2}$ and (c) $G = 2000/\sqrt{2}$; with $\alpha = \cot \theta = 1$). The top row of panels shows the cross-slope velocity field $(v, w)$ and out-of-plane displacement on the $(y, z)$-plane at $x = 0$. The lower row of panels shows contours of the out-of-plane displacement on the $(x, y)$-plane (with contour levels at intervals of 0.2 of the maximum).
the numerical data. It performs best at the lowest and the highest wavenumbers where the eigenfunction possesses short-wavelength structure in $y$ (as demanded by the local theory). For small $k$ and large $G$, $m \sim k^{-1}$ and the approximate long-wave cut-off is

$$k \approx \frac{4(\cot \theta/3)^{3/4}}{S_m(\alpha)} G^{-1/4},$$

reflecting an out-of-plane balance of forces between cross-stream bending stiffness, pressure and shear. At large $k$, $m \sim k$ and the short-wave cut-off is given by

$$k \approx (3/4)^{3/4} [G S_m(\alpha)/2]^{1/2},$$

via a balance between bending stiffness and shear. The critical value of $G$ is poorly predicted by the local analysis because $m$ is not large there.

For $k \gg 1$, the local analysis also predicts that the maximum growth rate occurs for $m = -k \text{sign } S$ (i.e. wavy perturbations with crests aligned at $45^\circ$ and orientated to take advantage of the destabilizing effect of the local shear), implying a most unstable wavenumber,

$$k \approx [G S_m(\alpha)/8]^{1/2},$$

which is also included in figure 6. Overall, the local analysis offers a useful guide to the stability characteristics, but is only quantitatively accurate at large $k$.

5. Qualitative experiments

To provide qualitative verification of the theoretical predictions, we performed a suite of experiments. A 7.6 cm wide, 2.6 cm deep and 1.2 m long channel was mounted on an inclinable table (the achievable fluid flux effectively limited us to an operating range of $(0^\circ, 35^\circ)$). ‘Thera-Band’ latex exercise bands were placed on top of the channel to form an elastic skin. Four bands of different thickness were used and were clamped into place using slats. An elongation test indicated that Young’s modulus for all bands was in the range $2 \pm 0.4$ GPa. The sheets used were colour coded: tan ($2d = 0.12 \pm 0.02$ mm), yellow ($2d = 0.15 \pm 0.02$ mm), red ($2d = 0.20 \pm 0.02$ mm) and green ($2d = 0.25 \pm 0.02$ mm). At the upper end of the channel, a reservoir and a solid top plate were inserted (the latter to help generate the desired base state). Golden syrup ($\rho = 1.4$ g cm$^{-3}$; $\mu = 2.6 \pm 0.2$ or $3.4 \pm 0.2$ Pa s at 20°C) was supplied to the reservoir, and the flux through the duct was controlled manually and made as constant as possible using a valve.

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Figure 10. Maximum magnitude of the shear $S$ over $y$ as a function of $\alpha$. 

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For $\theta \leq 10^\circ$, experiments could be maintained for several minutes with the help of a pump; for $\theta > 10^\circ$, experiments could be maintained for at least 30 s. Still images of a deflected laser beam at 1/3 s intervals permitted measurements of the wavelength and wavespeed. The experimental set-up is shown in figure 11a.

With this arrangement, we successfully observed wrinkling of the latex sheet due to the underlying fluid flow: a sample pattern is shown in figure 11b. Moreover, the wrinkles propagated downstream at speeds that were a small fraction of the average fluid speed. Overall, the observed properties of the wrinkling patterns and the trends on varying the inclination and the thickness of the latex sheet were consistent with the theoretical predictions: for experiments performed at angles below a critical value, the sheet appeared flat; at higher inclination, buckles were observed. As expected from theory, the critical value was of the order of tens of degrees and increased with increasing thickness of the elastic sheet. Similarly, the observed wavelengths were of the order of centimetres and decreased with increasing slope and with decreasing thickness of the elastic sheet, whereas the wavespeed increased with slope. Distinct chevron patterns much like the linear eigenfunctions were observed, although the maxima were generally located near the edges and rarely along the centre line. Frequently, the maxima on either side were not in phase suggesting a competition between even and odd modes of instability, exactly as in the theory.

