On the Riemann property of angular lattice sums and the one-dimensional limit of two-dimensional lattice sums

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We consider a general class of two-dimensional lattice sums consisting of complex powers of inverse quadratic functions. We consider two cases, one where the quadratic function is negative definite and another more restricted case where it is positive definite. In the former, we use a representation due to H. Kober, and consider the limit \( u \to \infty \), where the lattice becomes ever more elongated along one period direction (the one-dimensional limit). In the latter, we use an explicit evaluation of the sum due to Zucker and Robertson. In either case, we show that the one-dimensional limit of the sum is given in terms of \( \zeta(2s) \) if \( \text{Re}(s) > 1/2 \) and either \( \zeta(2s-1) \) or \( \zeta(2-2s) \) if \( \text{Re}(s) < 1/2 \). In either case, this leads to a Riemann property of these sums in the one-dimensional limit: their zeros must lie on the critical line \( \text{Re}(s) = 1/2 \). We also comment on a class of sums that involve complex powers of the distance to points in a two-dimensional square lattice and trigonometric functions of their angle. We show that certain of these sums can have their zeros on the critical line but not in a neighbourhood of it; others are identically zero on it, while still others have no zeros on it.

**Keywords:** lattice sums; Dirichlet \( L \)-functions; Riemann hypothesis

1. Introduction

This paper is concerned with the properties of two-dimensional lattice sums, and in particular with the distribution of their zeros, and the connection between their properties and those of one-dimensional sums. Such sums are of interest in their own right, but also arise in many areas of physics and cosmology, where they can be used in analytic continuation regularization methods (Elizalde 2008). They are then invaluable in studies of topics as diverse as the fluctuations of the vacuum energy, the Casimir effect, quantum fields pervading the universe and the cosmological constant. The article by Elizalde and its references, in particular

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the website of his second reference, give an idea of the variety of sums of this type, which are classified as generalizations of the Riemann zeta function, and the richness of the fields in which they find the applications.

We commence in §§2–7 with a discussion of lattice sums generated by complex powers of inverse quadratic functions, with successively the quadratic function being positive definite and then negative definite. Sections 8–10 discuss the relationship between sums of inverse powers of distances in the plane, and related sums involving powers of cosines or sines of angles in the plane as well as inverse distances. In the former case, the focus is on the study of the limiting behaviour of the zeros as the two-dimensional sums become ever more elongated along one period direction (the one-dimensional limit). It is shown that in this limit the zeros must lie on the critical line identified in the Riemann hypothesis for the zeros of the zeta function. In the latter case, sums are shown to have a variety of possible behaviours: they can have no zeros on the critical line, or be zero identically on the critical line and non-zero in a neighbourhood of it, or have zeros on the line but not in its neighbourhood.

There appears to be little in the literature on the zeros of two-dimensional lattice sums. Lorenz (1871) and Hardy (1920) derived an elegant formula for the sum over complex inverse powers of distance for the square lattice, from which the Riemann hypothesis or its generalization to Dirichlet L-functions predict that zeros lie only on the critical line. Davenport & Heilbronn (1936) showed that a particular lattice sum over a rectangular lattice had an infinity of zeros off the critical line, which may have dampened interest in this general class of sums.

In §2, we consider a lattice sums of hyperbolic type studied by Zucker & Robertson (1984) and provide a derivation of a functional equation for them, which had been put forward by Zucker & McPhedran (2008), based on numerical evidence. We next exhibit the behaviour of these sums near specific values for the complex power $s$, before studying their one-dimensional limit in §4. Here, we use results of Espinosa & Moll (2002) for definite integrals of the product of zeta functions to establish that in this limit all zeros must lie on the critical line. In §5, we consider the sequences of values of $s$ for which the hyperbolic sum is zero and illustrate their properties numerically. In §6, we give the Macdonald function representation due to Kober (1935) for a class of double sums containing inverse powers of a negative-definite quadratic form, which we go on to show again has the property that in the one-dimensional limit its zeros lie on the critical line. In §8, we consider the expansion of a sum of the Epstein zeta type away from the square lattice, which can be evaluated by introducing a class of angular lattice sums that form the subject of the rest of the paper. In §9, we study the lowest order set of angular lattice sums and show that independent sums do not share zeros. In §10, we study the functional equations for angular lattice sums, and use these to establish the variety of behaviours that these sums may have with regard to their zero values.

2. The functional equation for $T(1, 0, -r^2; s)$

Zucker & Robertson (1984) have proved the following result: if

$$T(1, 0, -r^2; s) = \sum_{m^2 \neq r^2 n^2} |m^2 - r^2 n^2|^{-s}$$

(2.1)
is a lattice sum of hyperbolic type over a rectangular lattice with integral periods 1 and \( r \), then

\[
T(1, 0, -r^2; s) = 4r^{-2s}(1 - 2^{1-s} + 2^{1-2s})L_1^2(s) + 2 \sum_{t=1}^{r-1} [(2r, t; s) + (2r, 2r - t; s)]^2,
\]

where

\[
(k, l; s) = \sum_{n=0}^{\infty} (kn + l)^{-s} = k^{-s}\zeta(s, l/k),
\]

\( L_1(s) = \zeta(s) \) is the Riemann zeta function; and \( \zeta(s, l/k) \) is the Hurwitz zeta function with rational second argument. They were able to evaluate \( T(1, 0, -r^2; s) \) in terms of squares of Dirichlet \( L \)-functions (see Zucker & Robertson 1976) of positive order for \( r = 1–6 \). (Note that in (2.1) and double sums of similar type, the sums over \( m \) and \( n \) runs from \( -\infty \) to \( \infty \), with the singular terms \( m^2 = r^2n^2 \) excluded.)

Let us rewrite (2.2) in terms of the following symmetrized bracket symbols or Hurwitz zeta functions

\[
(k, l; s)_+ = (k, l, s) + (k, k - l, s) = k^{-s}\zeta_+(s, l/k),
\]

\[
(k, l; s)_- = (k, l, s) - (k, k - l, s) = k^{-s}\zeta_-(s, l/k),
\]

where the symmetrized Hurwitz zeta functions \( \zeta_+, \zeta_- \) were discussed by McPhedran et al. (2007). Then

\[
T(1, 0, -r^2; s) = 4r^{-2s}(1 - 2^{1-s} + 2^{1-2s})\zeta^2(s) + 2 \sum_{t=1}^{r-1} (2r, t; s)^2_+.
\]

Using the result \( \zeta(s, 1/2) = (2^s - 1)\zeta(s) \), (2.5) may be expressed as

\[
T(1, 0, -r^2; s) = 4r^{-2s}(2^{1-s} - 1)\zeta^2(s) + 2 \sum_{t=1}^{r} (2r, t; s)^2_+.
\]

We wish to establish the functional equation satisfied by \( T(1, 0, r; s) \). We have found numerically that the following relationship holds:

\[
T(1, 0, -r^2; s) = T(1, 0, -r^2, 1-s)\left(\frac{\pi}{r}\right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)} \tan\left(\frac{s\pi}{2}\right).
\]

This is to be compared with the functional equation derived by Riemann (Titchmarsh & Heath-Brown 1987) for \( \zeta(s) \),

\[
\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin\left(\frac{s\pi}{2}\right)\zeta(1-s).
\]

Before commenting on how this may be established for arbitrary \( r \), we illustrate it for \( r = 1, 2 \). In the first of these,

\[
T(1, 0, -1; s) = 4(1 - 2^{1-s} + 2^{1-2s})\zeta^2(s)
\]

\[
= 2 \left[ \left( 1 + \frac{1}{\sqrt{2}} \right) N_2^2(s) + \left( 1 - \frac{1}{\sqrt{2}} \right) P_2^2(s) \right],
\]

using the basis for order 2

\[ N_2(s) = (1 - 2^{1/2-s})\zeta(s), \quad P_2(s) = (1 + 2^{1/2-s})\zeta(s). \] (2.10)

The square of each obeys the same functional equation, as then does their superposition, i.e. \( T(1, 0, -r; s) \). Additionally, by construction and assuming the Riemann hypothesis, each of these basis functions has all its zeros on the critical line \( \text{Re}(s) = 1/2 \).