Despite this qualitative success, we were unable to provide a quantitative validation of the theory because we found significant problems regarding reproducibility: experiments with apparently identical set-up had variations as much as $10^\circ$ in onset angle. We believe that the problem originates from some initial compression or tension of the sheet, which is almost impossible to avoid and is unquantifiable. A similar, although less serious, issue arose in our previous experiments with an isolated sheared elastic sheet (Balmforth et al. 2008). There, theory predicted that displacements of the plate in $y$ by fractions of a millimetre could appreciably shift onset. We therefore anticipate that the presence of tiny amounts of compression or tension can significantly alter the shear instability, with compression accentuating it and tension eliminating it.
6. Geological application

The surface of ‘ropy pahoehoe’ lava flows has a characteristic corrugated appearance consisting of folds a few centimetres in amplitude and a few centimetres to tens of centimetres in wavelength (Fink & Fletcher 1978), resulting from an imposition of stresses on the cooled, rheologically distinct upper crust. Based on field studies of solidified flows and movies of active ones, Fink & Fletcher (1978) suggest that wrinkles are formed where a flow encounters a constriction or a sudden change in slope, and the faster moving lava advects upstream crust into crust near the obstacle to compress and buckle it. Beyond the generation point, the flow rotates the wrinkles into their characteristic parabolic shape.

A possible alternative mechanism is suggested by the current study: the corrugations are a manifestation of shear-induced wrinkling. Although our model is a crude approximation of a real lava flow, it does capture the most fundamental features. Regular wrinkles are particularly formed on channelized lava flows bordered by ‘levees’ (Fink & Fletcher 1978; Garry et al. 2006), providing some justification for our geometry. From a rheological perspective, although lava is a heterogeneous fluid with significant non-Newtonian, temperature-dependent rheology (Griffiths 2000), a number of observations indicate that our approximation is reasonable at leading order. First, the formation of the upper boundary layer (solid or otherwise) insulates the underlying flow and the temperature in the interior is essentially constant and uniform (Hon et al. 1994). Second, ropes form preferentially relatively close to the vent where the bubble and crystal contents are comparatively low, and so the interior fluid may be approximately Newtonian. The overlying crust is observed to be a visco-elastic shell, with a brittle upper casing and a high-viscosity lower cushion (Hon et al. 1994). Our elastic plate provides one possible idealization of this crust (cf. Iverson 1990); we also provide a corresponding analysis for a very viscous fluid plate in appendix B (cf. Biot 1961; Fink & Fletcher 1978). The two models share many common features and we concentrate on the former here. We note that the most immediate shortcoming of the model is that it enforces conservation of crust material, whereas stretching and fracture in conjunction with solidification of freshly exposed lava can generate new crust in response to shear. Fracture at the plate edges also allows the underlying flow to advect the crust, modifying the basic flow and the lateral boundary conditions on the plate. This advection modifies the details of the base flow and the lateral boundary conditions on the plate; our model is not directly applicable to this case.

For the lava flows observed by Fink & Fletcher (1978), \( \theta = 5^\circ \), \( 2y_0 \geq 2.5 \text{ m} \), \( 2d \leq 5 \text{ cm} \) and the wavelength is of the order of 10 cm. Taking representative values \( \alpha = 0.5 \) (Calvari et al. 1994), \( E = 95 \text{ GPa} \), \( v = 0.27 \) (Schilling et al. 2003) and \( \rho = 3000 \text{ kg m}^{-3} \), we predict that a least-stable wavelength of 10 cm requires a skin thickness of order 0.2 mm. This prediction is appreciably thinner than estimated for the actual flow, but nevertheless is geologically reasonable (Hon et al. 1994). Thus, buckling by shear alone is a possible explanation for ropy pahoehoe formation, although the theoretically predicted requirements are at the limits of what is observed. The mode shapes are also in qualitative agreement with the field observations (compare figures 1 and 9).