For \( r = 2 \),

\[ T(1, 0, -4; s) = 2[1 + (\sqrt{2} - 1)2^{-s} + 2^{1-2s}] [1 - (\sqrt{2} + 1)2^{-s} + 2^{1-2s}]\zeta^2(s). \] (2.11)

There are three basis functions for \( 2r = 4 \) of positive order (with the function of negative order being of course \( L_{-4}(s) \))

\[
\begin{align*}
  P_4^{(1)}(s) &= (1 + 4^{1/2-s})\zeta(s), \\
  P_4^{(2)} &= (1 + 2^{1/2-s} + 4^{1/2-s})\zeta(s), \\
  N_4^{(1)} &= (1 - 4^{1/2-s})\zeta(s).
\end{align*}
\] (2.12)

In terms of these, we derive a form satisfying (2.7) for \( r = 2 \),

\[ T(1, 0, -4; s) = \frac{1}{4} (7 + 5\sqrt{2}) P_4^{(1)}(s)^2 + \frac{1}{4} (1 - \sqrt{2}) P_4^{(2)}(s)^2 - \sqrt{2} N_4^{(1)}(s)^2. \] (2.13)

Let us define two sums over symmetrized zeta functions—first, for the symmetric sum

\[ S_+(r, s) = \frac{1}{(2r)^2} \sum_{t=1}^{2r} [\zeta_+(s, t/(2r))]^2 \] (2.14)

and for the antisymmetric sum

\[ S_-(r, s) = \frac{1}{(2r)^2} \sum_{t=1}^{2r} [\zeta_-(s, t/(2r))]^2. \] (2.15)

Then we note that from (2.2),

\[ T(1, 0, -r^2; s) = S_+(r, s). \] (2.16)

We also introduce half-space lattice sums, first over the region above the lines \( m = \pm rn \)

\[ G(1, 0, -r^2; s) = \sum_{m^2 > r^2 n^2} (m^2 - r^2 n^2)^{-s} \] (2.17)

and then below the lines \( m = \pm rn \)

\[ L(1, 0, -r^2; s) = \sum_{m^2 < r^2 n^2} (m^2 - r^2 n^2)^{-s}. \] (2.18)

We next introduce a lattice sum of the Epstein zeta type, although not over a negative-definite quadratic function

\[ Z(1, 0, -r^2; s) = \sum_{m^2 \neq r^2 n^2} (m^2 - r^2 n^2)^{-s} \] (2.19)

and finally a sum representing a difference of the contributions in $\mathcal{G}$ and $\mathcal{L}$

$$
\mathcal{D}(1, 0, -r^2; s) = \sum_{m^2 > r^2n^2} (m^2 - r^2n^2)^{-s} - \sum_{m^2 < r^2n^2} (m^2 - r^2n^2)^{-s}.
$$

(2.20)

The four sums just defined are of course connected

$$
\begin{align*}
Z(1, 0, -r^2; s) &= \mathcal{G}(1, 0, -r^2; s) + \mathcal{L}(1, 0, -r^2; s), \\
\mathcal{D}(1, 0, -r^2; s) &= \mathcal{G}(1, 0, -r^2; s) - \mathcal{L}(1, 0, -r^2; s)
\end{align*}
$$

and as well

$$
T(1, 0, -r^2; s) = e^{i\pi s/2}
\left[ Z(1, 0, -r^2; s) \cos \left( \frac{s\pi}{2} \right) - i\mathcal{D}(1, 0, -r^2; s) \sin \left( \frac{s\pi}{2} \right) \right].
$$

(2.22)

Furthermore, if we look at Zucker & Robertson’s (1984) derivation of (2.2), we can easily express $\mathcal{G}_{+}(1, 0, -r^2; s)$ and $\mathcal{L}(1, 0, -r^2; s)$ in terms of $S_{+}(r, s)$ and $S_{-}(r, s)$

$$
\begin{align*}
\mathcal{G}(1, 0, -r^2; s) &= \frac{1}{2} [S_{+}(r, s) + S_{-}(r, s)], \\
\mathcal{L}(1, 0, -r^2; s) &= \frac{e^{-irs}}{2} [S_{+}(r, s) - S_{-}(r, s)].
\end{align*}
$$

(2.23)

Using (2.23) in (2.21), we find

$$
Z(1, 0, -r^2; s) = e^{-i\pi s/2}
\left[ S_{+}(r, s) \cos \left( \frac{s\pi}{2} \right) + iS_{-}(r, s) \sin \left( \frac{s\pi}{2} \right) \right]
$$

(2.24)

and

$$
\mathcal{D}(1, 0, -r^2; s) = e^{-i\pi s/2}
\left[ iS_{+}(r, s) \sin \left( \frac{s\pi}{2} \right) + S_{-}(r, s) \cos \left( \frac{s\pi}{2} \right) \right].
$$

(2.25)

Now, the functional equation for Epstein zeta functions is known (Mordell 1930; Potter 1934), but the proof does not apply to the case in hand, since it relies on the quadratic form in the denominator of the lattice sums not having real roots. Instead, we will derive the functional equation using the symmetrized basis functions introduced in McPhedran et al. (2007). The zeta functions $\zeta_{+}(s, t/(2r))$ arising in $S_{+}(r, s)$ can be written as a superposition of basis functions $\mathcal{P}, \mathcal{N}$ satisfying

$$
\mathcal{P}_{+2r}(s) = 2^{s} \pi^{s-1} (2r)^{1/2-s} \Gamma(1-s) \sin \left( \frac{s\pi}{2} \right) \mathcal{P}_{+2r}(1-s),
$$

(2.26)

a generalization of (2.8), and

$$
\mathcal{N}_{+2r}(s) = -2^{s} \pi^{s-1} (2r)^{1/2-s} \Gamma(1-s) \sin \left( \frac{s\pi}{2} \right) \mathcal{N}_{+2r}(1-s).
$$

(2.27)

There are $r+1$ functions in the basis for $S_{+}(r, s)$. McPhedran et al. (2007) introduced a vector notation to describe such bases. The subscript of the function $\mathcal{P}$ or $\mathcal{N}$ is the period, which gives the length of the required vector. The elements of the vector are then the coefficients of $1/n^s$ in expansions derived using $\text{Re}(s)$ large and positive, for $n$ running over the length of the vector. This notation enables us to write down the two of these 2r component vectors related to the

Riemann zeta function,
\[
\mathcal{P}^{(1)}_{+2r}(s) = [1, 1, 1, \ldots, 1, 1 + \sqrt{2r}], \quad \mathcal{N}^{(1)}_{+2r}(s) = [1, 1, 1, \ldots, 1, 1 - \sqrt{2r}].
\] (2.28)

Apart from these two, we can use vectors that have some of the properties of those associated with Dirichlet \(L\)-functions: they have first component 1; last component 0; are symmetric round the \(r\)th value; and are orthogonal to each other and to the vectors in (2.28) (which requires that their sum be zero).

The functional equation satisfied by the squares of these basis functions is the same
\[
\begin{bmatrix}
\mathcal{P}^2_{+2r}(s) \\
\mathcal{N}^2_{+2r}(s)
\end{bmatrix}
= 2^{2s} \pi^{2s-2} (2r)^{1-2s} \Gamma(1-s)^2 \sin^2 \left( \frac{s \pi}{2} \right) \begin{bmatrix}
\mathcal{P}^2_{+2r}(1-s) \\
\mathcal{N}^2_{+2r}(1-s)
\end{bmatrix},
\] (2.29)

or, replacing one of the factors of \(\Gamma(1-s)\) by \(\pi/(\sin(\pi s) \Gamma(s))\),
\[
\begin{bmatrix}
\mathcal{P}^2_{+2r}(s) \\
\mathcal{N}^2_{+2r}(s)
\end{bmatrix} \cos \left( \frac{s \pi}{2} \right)
= \frac{\pi^{2s-1} \Gamma(1-s)}{r^{2s-1} \Gamma(s)} \begin{bmatrix}
\mathcal{P}^2_{+2r}(1-s) \\
\mathcal{N}^2_{+2r}(1-s)
\end{bmatrix} \sin \left( \frac{s \pi}{2} \right).
\] (2.30)

Now, we can view the definition of \(S_+(r, s)\) as requiring us to form the dot product of a row vector of elements \(\zeta_+(s, t/(2r))\) (for \(t\) running from 1 to 2\(r\)) with its transpose. When we expand this using the basis of \(\mathcal{P}\) and \(\mathcal{N}\)'s, the important fact is that no cross terms arise due to the orthogonality of the basis. This then gives us the functional equations for \(S_+(r, s)\)
\[
S_+(r, s) \cos \left( \frac{s \pi}{2} \right) = \left( \frac{\pi}{r} \right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)} S_+(r, 1-s) \sin \left( \frac{s \pi}{2} \right).
\] (2.31)

The functional equation for \(S_-(r, s)\) follows with minor changes: the factor \(\sin^2(s \pi/2)\) in the numerator of (2.29) is replaced by \(\cos^2(s \pi/2)\) due to the differing form of the functional equation for \(\mathcal{P}_-\) and \(\mathcal{N}_-\) from that for \(\mathcal{P}_+\) and \(\mathcal{N}_+\). When this factor in the numerator is divided by a factor \(\cos(s \pi/2) \sin(s \pi/2)\) in the denominator arising from the functional equation for \(\Gamma(1-s)\), we arrive at
\[
S_-(r, s) \sin \left( \frac{s \pi}{2} \right) = \left( \frac{\pi}{r} \right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)} S_-(r, 1-s) \cos \left( \frac{s \pi}{2} \right).
\] (2.32)

Using these in (2.24), we find
\[
Z(1, 0, -r^2; s)e^{i\pi s/2} = \left( \frac{\pi}{r} \right)^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)} Z(1, 0, -r^2; 1-s)e^{i\pi(1-s)/2},
\] (2.33)
in accord with the form of the functional equation derived by Mordell (1930). These functional equations enable us to deduce the functional equations for \(T\), which agrees with (2.7), \(\mathcal{D}\), \(\mathcal{G}\) and \(\mathcal{L}\).

Zucker & McPhedran (2008) have given expressions in terms of squares of Dirichlet \(L\)-functions of positive order for \(S_+(r, s)\) with \(r\) ranging from 1 to 13. We give corresponding expressions in table 1 for \(S_-(r, s)\) with \(r\) ranging up to 12, this time of course in terms of Dirichlet \(L\)-functions of negative order. The expressions in table 1, when combined with those in Zucker & McPhedran (2008), enable the solution in terms of \(L\)-functions for the sums \(Z\), \(\mathcal{G}\), \(\mathcal{L}\) and \(\mathcal{D}\) with argument \((1, 0, -r^2; s)\) for \(r\) ranging up to 12.