It is also worth noting that there is a considerable experimental literature on laboratory analogue flows using cooling PEG wax (Griffiths et al. (2003) and Garry et al. (2006) and references therein). Qualitatively, the features produced by these
extruded, cooling flows of wax mimic those of real lava flows: they form levees, become channelized, and once a solidified skin forms may develop surface texture and regular folding. A second laboratory analogue flow is the intrusion of a salty gravity current into a less dense surfactant solution. At the interface between the two fluids, a micellar gel is formed that is contorted by the dynamics of the underlying flowing fluid in a manner resembling an active lava flow (Clayton 2004). Our predicted mode shapes are in qualitative agreement with both types of laboratory flows.

7. Conclusions

In this article, we have demonstrated that an elastic plate can be buckled through traction exerted by an underlying fluid shear flow. More precisely, we have explored linear buckling instabilities in an elastic plate clamped over an inclined duct filled with fluid flowing under gravity alone. The varying fluid traction across the top of the duct leads to a destabilizing in-plane shear in the plate, and we have mapped out the conditions under which this effect drives buckling and classified its character. Instability is most readily observed for thin sheets on steep slopes and deep channels, and the most unstable modes take the form of downslope-directed chevron patterns. These patterns resemble the structures seen on the crusts of pahoehoe lava flows, leading us to explore whether crustal shear due to underlying lava flow could be responsible for their formation.

We also successfully conducted a suite of simple experiments to confirm the theory qualitatively. Unfortunately, the experiments were plagued by technical complications. In particular, an uncontrollable amount of slack or tension was probably introduced in the elastic plate when it was initially clamped into place. The resulting lateral compression swamps the shear-induced instability and prevents any quantitative comparison with theory. To surmount this issue, a more sophisticated experiment is needed, which we leave for future work.

Although our main focus has been the buckling of an elastic plate, we have also given a brief discussion of the corresponding problem when the fluid is coated by a thin film of much more viscous fluid. Such a skin can also be buckled by an underlying fluid shear flow, giving another example of G. I. Taylor’s analogy between an elastic plate and a viscous sheet.

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Appendix A. Derivation of the Föppl–von Kármán equations

This appendix provides an asymptotic derivation of the Föppl–von Kármán equations for the elastic plate from the three-dimensional equilibrium equations. The analysis is similar to that of Ciarlet (1979; see also Antman 1995) and we omit details for brevity; novelties are the inclusion of non-dead loads and the discussion of larger displacements.
We return to dimensional variables. The plate is described by Lagrangian coordinates $\mathbf{X}$ for the unstressed reference configuration and Eulerian coordinates $\hat{\mathbf{x}} = \mathbf{X} + [\xi(\mathbf{X}), \zeta(\mathbf{X})]$ for the deformed configuration. The equations are derived in Lagrangian coordinates before being converted into an Eulerian form.

Conservation of momentum is given by

$$\nabla_{\mathbf{X}} \cdot \dot{\mathbf{S}} = 0, \quad \text{(A 1)}$$

where subscript $\dot{\mathbf{X}}$ denotes Lagrangian derivatives; $\dot{\mathbf{S}} = \dot{\Sigma} \hat{\mathbf{F}}^\dagger$ is the first Piola–Kirchhoff stress tensor, $\dot{\mathbf{F}} = \nabla_{\mathbf{X}} \hat{\mathbf{x}}$ is the deformation gradient tensor, and $\dot{\Sigma}$ is the second Piola–Kirchhoff stress tensor. We assume that the material is Hookean, with linear, isotropic constitutive relation

$$\dot{\Sigma} = 2\mu \dot{\varepsilon} + \lambda \operatorname{tr}(\dot{\varepsilon}) \mathbf{I}, \quad \text{(A 2)}$$

where $\mu$ and $\lambda$ are Lamé constants and $\dot{\varepsilon} = (\dot{\mathbf{F}}^\dagger \dot{\mathbf{F}} - \mathbf{I})/2$ is the strain tensor.