Table 1. Expansion of $S_\tau(r; s)$ in Dirichlet $L$-functions, with $\omega=\exp(i\pi/3)$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$S_\tau(r; s)$ = $\sum_{t=1}^r [(2r, t) - (2r, 2r - t)]^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2L_{-4}(s)$</td>
</tr>
<tr>
<td>3</td>
<td>$2(1 + 2^{-s} + 2^{1-2s})L_{-3}(s)$</td>
</tr>
<tr>
<td>4</td>
<td>$(1 + 2^{1-2s})L_{-4}(s) + L_{-2}(s)$</td>
</tr>
<tr>
<td>5</td>
<td>$2(1 + 2^{1-2s})L_{-3}(s)L_{-5}(s)$</td>
</tr>
<tr>
<td>6</td>
<td>$[1 + (\sqrt{2} - 1)2^{-s} + 2^{1-2s}] [1 - (\sqrt{2} + 1)2^{-s} + 2^{1-2s}]L_{-3}(s) + (1 + 2.3^{-s} + 3^{1-2s})L_{-4}(s)$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{3}{2} (1 - 2^{-s} + 2^{1-2s})L_{-7}(s) + \frac{3}{2} (1 + 2^{-s} + 2^{1-2s})L_{-9}(s) L_{-7}(s)$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{1}{2} (1 - 2^{-2s} + 2^{2-4s})L_{-4}(s) + \frac{1}{2} (1 + 2^{1-2s})L_{-8}(s) + L_{-16}(s) L_{-16}(s)$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{3}{2} (1 + 2^{-s} + 2^{1-2s})(1 + 3^{1-2s})L_{-3}(s) + \frac{3}{2} (1 - 2^{-s} + 2^{1-2s})L_{-9}(s) L_{-7}(s)$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{1}{2} (1 - 2.5^{-s} + 5^{1-2s})L_{-4}(s) + \frac{1}{2} L_{-20}(s) + (1 + 3.2^{-2s} + 2^{-4s})L_{-5}(s) L_{-3}(s)$</td>
</tr>
<tr>
<td>12</td>
<td>$\left{ [(1 + 2^{-s})(1 + 2^{1-2s} + 2^{-2s}) + 2^{-6s}]L_{-3}(s) + [(1 + 2^{1-2s})(1 + 2^{-3s} + 3^{1-2s})]L_{-4}(s)$</td>
</tr>
<tr>
<td></td>
<td>$+ (1 - 2.3^{-s} + 3^{1-2s})L_{-8}(s) + L_{-24}(s)/2 \right}$</td>
</tr>
</tbody>
</table>

3. Behaviour of $T$ near special points

It is obvious from the functional equation that $T$ has a second-order pole at $s=1$ for any finite $r$. Its expansion near $s=1$ for fixed $r$ is (Zucker & McPhedran 2008)

$$T(1, 0, -r^2; s) = \frac{2}{rs^2} + \frac{4\gamma}{rs} + \frac{2}{r} (\gamma^2 + 2\gamma_1) + C(r) + O(s),$$

(3.1)

where $\gamma_1$ is the first Stieltjes constant and $C(r)$ is a constant depending on $r$. The first three values of $C(r)$ are

$$C(1) = 2 \log^2(2), \quad C(2) = \frac{3}{2} \log^2(2), \quad C(3) = \frac{2}{3} \log^2(2) + \frac{1}{3} \log^2(3).$$

(3.2)

Using (3.1) in the functional equation (2.7), we deduce the form of $T$ for $r$ fixed and $s$ tending to zero

$$T(1, 0, -r^2; s) = 1 + 2 \log \left( \frac{\pi}{r} \right) s + O(s^2).$$

(3.3)

Note also that $T(1, 0, -r^2; s)$ has second-order zeros at $s=-2n$, $n=1, 2, 3, \ldots$, where there is one contribution from $1/\Gamma(s)$ and one from $\tan(s\pi/2)$. For $s=1-2n$, the zero of $1/\Gamma(s)$ is cancelled by the pole of $\tan(s\pi/2)$. Because these zeros are of second order, and the pole at $s=1$ is second order, $T(1, 0, -r^2; s)$ is non-negative on $\text{Im}(s)=0$.

Another useful result is that, certainly in $\text{Re}(s)>1$,

$$\lim_{r \to \infty} T(1, 0, -r^2; s) = 2\zeta(2s).$$

(3.4)

The result (3.4) is in agreement with the results of Zucker & Robertson (1984),

$$T(1, 0, -r^2; 2) = \sum_{m^2 \neq r^2 n^2} |m^2 - r^2 n^2|^{-2} = \frac{\pi^4}{45} + \frac{\pi^4}{18r^2} - \frac{\pi^4}{120r^4}$$

(3.5)

and

$$T(1, 0, -r^2; 4) = \sum_{m^2 \neq r^2 n^2} |m^2 - r^2 n^2|^{-4} = \frac{\pi^8}{4725} + \frac{\pi^8}{16200r^4} + \frac{\pi^8}{4536r^6} - \frac{\pi^8}{17280r^8}.$$  

(3.6)
(Here, we have corrected an evident error in the expression in Zucker & Robertson (1984).) Thus, we write
\[
\lim_{r \to \infty} T(1, 0, -r^2; n) = 2\zeta(2n) = \frac{(2\pi)^{2n}}{(2n)!}|B_{2n}|,
\]
for integers \(n > 1\), where the \(B_n\) denote the Bernoulli numbers. Using the functional equation (2.7), we find from (3.7) that
\[
\lim_{r \to \infty} \left[ \frac{T(1, 0, -r^2; 1-n)}{r^{2n-1}} \right] = \frac{2\Gamma(n)\sqrt{\pi}}{n\Gamma(n + 1/2)}|B_{2n}|\cos^2(\pi n/2),
\]
again for \(n > 1\).

Note that the zeros of \(T(1, 0, -1; s)\) are those of \(\zeta^2(s)\), and the following zeros of the prefactor
\[
s = \frac{1}{2} \pm \frac{\pi i}{4 \log 2} + \frac{2\pi im}{\log 2},
\]
where \(m\) is any integer. The prefactor zeros are of first order.

The prefactor zeros in (2.11) are
\[
s = \frac{1}{2} \pm \frac{\pi i}{\log 2} \pm \frac{i}{\log 2} \tan^{-1} \left[ \frac{\sqrt{10 + 4\sqrt{2}}}{-2 + \sqrt{2}} \right] + \frac{2\pi im}{\log 2}
\]
and
\[
s = \frac{1}{2} \pm \frac{i}{\log 2} \tan^{-1} \left[ \frac{\sqrt{10 - 4\sqrt{2}}}{2 + \sqrt{2}} \right] + \frac{2\pi im}{\log 2}.
\]

For \(r=3\),
\[
T(1, 0, -3; s) = 2(1 + 2^{1-s} + 2^{1-2s})(1 - 2.3^{-s} + 3^{1-2s})\zeta^2(s).
\]
From (3.12), we see that (2.7) holds for \(r=3\). The prefactor zeros in (3.12) are given by (3.9) and by
\[
s = \frac{1}{2} \pm \frac{i \tan^{-1}(\sqrt{2})}{\log 3} + \frac{2m\pi}{\log 3}.
\]

4. The one-dimensional limit of the two-dimensional sums

We now consider the one-dimensional limit \(r \to \infty\) of the sums \(D, Z, T, G\) and \(L\). In this limit, we can replace appropriate discrete sums by integrals, where the sums are identified with the Riemann sum representation of the integrals. For the integrals, we use the following expressions due to Espinosa & Moll (2002):
\[
\int_0^1 \zeta^2(z, q) dq = 2\Gamma^2(1-z)(2\pi)^{2z-2}\zeta(2-2z)
\]
and
\[
\int_0^1 \zeta(z, q)\zeta(z, 1-q) dq = 2\Gamma^2(1-z)(2\pi)^{2z-2}\zeta(2-2z).
\]
Espinosa & Moll (2002) derived these integrals for $z$ real, with $z<0$, and later commented that analytic continuation means they converge for $z$ real with $z<1/2$. We rely on the same analytic continuation argument to apply (4.1) and (4.2) for $z$-complex and in the region $\text{Re}(z)<1/2$. We then write

$$\lim_{r \to \infty} \left( \frac{1}{2r} \right) \sum_{t=1}^{2r} \zeta \left( s, \frac{t}{2r} \right)^2 = \int_0^1 \zeta(s, q)^2 dq = 2I^2(1-s)(2\pi)^{2s-2}\zeta(2-2s)$$ (4.3)

and

$$\lim_{r \to \infty} \left( \frac{1}{2r} \right) \sum_{t=1}^{2r} \zeta \left( s, \frac{t}{2r} \right) \zeta \left( s, 1 - \frac{t}{2r} \right) = \int_0^1 \zeta(s, q)\zeta(s, 1-q) dq$$

$$= -2I^2(1-s)(2\pi)^{2s-2}\zeta(2-2s)\cos(\pi s).$$ (4.4)

Hence,

$$\lim_{r \to \infty} \left( \frac{1}{2r} \right) \sum_{t=1}^{2r} \zeta^+ \left( s, \frac{t}{2r} \right)^2 = 8I^2(1-s)(2\pi)^{2s-2}\zeta(2-2s)\sin^2(\pi s/2)$$ (4.5)

and

$$\lim_{r \to \infty} \left( \frac{1}{2r} \right) \sum_{t=1}^{2r} \zeta^- \left( s, \frac{t}{2r} \right)^2 = 8I^2(1-s)(2\pi)^{2s-2}\zeta(2-2s)\cos^2(\pi s/2).$$ (4.6)