Boundary conditions describing fluid forcing of the lower surface and a free upper surface are

$$\mathbf{S}^\dagger \cdot \mathbf{N} = \begin{cases} 0, & \hat{Z} = d, \\ \hat{\mathbf{r}}(\hat{\mathbf{x}}) \frac{d\mathbf{s}}{d\mathbf{S}}, & \hat{Z} = -d, \end{cases}, \quad \text{(A 3)}$$

where $d\mathbf{s}/d\mathbf{S} = |(\dot{\mathbf{F}}^{-1})^\dagger \cdot \mathbf{N}|$ and $\mathbf{N} = (0, 0, 1)$ is the normal to the plate in its undeformed configuration.

We pose the following non-dimensionalization and scalings (emphasizing that the new unhatted variables are distinct from those in the main text):

$$(\hat{X}, \hat{Y}, \hat{Z}) = (LX, LY, dZ), \quad (\xi, \zeta) = L'(\delta\xi, \delta\zeta), \quad (\hat{r}_h, \hat{r}_z) = \mu \delta^2(\tau_h, \delta\tau_z), \quad \text{(A 4)}$$

$$(\Sigma_{XX}, \Sigma_{XY}, \Sigma_{YY}, \Sigma_{XZ}, \Sigma_{YZ}, \Sigma_{ZZ}) = \mu \delta^2(\Sigma_{XX}, \Sigma_{XY}, \Sigma_{YY}, \delta\Sigma_{XZ}, \delta\Sigma_{YZ}, \delta^2\Sigma_{ZZ}),$$

where $\delta = L'/L \ll 1$, $L'$ is a typical out-of-plane displacement, and $L$ is a characteristic cross-stream length scale. The scalings for components of $\Sigma$ are forced by the governing equations and boundary conditions; we also scale and non-dimensionalize $\mathbf{S}$ in the same way. We approximate the equations to leading order in $\delta$, assuming $\Delta = d/L'$ is order unity.

From (A 2) and the scalings for $\Sigma$, we find at leading order

$$\xi = -\Delta Z \nabla_{\mathbf{H}} \hat{\xi} + \hat{\xi}(X, Y), \quad \partial \xi/\partial Z = 0, \quad \text{(A 5)}$$

where $\hat{\xi}$ is the in-plane displacement of the centre plane $(Z=0)$ and subscript $H$ denotes components in the $(X, Y)$ plane, and

$$\Sigma_{XX} = \frac{2(\lambda + \mu)}{\lambda + 2\mu} \left( \frac{2 \partial \xi}{\partial X} + \left[ \frac{\partial \zeta}{\partial X} \right]^2 \right) + \frac{\lambda}{\lambda + 2\mu} \left( \frac{2 \partial \eta}{\partial Y} + \left[ \frac{\partial \zeta}{\partial Y} \right]^2 \right), \quad \text{(A 6)}$$

$$\Sigma_{XY} = 2 \left( \frac{\partial \xi}{\partial Y} + \frac{\partial \eta}{\partial X} + \frac{\partial \zeta}{\partial X} \frac{\partial \zeta}{\partial Y} \right), \quad \text{(A 7)}$$

$$\Sigma_{YY} = \frac{\lambda}{\lambda + 2\mu} \left( \frac{2 \partial \xi}{\partial Y} + \left[ \frac{\partial \zeta}{\partial Y} \right]^2 \right) + \frac{2(\lambda + \mu)}{\lambda + 2\mu} \left( \frac{2 \partial \eta}{\partial Y} + \left[ \frac{\partial \zeta}{\partial Y} \right]^2 \right). \quad \text{(A 8)}$$
Substituting (A 4) into (A 1) and (A 3), we obtain

\[
\frac{\partial \Sigma_{XY}}{\partial X} + \frac{\partial \Sigma_{XY}}{\partial Y} + \frac{1}{\Delta} \frac{\partial \Sigma_{XZ}}{\partial Z} = 0, \tag{A 9}
\]