We can then find

$$\lim_{r \to \infty} \left[ \mathcal{S}_+(r, s) \left( \frac{r}{\pi} \right)^{2s-1} \right] = \frac{4}{\pi} I^2(1-s)\zeta(2-2s)\sin^2(s\pi/2)$$ (4.7)

and

$$\lim_{r \to \infty} \left[ \mathcal{S}_-(r, s) \left( \frac{r}{\pi} \right)^{2s-1} \right] = \frac{4}{\pi} I^2(1-s)\zeta(2-2s)\cos^2(s\pi/2),$$ (4.8)

for $\text{Re}(s)>1/2$. We next apply the functional equations for $\mathcal{S}_+$ and $\mathcal{S}_-$ and replace $s$ by $1-s$, to obtain

$$\lim_{r \to \infty} \mathcal{S}_+(r, s) = 2\zeta(2s) = \lim_{r \to \infty} \mathcal{S}_-(r, s),$$ (4.9)

in $\text{Re}(s)>1/2$. We thus have

$$\lim_{r \to \infty} \mathcal{G}(1, 0, -r^2; s) = 2\zeta(2s) = \lim_{r \to \infty} T(1, 0, -r^2; s)$$

$$= \lim_{r \to \infty} Z(1, 0, -r^2; s) = \lim_{r \to \infty} \mathcal{D}(1, 0, -r^2; s),$$ (4.10)

while

$$\lim_{r \to \infty} \mathcal{L}(1, 0, -r^2; s) = 0.$$ (4.11)

We give an example of the convergence of the sums $\mathcal{D}$, $Z$ and $T$ towards the analytic limit in (4.10) in figure 1, for a case when $0.5<\text{Re}(s)<1$. The sum $T$ converges more rapidly than the other two sums, but of course the convergence for all three becomes more rapid as $\text{Re}(s)$ increases away from 1/2.
5. Zeros and null sequences

Given that for fixed $s$, as $r \to \infty$ the functions $S_+, S_-,$ $G(1, 0, -r^2; s)$, $T(1, 0, -r^2; s)$, $Z(1, 0, -r^2; s)$ and $D(1, 0, -r^2; s)$ all tend to $\zeta(2s)$, these one-dimensional limits can have no zeros in $\text{Re}(s) > 1/2$ (as $\zeta(s)$ has no zeros in $\text{Re}(s) > 1$). From the functional equations they satisfy, the same comment applies to the corresponding functions (with scale factors $r^{2s-1}$ incorporated) in $\text{Re}(s) < 1/2$.

Similarly, if we suppose we can construct null sequences $s_n(r)$ for the six functions such that the function value is zero, then any null sequence that converges to a regular point $s_n(\infty)$ in the finite part of the plane must have $\text{Re}[s_n(\infty)] = 1/2$. (If it converged to a point not on $\text{Re}[s_n(\infty)] = 1/2$, this would be a regular point, and the limit point of a sequence of zeros of the relevant function, and thus a zero of that function—a contradiction.)

The exceptions to this statement are found in sequences that tend to the irregular points $s=0$ and $s=1$. These exceptions arise owing to the fact that the limits $s \to s_0$ and $r \to \infty$ do not commute for $s_0=0,1$. Such non-commuting limits have been encountered before in the literature on lattice sums: for example, the papers by Rayleigh (1892), McPhedran & McKenzie (1978), Perrins et al. (1979), Borwein et al. (1988, 1989) and Shail (1995).

We discuss the non-commuting limits in the case of the point $s=1$. From (3.1) and (4.10),

$$\lim_{s \to 1} \lim_{r \to \infty} T(1, 0, -r^2; s) = 2\zeta(2) = \frac{\pi^2}{3}, \quad \lim_{r \to \infty} \lim_{s \to 1} T(1, 0, -r^2; s) = \infty.$$  \hfill (5.1)

The mechanism for this behaviour can be seen in the zero on the lower right of the graphs in figures 2–4; this zero, together with its conjugated zero, come together as $r \to \infty$ and quench the second-order pole of $T$ that exists for any finite $r$, but which has residue zero for infinite $r$ (see (3.1)).

Figures 2–4 show both zeros on the critical line and zeros to the right of it. The tendency is for the latter to drift towards the critical line with increasing $r$ (with the mean value of $\text{Re}(s)$ for the set excluding the zero heading towards $s=1$ decreasing slowly with increasing $r$). Another interesting feature is the evident
correlation between the off-axis zeros and gaps in the distribution of on-axis zeros (calling the axis the critical line). The number of zeros in the interval shown of the critical line increases slowly with \( r \): for \( r = 1, 2, 3 \), the zeros consist of double zeros of \( \zeta(s) \) and single prefactor zeros, with there being 13 of the former in the

Figure 2. Zeros of \( T(1, 0, -r^2; s) \) with \( \text{Re}(s) \geq 0.5 \) (horizontal axis) and \( \text{Im}(s) < 60 \) (vertical axis) for (a) \( r = 5 \) and (b) \( r = 6 \).

Figure 3. Zeros of \( T(1, 0, -r^2; s) \) with \( \text{Re}(s) \geq 0.5 \) and \( \text{Im}(s) < 60 \) for (a) \( r = 7 \) and (b) \( r = 8 \).

Figure 4. Zeros of \( T(1, 0, -r^2; s) \) with \( \text{Re}(s) \geq 0.5 \) and \( \text{Im}(s) < 60 \) for (a) \( r = 9 \) and (b) \( r = 10 \).
graphical range, and respectively 13, 26 and 32 of the latter. For \( r \geq 4 \), the zeros on the critical line are of first order, with their number in the graphical range being 40, 36, 45, 56, 61, 65 and 63 for \( r = 4, \ldots, 10 \). For \( r = 20 \), there are 78 zeros on the critical line in the graphical interval, and 9 off it to its right. The increase in the number of zeros on the critical line is associated with a decrease in the frequency of gaps in their distribution.

For the sum \( S_- \) (figure 5), there is not a sequence of zeros tending towards \( s = 1 \) with increasing \( r \), and the sums for \( r = 1, \ldots, 5 \) have no zeros off the critical line. Otherwise, the behaviour for \( S_- \) and \( S_+ = T \) is similar: when off-axis zeros first appear for low \( r \), they are relatively numerous in comparison with on-axis zeros, but with increasing \( r \) their relative number decreases and the gaps in the distribution of on-axis zeros become less pronounced.

6. A two-dimensional lattice sum and its Bessel function representation

We consider two-dimensional lattice sums of the Epstein zeta type

\[
Z(a, b, c; s) = \sum_{m_1, m_2 = -\infty}^{\infty} \frac{1}{(am_1^2 + 2bm_1 m_2 + cm_2^2)^s},
\]

with \( a > 0, b, c > 0 \) being real chosen such that the discriminant \( D \) is negative (so the quadratic form in (6.1) is positive definite)

\[
D = b^2 - ac = -a^2 u^2 < 0, \quad \Delta = -D = a^2 u^2 > 0.
\]

In (6.1), the prime indicates that the singular term coming from the origin is omitted.

Kober (1935) provided the following Bessel function representation for the series \( Z \), written in terms of \( w = s - 1/2 \):

\[
\begin{align*}
&u^{-1/2} \Delta^{w/2 + 1/4} \frac{\Gamma(w + 1/2)}{8\pi^{w+1/2}} Z(a, b, c; w + 1/2) \\
= &\sum_{m_1, m_2 = 1}^{\infty} \left( \frac{m_1}{m_2} \right)^w K_w(2\pi u m_1 m_2) \cos 2\pi v m_1 m_2 \\
&+ \frac{1}{4} u^{-w} \frac{\Gamma(w)}{\pi^w} \zeta(2w) + \frac{1}{4} u^w \frac{\Gamma(w + 1/2)}{\pi^{w+1/2}} \zeta(2w + 1).
\end{align*}
\]
Here \( v=b/a \). Kober denotes the right-hand side of (6.3) by \( \Theta(u, v, w) \) and comments that it is a regular function of \( w \), apart from the poles at \( w=1/2 \) and \( w=-1/2 \). Note that the Macdonald function satisfies \( K_w(z) = K_{-w}(z) \), so that the sum on the right-hand side of (6.3) is even in \( w \). The same comment applies to the axial sum term

\[
A(u, w) = \frac{1}{4} u^{-w} \frac{\Gamma(w)}{\pi^w} \zeta(2w) + \frac{1}{4} u^w \frac{\Gamma(w + 1/2)}{\pi^{w+1/2}} \zeta(2w + 1),
\]

(6.4)

from the functional equation satisfied by the Riemann zeta function. The left-hand side is thus even, which gives the known functional equation for \( Z \) (Potter 1934),

\[
\left( \frac{\sqrt{4}}{\pi} \right)^s \Gamma(s) Z(s) = \left( \frac{\sqrt{4}}{\pi} \right)^{1-s} \Gamma(1-s) Z(1-s).
\]

(6.5)

It has been shown by Potter & Titchmarsh (1935) and also by Kober (1936) that \( Z(a, b, c; s) \) has an infinity of zeros on the critical line \( \text{Re}(s) > 1/2 \). Potter & Titchmarsh (1935) also exhibited numerically two zeros of \( Z(1, 0, 5; s) \) not lying on the critical line. Davenport & Heilbronn (1936) proved that if the class number of the determinant of the quadratic form is even, or is odd and different from 1, then \( Z(a, b, c; s) \) has an infinite number of zeros off the critical line, in addition to those on it.

7. The one-dimensional limit of \( Z(a, b, c; s) \)

We consider the one-dimensional limit \( u \to \infty \) of \( Z(a, b, c; s) \). In (6.3), we note that the Macdonald function gives the following asymptotic estimate for the terms of the series:

\[
\left( \frac{m_1}{m_2} \right)^w K_w(2\pi um_1 m_2) \cos 2\pi v m_1 m_2 \sim \left( \frac{m_1}{m_2} \right)^w e^{-2\pi um_1 m_2 / \sqrt{4 um_1 m_2}} \cos 2\pi v m_1 m_2,
\]

(7.1)

so that the series is absolutely and uniformly convergent and tends to zero exponentially fast as \( u \to \infty \). Note that the onset of the exponential behaviour in (7.1) occurs when \( 2\pi um_1 m_2 > |w| \). Thus, we neglect the series and retain only the axial terms, to arrive at

\[
a^s Z(a, b, c; s) \sim 2^{\zeta(2s)} + 2\sqrt{\pi} u^{-2s} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \zeta(2s - 1).
\]

(7.2)

This expression is useful in \( \text{Re}(s) > 1/2 \) and shows that for any fixed \( s \) in this region we have that

\[
\lim_{u \to \infty} a^s Z(a, b, c; s) = 2^{\zeta(2s)}.
\]

(7.3)

Since the non-trivial zeros of \( \zeta(s) \) are known (Titchmarsh & Heath-Brown 1987) to lie in the critical strip \( 0 < \text{Re}(s) < 1 \), we deduce that \( \lim_{u \to \infty} a^s Z(a, b, c; s) \) has no zeros in \( \text{Re}(s) > 1/2 \).
In \( \text{Re}(s) < 1/2 \), we define
\[
\tilde{Z}(a, b, c; s) = \left( \frac{u^{2s-1} \Gamma(s)}{\sqrt{\pi} \Gamma(s - 1/2)} \right) Z(a, b, c; s),
\]
so that from (7.2)
\[
a^s \tilde{Z}(a, b, c; s) \sim 2\zeta(2s - 1) + \frac{2}{\sqrt{\pi}} \frac{u^{2s-1}}{\Gamma(s - 1/2)} \zeta(2s)
\]
and in \( \text{Re}(s) < 1/2 \),
\[
\lim_{u \to \infty} a^s \tilde{Z}(a, b, c; s) = 2\zeta(2s - 1).
\]
We can then say that \( \lim_{u \to \infty} a^s \tilde{Z}(a, b, c; s) \) has no zeros in \( \text{Re}(s) < 1/2 \).

Let us suppose next that we can construct a sequence of zeros \( s_0(u) \) of \( Z(a, b, c; s) \) in \( \text{Re}(s) > 1/2 \). From the functional equation (6.5), and the analytic nature of \( Z \) and \( \tilde{Z} \), it follows that \( s_0(u) \) is also a sequence of zeros of \( Z(a, b, c; s) \), while \( 1 - s_0(u) \) and \( 1 - s_0(u) \) are sequences of zeros of \( Z(a, b, c; s) \). If the sequence \( s_0(u) \) has a limit \( s_0(\infty) \), then this limit point must satisfy \( \text{Re}[s_0(\infty)] = 1/2 \). For suppose it were off the critical line: without loss of generality, we may take it to be in \( \text{Re}[s_0(\infty)] > 1/2 \), and thus in a neighbourhood where \( Z \) is regular, leading to \( Z(a, b, c; s_0(\infty)) = 0 \)—a contradiction. Hence, any sequence of zeros of \( Z(a, b, c; s) \) in \( \text{Re}(s) > 1/2 \) that has a limit as \( u \to \infty \) must tend to the critical line, and any sequence of zeros of \( \tilde{Z}(a, b, c; s) \) in \( \text{Re}(s) < 1/2 \) that has a limit as \( u \to \infty \) must also tend to the critical line.

8. The expansion of \( Z(1, b, c; s) \) and angular lattice sums

We now consider
\[
Z(1, b, c; s) = \sum_{m_1, m_2} \frac{1}{(m_1^2 + 2bm_1m_2 + cm_2^2)^s},
\]
which we rewrite as
\[
Z(1, b, c; s) = \sum_{m_1, m_2} \frac{1}{(m_1^2 + m_2^2 + 2bm_1m_2 + (c - 1)m_2^2)^s}.
\]
Hence, we seek an expansion for
\[
Z(1, b, c; s) = \sum_{m_1, m_2} \frac{1}{(m_1^2 + m_2^2)^s} \left[ 1 + \frac{2bm_1m_2}{m_1^2 + m_2^2} + \frac{(c - 1)m_2^2}{m_1^2 + m_2^2} \right]^{-s}.
\]
We write the last factor in (8.3) in the form corresponding to the generating function for Gegenbauer polynomials \( \text{Abramowitz & Stegun 1972} \), putting
\[
R^2 = 1 - 2xz + z^2 = 1 + \frac{2bm_1m_2}{m_1^2 + m_2^2} + \frac{(c - 1)m_2^2}{m_1^2 + m_2^2},
\]
which gives
\[
z = m_2 \sqrt{\frac{(c - 1)}{m_1^2 + m_2^2}}, \quad x = -\frac{2bm_1}{\sqrt{c - 1}\sqrt{m_1^2 + m_2^2}}.
\]

Hence, we obtain

\[ Z(1, b, c; s) = \sum_{n=0}^{\infty} \sum_{m_1, m_2}^t C_n^{(s)} \left[-\frac{2bm_1}{\sqrt{c-1}\sqrt{m_1^2 + m_2^2}}\right] \frac{(m_2\sqrt{c-1})^n}{(m_1^2 + m_2^2)^{s+n/2}}. \] (8.6)

Now \( n \) must be even for the sum over \( m_2 \) to be non-zero, so (8.6) becomes

\[ Z(1, b, c; s) = \sum_{n=0}^{\infty} \sum_{m_1, m_2}^t C_{2n}^{(s)} \left[-\frac{2bm_1}{\sqrt{c-1}\sqrt{m_1^2 + m_2^2}}\right] \frac{m_2^{2n}(c-1)^n}{(m_1^2 + m_2^2)^{s+n}}. \] (8.7)

We now concentrate on the particular case where \( b = 0 \). We have

\[ C_{2n}^{(s)}(0) = \frac{(-1)^n\Gamma(s+n)}{\Gamma(n+1)\Gamma(s)}, \] (8.8)

so that

\[ Z(1, 0, c; s) = \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{\Gamma(n+1)\Gamma(s)} (1-c)^n \sum_{m_1, m_2}^t \frac{m_2^{2n}}{(m_1^2 + m_2^2)^{s+n}}. \] (8.9)

The sums over \( m_1 \) and \( m_2 \) occurring in (8.9) may be expressed as angular lattice sums. Let us write

\[ m_1 = \cos(\theta_{m_1, m_2})\sqrt{m_1^2 + m_2^2}, \quad m_2 = \sin(\theta_{m_1, m_2})\sqrt{m_1^2 + m_2^2}. \] (8.10)

Then,

\[ \sum_{m_1, m_2}^t \frac{m_2^{2n}}{(m_1^2 + m_2^2)^{s+n}} = \sum_{m_1, m_2}^t \frac{\sin^{2n}(\theta_{m_1, m_2})}{(m_1^2 + m_2^2)^s} = \sum_{m_1, m_2}^t \frac{\cos^{2n}(\theta_{m_1, m_2})}{(m_1^2 + m_2^2)^s}, \] (8.11)

using the square symmetry of the array over which we sum.

We introduce a general set of angular lattice sums for the square array

\[ C(n, m; s) = \sum_{p, q}^t \frac{\cos^n(m\theta_{p, q})}{(p^2 + q^2)^s} \quad \text{and} \quad S(n, m; s) = \sum_{p, q}^t \frac{\sin^n(m\theta_{p, q})}{(p^2 + q^2)^s}. \] (8.12)

Of these, certain are zero, while others can be evaluated easily

\[ C(0, m; s) = S(0, m; s) = 4\xi(s)L_{-1}(s), \quad S(2n-1, m; s) = 0, \] (8.13)

\[ C(2, m; s) + S(2, m; s) = C(0, m; s), \] (8.14)

while for the square array, both \((p, q)\) and \((q, p)\) are in the array, so that \(\theta_{p, q}\) and \(\pi/2-\theta_{p, q}\) can be interchanged without altering the sums, so that

\[ C(2n, 2m-1; s) = S(2n, 2m-1; s). \] (8.15)

Combining (8.14) and (8.15),

\[ C(2, 2m-1; s) = S(2, 2m-1; s) = \frac{1}{2}C(0, m; s). \] (8.16)

One such angular lattice sum has been investigated by McPhedran et al. (2004)

\[ C(1, 4m; s) = \sum_{p, q}^t \frac{\cos(4m\theta_{p, q})}{(p^2 + q^2)^s} = C(2, 2m; s) - S(2, 2m; s), \] (8.17)
so that, again using (8.14),
\[
C(2, 2m; s) = \frac{1}{2} [C(0, 2m; s) + C(1, 4m; s)],
\]
\[
S(2, 2m; s) = \frac{1}{2} [C(0, 2m; s) - C(1, 4m; s)].
\]

The functional equation is known (see McPhedran et al. 2004, eqns (32) and (59)) for \(C(1, 4m; s)\)
\[
G_{4m}(s) = C(1, 4m; s) \frac{\Gamma(s + 2m)}{\pi^s} = G_{4m}(1 - s).
\]