\[
\frac{\partial \Sigma_{XY}}{\partial X} + \frac{\partial \Sigma_{YY}}{\partial Y} + \frac{1}{\Delta} \frac{\partial \Sigma_{YZ}}{\partial Z} = 0, \tag{A 10}
\]

\[
\frac{\partial S_{XZ}}{\partial X} + \frac{\partial S_{YZ}}{\partial Y} + \frac{1}{\Delta} \frac{\partial S_{ZZ}}{\partial Z} = 0, \tag{A 11}
\]

subject to

\[
(S_{XZ}, S_{YZ}) = \begin{cases} 0, & Z = 1, \\ \tau_h, & Z = -1, \end{cases} \quad S_{ZZ} = \begin{cases} 0, & Z = 1, \\ \tau_Z, & Z = -1, \end{cases}
\tag{A 12}
\]

at leading order. Integrating (A 9)–(A 11) across the thickness of the plate, we obtain the non-dimensional Föppl–von Kármán equations in Lagrangian coordinates as

\[
\tau_h = \nabla_h \cdot \mathbf{N}, \quad \frac{8}{3} \Delta \frac{\lambda + \mu}{\lambda + 2\mu} \nabla_h^4 \zeta = -\tau_z + \Delta \nabla_h \cdot \tau_h + \nabla_h \cdot (\mathbf{N} \cdot \nabla_h \zeta), \tag{A 13}
\]

where \( \mathbf{N} = 4\Delta \mathbf{e} + 4\Delta \lambda \text{tr}(\mathbf{e}) \mathbf{l}/(\lambda + 2\mu) \) and \( 2\mathbf{e} = \nabla_h \hat{\zeta} + \nabla_h \hat{\xi} + \nabla_h \hat{\zeta} \mathbf{p} \). The term \( \Delta \nabla_h \cdot \tau_h \) in (A 13) can be omitted on the grounds that, if the pressure perturbation scales as \( \mu_0 \delta^4 \), then the in-plane traction perturbations also scale as \( \mu_0 \delta^4 \) but our choice in (A 4) is one order lower (to accommodate larger in-plane tractions in the base state, but for which \( \Delta \nabla_h \cdot \tau_h = 0 \)).

Finally, we convert to Eulerian coordinates by noting that \( \nabla_h = \nabla_h + \mathcal{O}(\delta^2) \) and the leading order equations remain valid. The system (A 13) is now equivalent to the system (2.3) on redimensionalizing and identifying \( \nu = \lambda / 2(\lambda + \mu) \) and \( E = \mu(3\lambda + \mu)/(\lambda + \mu) \).

The terms in (A 13) all remain at the same leading order provided \( \delta^{1/3} \ll \Delta \ll \delta^{-1/3} \), which limits the magnitude of \( \Delta \). In turn, this limits how large we may take \( G \) in order for the plate equations to remain valid. Specifically, we find \( G \ll y_0 / d \) which allows for relatively large numerical values of \( G \) provided the plate is thin enough.

### Appendix B. A viscous fluid plate

In this appendix, we present complementary results for the buckling of a thin, very viscous fluid overlying the channel. Such skins have similar behaviour to elastic plates (Ribe 2002; Teichman 2002) and we find an analogous shear instability to that explored in the main text.

We derive the fluid equivalent of the Föppl–von Kármán equations; the approach is similar to that given in appendix A except we remain in Eulerian coordinates. Our derivation is similar to that of Teichman (2002), but allows for interaction between the ‘plate’ fluid and the underlying fluid reservoir. We commence with a dimensional formulation. The skin is initially uniform, flat and is contained between \( \hat{h}^- (\hat{x}, \hat{y}, \hat{t}) \leq \hat{z} \leq \hat{h}^+ (\hat{x}, \hat{y}, \hat{t}) \).
Conservation of momentum (ignoring gravity) and mass in the threedimensional skin are given by

\[ \mathbf{\nabla} \cdot \mathbf{\hat{\sigma}}_p = 0, \quad \mathbf{\nabla} \cdot \mathbf{\hat{u}}_p = 0, \quad \mathbf{\hat{\sigma}}_p = -\mathbf{\hat{p}}_p \mathbf{1} + \mu_p (\mathbf{\nabla} \mathbf{\hat{u}}_p + \mathbf{\nabla} \mathbf{\hat{u}}_p^\top), \]  