We compare this with the functional equation for \(G_0(s)\)
\[
G_0(s) = C(1, 0; s) \frac{\Gamma(s)}{\pi^s} = G_0(1 - s) = 4\zeta(s)L_4(s) \frac{\Gamma(s)}{\pi^s}
\]
and write
\[
G_{4m}(s) = (e_{4m}(s) + o_{4m}(s))(E_{4m}(s) + O_{4m}(s)),
\]
where
\[
e_{4m}(s) = \frac{1}{2} \left( \frac{\Gamma(s + 2m)}{\Gamma(s)} + \frac{\Gamma(1 - s + 2m)}{\Gamma(1 - s)} \right),
\]
\[
o_{4m}(s) = \frac{1}{2} \left( \frac{\Gamma(s + 2m)}{\Gamma(s)} - \frac{\Gamma(1 - s + 2m)}{\Gamma(1 - s)} \right)
\]
and
\[
E_{4m}(s) = \frac{\Gamma(s)}{2\pi^s} C(1, 4m; s) + \frac{\Gamma(1 - s)}{2\pi^{1-s}} C(1, 4m; 1 - s),
\]
\[
O_{4m}(s) = \frac{\Gamma(s)}{2\pi^s} C(1, 4m; s) - \frac{\Gamma(1 - s)}{2\pi^{1-s}} C(1, 4m; 1 - s).
\]

The equation (8.21) has thus had its \(m\)-dependent prefactor split off and symmetrized, and the residue also symmetrized, with \(e, E\) and \(o, O\) being, respectively, even and odd under replacement of \(s\) by \(1 - s\). This means they are also, respectively, real and pure imaginary on the critical line \(\text{Re}(s) = 1/2\). The requirement that \(G_{4m}(s)\) be even enables us to relate \(E_{4m}(s)\) and \(O_{4m}(s)\)
\[
o_{4m}(s)E_{4m}(s) + e_{4m}(s)O_{4m}(s) = 0, \quad O_{4m}(s) = \frac{o_{4m}(s)}{e_{4m}(s)} E_{4m}(s).
\]

We thus find
\[
\frac{\Gamma(s)}{\pi^s} C(2, 2m; s) = \frac{1}{2} \left[ G_0(s) + E_{4m}(s) - \frac{o_{4m}(s)}{e_{4m}(s)} E_{4m}(s) \right]
\]
and
\[
\frac{\Gamma(s)}{\pi^s} S(2, 2m; s) = \frac{1}{2} \left[ G_0(s) - E_{4m}(s) + \frac{o_{4m}(s)}{e_{4m}(s)} E_{4m}(s) \right].
\]
From equations (8.26) and (8.27) we can conclude that for either \( \mathcal{C}(2, 2m; s) \) or \( \mathcal{S}(2, 2m; s) \) to be zero on the critical line, the imaginary part being zero tells us that \( \mathcal{E}_{4m}(s) = 0 \) and then the real part being zero gives \( G_0(s) = 0 \). Thus, for one to be zero on that line, so must the other, the zero must also be a common zero of \( G_0(s) \) and \( \mathcal{E}_{4m}(s) \).

We will show in \$9\) that independent lattice sums such as \( G_0(s) \) and \( \mathcal{E}_{4m}(s) \) do not in fact have common zeros. Hence, we see that \( \mathcal{C}(2, 2m; s) \) and \( \mathcal{S}(2, 2m; s) \) do not have zeros on the critical line \( \text{Re}(s) = 1/2 \).

9. The system of angular sums of order 4

The lowest order system of angular sums is that of order 4 and is of particular interest since it contributes the first angular sum in the expansion (8.9)

\[
Z(1, 0, c; s) = 4\zeta(s)L_{-4}(s) \left[ 1 + \frac{s}{2} (1-c) \right] + \frac{s(s+1)}{2!} (1-c)^2 \mathcal{C}(4, 1; s) + O(1-c)^3.
\]

(9.1)

The system, in common with all higher order systems, can be generated from any two independent angular sums, or from the two sums occurring in (9.1): \( \mathcal{C}(0, 1; s) \) and \( \mathcal{C}(4, 1; s) = \mathcal{S}(4, 1; s) \). It contains the other elements

\[
\mathcal{S}(2, 2; s) = 4 \sum_{p,q} \frac{p^2 q^2}{(p^2 + q^2)^{s+2}} = 2\mathcal{C}(0, 1; s) - 4\mathcal{C}(4, 1; s),
\]

(9.2)

\[
\mathcal{C}(2, 2; s) = \sum_{p,q} \frac{(p^2 - q^2)^2}{(p^2 + q^2)^{s+2}} = -\mathcal{C}(0, 1; s) + 4\mathcal{C}(4, 1; s)
\]

(9.3)

and

\[
\mathcal{C}(1, 4; s) = \sum_{p,q} \cos 4\theta_{p,q} \frac{1}{(p^2 + q^2)^s} = -3\mathcal{C}(0, 1; s) + 8\mathcal{C}(4, 1; s).
\]

(9.4)

Suppose now that two independent sums \( \mathcal{C}(4, 1; s) \) and \( \mathcal{C}(0, 1; s) \) shared a zero \( s_0 \). Then this means that all other sums in the system share the zero \( s_0 \), and that

\[
\sum_{p,q} \frac{P_4(p,q)}{(p^2 + q^2)^{s_0}} = 0,
\]

(9.5)

for all polynomials \( P_4(p,q) \) of the form

\[
P_4(p, q) = c_{4,0} p^4 + c_{4,2} p^2 q^2 + c_{4,4} q^4 = c_{4,0} (p^2 - \alpha_4 q^2)(p^2 - \beta_4 q^2),
\]

(9.6)

where \( \alpha_4 \) and \( \beta_4 \) are arbitrary complex or real quantities. As we can choose \( \alpha_4 \) and \( \beta_4 \) at will to alter the relative contributions of any lines in the array to the sum, and this according to the hypothesis does not alter the sum, the hypothesis is clearly false. This argument will also apply to any system of order \( 4n, n > 2 \), and so the angular sums not trivially related to \( \zeta(s)L_{-4}(s) \) do not share any zeros with it.

This property, applied to the expansion (9.1), shows that if \( Z(0, 1, s_0) = 0 \), then

\[
Z(1, 0; s_0) = \frac{s_0(s_0 + 1)}{2} (1 - c)^2 C(4, 1; s_0) + O(1 - c)^3. \tag{9.7}
\]

As \( C(4, 1; s_0) \) cannot be zero, \( Z(0, 1, c; s_0) \) varies as \((1 - c)^2\) in the neighbourhood of any zero at \( c = 1 \), and construction of null trajectories leading to any such zero requires the variation of both \( c \) and \( s \).

The properties demonstrated here are evident in figures 6 and 7. Figure 6a shows \(|C(2, 2; s)|\) and \(|S(2, 2; s)|\) in the range of \( \text{Im}(s) \) from 0 to 20. While the curves have clear minima approaching zero, they do not reach it, in keeping with the discussion at the end of §7. Comparing figure 6a, b, we see that the crossing points of the two curves in figure 6a correspond to the zeros of \( C(0, 4; s) \) and \( C(1, 4; s) \) in figure 6b, in keeping with (8.18). In figure 7, we show the behaviour of

\[
C(4, 1; s) = \frac{1}{8} C(1, 4; s) + \frac{3}{8} C(0, 4; s), \tag{9.8}
\]

for which

\[
\frac{\Gamma(s)}{\pi^s} C(4, 1; s) = \frac{3}{8} G_0(s) + \frac{1}{8} E_4(s) - \frac{1}{8} \frac{\phi_4(s)}{e_4(s)} E_4(s). \tag{9.9}
\]
As we can see, the real part has a combination of two contributions (the first two terms on the right-hand side in (9.9)), and varies in a smaller range than the imaginary part (which comes from \( C(1, 4; s) \) alone). The graph on the right-hand side of \( |C(4, 1; s)| \) shows pronounced minima, some of which seem to attain zero. However, we have confirmed that the value at the minima is small but non-zero. This conclusion follows from the representation (9.9), exactly in the same fashion as for \( |C(2, 2; s)| \) and \( |S(2, 2; s)| \).

We can generalize these results to systems of trigonometric sums of higher order. Such systems contain more elements than the system of order 4 studied in this section, but can still be generated by linear combinations of any two independent members. There will be only two elements of each system having zeros on the critical line \( \text{Re}(s) > 1/2 \): \( C(0, 4m; s) \) and \( C(1, 4m; s) \). By virtue of the differing functional equations for these two sums, all other elements in the system of order \( 4m \) cannot have zeros on the critical line. However, they may be grouped into appropriately scaled pairs, and the equality in magnitude of the elements of such pairs will indicate the zeros of \( C(0, 4m; s) \) and \( C(1, 4m; s) \) (in the same way as seen in figure 6).