where subscript \( p \) denotes quantities pertaining to the plate fluid. Boundary conditions are

\[ \mathbf{\hat{\sigma}}_p \cdot \mathbf{\hat{n}} = \begin{cases} 0, & \hat{z} = \hat{h}^+, \\ \mathbf{\hat{t}}, & \hat{z} = \hat{h}^- \end{cases} \quad \mathbf{\hat{u}}_p = \mathbf{\hat{u}} \quad \text{on} \quad \hat{z} = \hat{h}^-, \]

\[ \frac{\partial \hat{h}^-}{\partial t} + \dot{u}_p \frac{\partial \hat{h}^+}{\partial x} + \dot{v}_p \frac{\partial \hat{h}^+}{\partial y} = \dot{w}_p \quad \text{on} \quad \hat{z} = \hat{h}^+. \]  

(B 2)

We non-dimensionalize and scale these equations similarly to (A 4)

\[ (\mathbf{x}, \mathbf{y}, \mathbf{z} - \alpha y_0, \hat{h}^+) = (Lx, Ly, dz, dh^+), \quad \mathbf{\hat{p}}_p = \mu_p \frac{U}{L} \epsilon^2 p_p, \]

\[ \dot{u}_p = U \epsilon (\epsilon u_p, \epsilon v_p, w_p), \quad (\mathbf{\hat{t}}, \mathbf{\hat{t}}_z) = \mu_p \frac{U}{L} \epsilon^3 (\tau_h, \tau_z), \]

\[ (\mathbf{\hat{\sigma}}_{pxx}, \mathbf{\hat{\sigma}}_{pxy}, \mathbf{\hat{\sigma}}_{pyy}, \mathbf{\hat{\sigma}}_{pxz}, \mathbf{\hat{\sigma}}_{pyz}, \mathbf{\hat{\sigma}}_{pzz}) = \mu_p \frac{U}{L} \epsilon^2 (\sigma_{pxx}, \sigma_{pxy}, \sigma_{pyy}, \epsilon \sigma_{pxz}, \epsilon \sigma_{pyz}, \epsilon^2 \sigma_{pzz}), \]  

(B 3)

where \( L \) is the length-scale of horizontal variation, \( \epsilon = d/L \ll 1 \), and the velocity scale \( U \) is assumed order unity, which must be verified \textit{a posteriori}. We approximate the equations to leading order in \( \epsilon \).

From (B 1) and the scalings of \( \mathbf{\hat{\sigma}}_p \), we find to leading order

\[ u_p = -z \frac{\partial w_p}{\partial x} + \bar{u}_p(x, y), \quad v_p = -z \frac{\partial w_p}{\partial y} + \bar{v}_p(x, y), \quad \frac{\partial w_p}{\partial z} = 0 \]  

(B 4)

and

\[ \sigma_{pxx} = 4 \frac{\partial u_p}{\partial x} + 2 \frac{\partial v_p}{\partial y}, \quad \sigma_{pxy} = \frac{\partial u_p}{\partial y} + \frac{\partial v_p}{\partial x}, \quad \sigma_{pyy} = 2 \frac{\partial u_p}{\partial x} + 4 \frac{\partial v_p}{\partial y}, \]  

(B 5)

where the barred velocities are the analogues of the centre-plane displacement for the elastic plate. The governing equations become

\[ \frac{\partial \sigma_{pxx}}{\partial x} + \frac{\partial \sigma_{pyy}}{\partial y} + \frac{\partial \sigma_{pz}}{\partial z} = 0, \]  

subject to

\[ \sigma_{pz} - \frac{\partial h^+}{\partial x} \sigma_{pzz} - \frac{\partial h^+}{\partial y} \sigma_{pyy} = \begin{cases} 0, & z = h^+, \\ \tau_j, & z = h^-, \end{cases} \]  