We conclude with an evaluation in closed form of \( S(2, 2; 2) \) and \( C(4, 1; 2) \). We rely on results concerning Weierstrassian elliptic functions (e.g. Whittaker & Watson 1965, ch. 20). The results we need concern two invariants \( g_2 \) and \( g_3 \)

\[
\frac{g_2}{60} = \frac{1}{16} \sum_{p,q} \frac{1}{(p\omega + q\omega')^4}, \quad \frac{g_3}{140} = \frac{1}{64} \sum_{p,q} \frac{1}{(p\omega + q\omega')^6},
\]

where \( 2\omega \) and \( 2\omega' \) are the periods of the fundamental period parallelograms in the complex plane of the Weierstrassian elliptic functions. For rectangles and putting \( \omega = 1, \omega' \) will be pure imaginary, and then

\[
\omega' = \frac{1}{K} K', \quad g_2 = \frac{4}{3} (1-k^2 + k^4) K^4, \quad g_3 = \frac{4}{27} (1-2k^2)(1+k^2)(2-k^2) K^6,
\]

where \( K = K(k^2) \) is the complete elliptic integral of the first kind and \( K' \) is the complementary function, i.e. \( K'(k^2) = K(1-k^2) \) (Whittaker & Watson 1965). Now, if \( \omega' = i \), then \( K = K' \), \( k^2 = 1-k^2 \) so \( k^2 = 1/2 \), and \( g_2 = K^2(1/2) \), \( g_3 = 0 \). We can then deduce that

\[
\sum_{p,q} \frac{1}{(p + i q)^4} = \frac{16}{60} g_2 = \frac{4}{15} K^4 \left( \frac{1}{2} \right) = \sum_{p,q} \frac{p^4 + q^4 - 6p^2 q^2}{(p^2 + q^2)^4} - 4i \sum_{p,q} \frac{pq(p^2 - q^2)}{(p + i q)^4}.
\]

By symmetry, the imaginary term vanishes and so

\[
\frac{4}{15} K^4 = \sum_{p,q} \frac{p^4 + q^4 - 6p^2 q^2}{(p^2 + q^2)^4} = \sum_{p,q} \left[ \frac{1}{(p^2 + q^2)^2} - \frac{8p^2 q^2}{(p^2 + q^2)^4} \right].
\]

This leads to an expression of the desired form,

\[
S(2, 2; 2) = 2\zeta(2)L_{-4}(2) - \frac{2}{15} K^4 \left( \frac{1}{2} \right).
\]
Now,
\[ \zeta(2) = \frac{\pi^2}{6}, \quad L_{-1}(2) = G, \quad K\left(\frac{1}{2}\right) = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}}, \]  

(9.15)
where \( G \) is Catalan’s constant, so
\[ S(2, 2; 2) = \frac{\pi^2 G}{3} - \frac{\Gamma^8(1/4)}{1920\pi^2} \]  

(9.16)
and
\[ C(4, 1; 2) = \frac{\pi^2 G}{4} + \frac{\Gamma^8(1/4)}{7680\pi^2}. \]  

(9.17)
Such exact expressions may prove useful in attempts to derive more general forms for the trigonometric sums.

10. The functional equation for \( C(2, 2m; s) \) and \( S(2, 2m; s) \)

We now convert (8.26) and (8.27) into functional equations for \( C(2, 2m; s) \) and \( S(2, 2m; s) \). To do this, we express the terms on the right-hand side as combinations of \( C(2, 2m; 1 - s) \) and \( S(2, 2m; 1 - s) \), using the functional equation (8.19) and the equations (8.18). Let us define

\[
\hat{C}(2, 2m; s) = \frac{\Gamma(s)}{\pi^s} C(2, 2m; s), \quad \hat{S}(2, 2m; s) = \frac{\Gamma(s)}{\pi^s} S(2, 2m; s)
\]

and

\[
\mathcal{F}_{2m}(s) = \frac{\Gamma(1 - s + 2m)\Gamma(s)}{\Gamma(1 - s)\Gamma(s + 2m)}.
\]

(10.1)

We note that \( \mathcal{F} \) has the properties

\[
\mathcal{F}_{2m}(1 - s) = \frac{1}{\mathcal{F}_{2m}(s)}, \quad [\mathcal{F}_{2m}(\bar{s})\mathcal{F}_{2m}(s)]_{s=1/2+i\ell} = 1.
\]

(10.2)
The functional equations we derive are then a coupled pair, which may be written as

\[
\begin{bmatrix}
\hat{C}(2, 2m; s) \\
\hat{S}(2, 2m; s)
\end{bmatrix}
= \frac{1}{2} \begin{bmatrix}
1 + \mathcal{F}_{2m}(s), & 1 - \mathcal{F}_{2m}(s) \\
1 - \mathcal{F}_{2m}(s), & 1 + \mathcal{F}_{2m}(s)
\end{bmatrix}
\begin{bmatrix}
\hat{C}(2, 2m; 1 - s) \\
\hat{S}(2, 2m; 1 - s)
\end{bmatrix}
\]

(10.3)
or, introducing a transformation matrix \( \hat{T}_{2m}(s) \),

\[
\begin{bmatrix}
\hat{C}(2, 2m; s) \\
\hat{S}(2, 2m; s)
\end{bmatrix}
= \hat{T}_{2m}(s)
\begin{bmatrix}
\hat{C}(2, 2m; 1 - s) \\
\hat{S}(2, 2m; 1 - s)
\end{bmatrix}.
\]

(10.4)
The eigenvalues and eigenvectors of the transformation matrix are

\[
1, \quad e_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \mathcal{F}_{2m}(s), \quad e_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

(10.5)
We write the transformation matrix using the spectral decomposition theorem as

$$\hat{T}_{2m}(s) = e_1 \cdot e_1^T + \mathcal{F}_{2m}(s) e_2 \cdot e_2^T,$$

where the superscript $T$ denotes the transpose.

We may force the transformation equations (10.3) into a more symmetric form by a rescaling of variables

$$\begin{bmatrix} \check{C}(2, 2m; s) \\ \check{S}(2, 2m; s) \end{bmatrix} = \frac{1}{\sqrt{\mathcal{F}_{2m}(s)}} \begin{bmatrix} \hat{C}(2, 2m; s) \\ \hat{S}(2, 2m; s) \end{bmatrix},$$

leading to

$$\begin{bmatrix} \check{C}(2, 2m; s) \\ \check{S}(2, 2m; s) \end{bmatrix} = \frac{1}{2\sqrt{\mathcal{F}_{2m}(s)}} \begin{bmatrix} \frac{1}{\sqrt{\mathcal{F}_{2m}(s)}} + \sqrt{\mathcal{F}_{2m}(s)} & \frac{1}{\sqrt{\mathcal{F}_{2m}(s)}} - \sqrt{\mathcal{F}_{2m}(s)} \\ \frac{1}{\sqrt{\mathcal{F}_{2m}(s)}} - \sqrt{\mathcal{F}_{2m}(s)} & \frac{1}{\sqrt{\mathcal{F}_{2m}(s)}} + \sqrt{\mathcal{F}_{2m}(s)} \end{bmatrix} \begin{bmatrix} \hat{C}(2, 2m; 1 - s) \\ \hat{S}(2, 2m; 1 - s) \end{bmatrix}.$$  \hfill (10.8)

The transformation matrix $\check{T}_{2m}(s)$ in (10.8) may be placed into a more aesthetic form if we introduce a quantity $\phi_{2m}(s)$ using

$$\mathcal{F}_{2m}(s) = e^{2i\phi_{2m}(s)},$$

where, from (10.2), $2\phi_{2m}(1/2 + it)$ is a real-valued function, the phase of $\mathcal{F}_{2m}(1/2 + it)$. Then we have

$$\hat{T}_{2m}(s) = e^{-i\phi_{2m}(s)} \begin{bmatrix} \cos(\phi_{2m}(s)) & -i \sin(\phi_{2m}(s)) \\ -i \sin(\phi_{2m}(s)) & \cos(\phi_{2m}(s)) \end{bmatrix}.$$  \hfill (10.10)

In (10.10), the matrix in square brackets has determinant unity; the exponential multiplying it has unit magnitude if $\phi(s)$ is real, which occurs only if $s = 1/2 + it$. We also have, corresponding to (10.6),

$$\check{T}_{2m}(s) = e^{-i\phi_{2m}(s)} \left[ e^{-i\phi_{2m}(s)} e_1 \cdot e_1^T + e^{i\phi_{2m}(s)} e_2 \cdot e_2^T \right].$$  \hfill (10.11)

From (8.18), if $1 - s_0$ is a zero of $\mathcal{C}(0, 2m; s)$, then $\mathcal{C}(2, 2m; 1 - s_0) = -\mathcal{S}(2, 2m; 1 - s_0)$, so that the vectors

$$\begin{bmatrix} \check{C}(2, 2m; 1 - s_0) \\ \check{S}(2, 2m; 1 - s_0) \end{bmatrix}$$

are parallel. The matrix $\check{T}$ preserves this parallelism, as $e_2$ is one of its eigenvectors. This corresponds to the property that, if $1 - s_0$ is a zero of $\mathcal{C}(0, 2m; s)$, then so is $s_0$. Similarly, $1 - s_0$ is a zero of $\mathcal{C}(1, 4m; s)$, then $\mathcal{C}(2, 2m; 1 - s_0) = \mathcal{S}(2, 2m; 1 - s_0)$,
so that the vectors
\[
\begin{bmatrix}
\tilde{C}(2, 2m; 1 - s_0) \\
\tilde{S}(2, 2m; 1 - s_0)
\end{bmatrix}
\text{ and } e_1
\]
are parallel. The matrix \( \hat{T} \) again preserves this property, corresponding to the statement that if \( 1 - s_0 \) is a zero of \( \tilde{C}(1, 4m; s) \), then so is \( s_0 \).

If \( s_0 \) is a zero of \( \tilde{C}(1, 4m; s) \) or \( \tilde{C}(0, 2m; s) \), then so are \( \bar{s}_0 \) and \( 1 - \bar{s}_0 \). We have that
\[
\phi_{2m}(1 - s) = -\phi_{2m}(s), \quad \phi_{2m}(1 - \bar{s}) = -\phi_{2m}(\bar{s}) = \phi_{2m}(s),
\]
so that
\[
\hat{T}_{2m}(\bar{s}) = e^{-i\phi_{2m}(\bar{s})} \left[ e^{-i\phi_{2m}(s)} e_1 \cdot (e_1)^T + e^{i\phi_{2m}(\bar{s})} e_2 \cdot (e_2)^T \right].
\]
The relation (10.15) ensures that \( \hat{T} \) preserves the null sum values in going from \( 1 - \bar{s}_0 \) to \( \bar{s}_0 \).

The Riemann hypothesis asserts that the non-trivial zeros of \( \zeta(s) \) have \( \text{Re}(s) = 1/2 \), while a particular case of the generalized Riemann hypothesis asserts that \( L_{-4}(s) \) has its non-trivial zeros on the same critical line. The results we have given here suggest that the same property holds for \( \tilde{C}(1, 4m; s) \) for arbitrary integer \( m \). This in turn carries over to the assertion that \( \tilde{C}(2, 2m; s) = \pm \tilde{S}(2, 2m; s) \) only on the critical line. In this context, it is interesting to note the properties of three functions that combine the \( \tilde{C}(2, 2m; s) \) and \( \tilde{S}(2, 2m; s) \). The first of these is
\[
\Delta_1(2, 2m; s) = \tilde{C}(2, 2m; s) \tilde{S}(2, 2m; 1 - \bar{s}) - \tilde{C}(2, 2m; 1 - \bar{s}) \tilde{S}(2, 2m; s),
\]
which satisfies the functional equation
\[
\Delta_1(2, 2m; 1 - s) = -\Delta_1(2, 2m; \bar{s}).
\]
Since \( 1 - \bar{s} = s \) on the critical line, \( \Delta_1(2, 2m; s) \) is identically zero there. The second is
\[
\Delta_2(2, 2m; s) = \tilde{C}(2, 2m; s) \tilde{C}(2, 2m; 1 - \bar{s}) - \tilde{S}(2, 2m; 1 - \bar{s}) \tilde{S}(2, 2m; s),
\]
which satisfies the functional equation
\[
\Delta_2(2, 2m; 1 - s) = \Delta_2(2, 2m; \bar{s}).
\]
Putting \( s = \sigma + it \), (10.17) and (10.19) can be written as
\[
\begin{aligned}
\Delta_1(2, 2m; \sigma + it) = -\Delta_1(2, 2m; 1 - \sigma + it), \\
\Delta_2(2, 2m; \sigma + it) = \Delta_2(2, 2m; 1 - \sigma + it),
\end{aligned}
\]
with these functions being, respectively, odd and even functions of \( \sigma - 1/2 \). The third is
\[
\Delta_3(2, 2m; s) = C(2, 2m; s)^2 - S(2, 2m; s)^2,
\]
which has the property that on the critical line its values coincide with those of \( \Delta_2(2, 2m; s) \). The behaviour of the common modulus of these two functions on the critical line is shown in figure 8, where, comparing with figure 6, we observe the confluence of the sets of zeros of \( 4\zeta(s)L_{-4}(s) \) and \( \tilde{C}(1, 4; s) \).
We can use the functional equation (10.8) together with the expression (10.10) to derive convenient expressions involving $D_1(2m; s)$ and $D_1(2m; 1 - s)$

$$D_1(2m; s) + D_2(2m; s) = |\tilde{C}(2m; 1 - s)|^2 - |\tilde{S}(2m; 1 - s)|^2$$

$$+ 2i \text{Im} \{\tilde{S}(2m; 1 - s)\tilde{C}(2m; 1 - s)\}$$

and

$$D_1(2m; s) - D_2(2m; s) = -e^{-2i\phi_{2m}(s)}\{|\tilde{C}(2m; 1 - s)|^2$$

$$-|\tilde{S}(2m; 1 - s)|^2 - 2i \text{Im} \{\tilde{S}(2m; 1 - s)\tilde{C}(2m; 1 - s)\}\}.$$  

We now use (10.22) and (10.23) to analyse the possible existence of zeros of $D_1$, $D_2$ and $D_3$ off the critical line. We note first that if $s_0$ is a zero of one of (10.22) or (10.23), it must be a zero of the other and also of $D_3$. In the second place, we see that

$$|D_1(2m; s) - D_2(2m; s)| = |D_1(2m; s) + D_2(2m; s)||e^{-2i\phi_{2m}(s)}|,$$

where the last function is monotonic in its variation with respect to $\sigma = \text{Re}(s)$. So, for a given value of $s$ off the critical line, $|D_1 - D_2| \neq |D_1 + D_2|$ unless $D_1 = 0 = D_2$. Furthermore, if either of $D_1$ or $D_2$ (but not both) is zero for $s = s_0$, then

$$|D_1(2m; s_0) - D_2(2m; s_0)| = |D_1(2m; s_0) + D_2(2m; s_0)|,$$

so $|e^{-2i\phi_{2m}(s_0)}| = 1$

and $s_0$ must lie on the critical line.

We thus deduce that zeros of $D_1$ and $D_2$ not on the critical line must be simultaneous zeros of both, and indeed must be zeros of the same order. (If they were zeros of differing order, the argument given above would still apply.) We also can say that such zeros cannot lie off the critical line but in its
neighbourhood, using the functional equation (10.17). (Indeed, $\Delta_1$ for fixed $t$ is an odd function of $\sigma$ about $\sigma=1/2$, and thus can only be zero when the first and second non-zero terms in its Taylor series have comparable moduli; these terms differ by at least two in their power of $(\sigma-1/2)$.)

We can solve (10.23) and (10.24) for $\Delta_1$ and $\Delta_2$, to obtain

$$
\begin{align*}
\Delta_2&(2, 2m; s) = e^{-i\phi_{2m}(s)} \begin{bmatrix}
\cos(\phi_{2m}(s)), & -\sin(\phi_{2m}(s)) \\
\sin(\phi_{2m}(s)), & \cos(\phi_{2m}(s))
\end{bmatrix} \\
-\imath \Delta_1&(2, 2m; s)
\end{align*}
$$

\times \left[ |\tilde{C}(2, 2m; 1-s)|^2 - |\tilde{S}(2, 2m; 1-s)|^2 \right] - 2 \imath \text{Im}[\tilde{S}(2, 2m; 1-s)\tilde{C}(2, 2m; 1-s)]. \tag{10.26}
$$

This relation expresses the vector in the left-hand side as a scaling and phase factor (in the exponential prefactor), multiplying a rotation matrix corresponding to a complex angle, operating on the right-hand vector.

From (10.26), we see that the requirement that $\Delta_1$ be zero everywhere on the critical line translates to the identity

$$
\sin(\phi_{2m}(s))[|\tilde{C}(2, 2m; 1-s)|^2 - |\tilde{S}(2, 2m; 1-s)|^2] + 2 \cos(\phi_{2m}(s))\text{Im}[\tilde{S}(2, 2m; 1-s)\tilde{C}(2, 2m; 1-s)] = 0, \tag{10.27}
$$

for $s=1/2+it$. This in turn gives us the elegant expression

$$
\Delta_2(2, 2m; s) = [1 - \imath \tan(\phi_{2m}(s))][|\tilde{C}(2, 2m; s)|^2 - |\tilde{S}(2, 2m; s)|^2], \tag{10.28}
$$

again for $s=1/2+it$. Referring to figure 6, we see that every point where $|\tilde{C}(2, 2; s)|^2 = |\tilde{S}(2, 2; s)|^2$ corresponds to a zero of either $\zeta(s)L_{-4}(s)$ or $\tilde{C}(1, 4; s)$ except the first. This point in fact has $s=1/2 + \imath \sqrt{3}/2$ and $\phi_{2m}=\pi/2$. Note also that

$$
\Re[\tilde{C}(2, 2m; 1/2 + it)] \equiv \Re[\tilde{S}(2, 2m; 1/2 + it)]. \tag{10.29}
$$

In figure 9, we show the two functions $\Delta_1(2, 2; \sigma + it)$ and $\Delta_2(2, 2; \sigma + it)$ in the region to the right of the critical line. The former of these varies in comparatively smooth fashion compared with the latter. The features evident for $\Delta_1$ near $t=9.07$ and $t=13.54$ have been confirmed to be caused by minima there in $\partial \Delta_1(2, 2; \sigma + it)/\partial \sigma$ for $\sigma=0.5$.

So, in summary, we have demonstrated the following properties of angular lattice sums:

— trigonometric sums such as $\Delta_2(2, 2m; 1-s)$ and $\Delta_3(2, 2m; 1-s)$ have zeros on the critical line, but cannot have zeros in a neighbourhood of it;
— trigonometric sums such as $\tilde{C}(2, 2m; s)$ and $\tilde{S}(2, 2m; s)$ have no zeros on the critical line;
— trigonometric sums such as $\Delta_1(2, 2m; 1-s)$ have no zeros in a neighbourhood of the critical line and are zero identically on it; and
— from the result that $\Delta_3(2, 2m; 1-s)$ has zeros on the critical line but cannot have them in a neighbourhood of it, then $\tilde{C}(2, 2m; s) = \pm \tilde{S}(2, 2m; s)$ has no
zeros in the neighbourhood of the critical line, a result relevant to the Riemann hypothesis for \( \zeta(s) \), and its generalization to \( L_{-4}(s) \), and its further generalization to the infinite class of sums \( C(1,4m; s) \) for integer \( m \).

There may be further properties of these sums that could be added to this list.

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References