(B 7)

where \( j \) denotes \( x, y \) or \( z \). The kinematic conditions imply

\[ \frac{\partial h^+}{\partial t} = \frac{\partial h^-}{\partial t} = \dot{w}_p(x, y, t), \]  

(B 8)

and we have \( h^+ - h^- = 2 \) at leading order. We define \( 2h = h^+ + h^- \).
Now integrating (B 6) across the plate and imposing (B 7), we obtain the fluid equivalents of the Föppl–von Kármán equations
\[ \tau_h = \Phi_h \cdot N, \quad \frac{8}{3} \Phi_h \cdot \nabla N = -\tau_z + \Phi_h \cdot \tau_h + \Phi_h \cdot (N \cdot \nabla h), \] (B 9)
where \( N = 4[e + \text{tr}(e)]_h \) with \( 2e = \Phi_h(\hat{u}_p, \hat{v}_p) + \Phi_h(\hat{u}_p, \hat{v}_p)^T - 2h \Phi_h \Phi_h w_p \). As for the elastic case, we neglect \( \Phi_h \cdot \tau_h \).

Re-dimensionalizing we obtain the governing equations
\[ \dot{\tau}_h = \Phi_h \cdot \dot{N}, \quad \frac{8}{3} \mu_p d^3 \Phi_h \cdot \nabla \dot{N} = -\dot{\tau}_z + \Phi_h \cdot (\dot{N} \cdot \nabla \dot{h}), \] (B 10)
where \( \dot{N} = 4\mu_p d^3[e + \text{tr}(e)]_h \) and \( 2\dot{e} = \Phi_h(\hat{\dot{u}}_p, \hat{\dot{v}}_p) + \Phi_h(\hat{\dot{u}}_p, \hat{\dot{v}}_p)^T - 2h \Phi_h \Phi_h \dot{w}_p \). At the side-walls, \( \hat{y} = \pm y_0 \), we have the equivalent of the clamped boundary conditions, \( \hat{u}_p = \hat{v}_p = \hat{w}_p = \partial \hat{w}_p / \partial \hat{y} = 0 \). The kinematic conditions and continuity of velocity imply \( \partial \dot{h} / \partial t = \dot{w}_p = \dot{w} \) and \( \hat{u}_p = \hat{v}_p = 0 \) on the base of the plate.

Using (B 10) and the associated boundary conditions in place of (2.3) in the governing equations (2.1)–(2.7), we obtain the equivalent system for a viscous plate (identifying \( z \) with \( h \)). On non-dimensionalizing according to (2.8) with \( \dot{u} = \mu_p d^3 / \mu_p d \), we arrive at identical stability equations to (4.1)–(4.8) except that the biharmonic operator acts on \( w \) rather than \( \zeta \) and
\[ G = \frac{3 \mu_p y_0^5 \sin \theta}{8 \mu_p d^3}. \]

We note that, for the viscous plate equation (B 10) to be valid, the base state velocity within the plate must scale according to (B 3) and so \( \mu_p d^3 \leq \rho g y_0^5 \).

Sample stability results are shown in figure 12. Figure 12a plots the eigenvalue spectrum for a particular set of parameter values. The two most unstable modes have a similar perturbation profile to those plotted in figures 5 and 9 with even and odd chevrons directed downslope. The origin of the complex plane forms a limit point of the spectrum; as one approaches this limit point, more cross-slope oscillations are accommodated in the eigenfunctions. The dispersion relation for the most unstable mode is plotted in figure 12b. As for the elastic case, a window of intermediate wavelengths is unstable.

**Figure 12.** The fluid plate. (a) The spectrum of \( \omega \) for \( G = 50 / \sqrt{2} \), \( \alpha = \cot \theta = 1 \) and \( k = 1 \). Modes are converged except very close to the origin. (b) The growth rate (solid contours at intervals of 0.02 from 0.01) and wavespeed (dashed contours at intervals of 0.002 from 0.001) of the most unstable mode on the \((k, G)\)-plane for \( \alpha = \cot \theta = 1 \).
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