Fast-forward of adiabatic dynamics in quantum mechanics

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We propose a method to accelerate adiabatic dynamics of wave functions (WFs) in quantum mechanics to obtain a final adiabatic state except for the spatially uniform phase in any desired short time. In our previous work, acceleration of the dynamics of WFs was shown to obtain the final state in any short time by applying driving potential. We develop the previous theory of fast-forward to derive a driving potential for the fast-forward of adiabatic dynamics. A typical example is the fast-forward of adiabatic transport of a WF, which is the ideal transport in the sense that a stationary WF is transported to an aimed position in any desired short time without leaving any disturbance at the final time of the fast-forward. As other important examples, we show accelerated manipulations of WFs, such as their splitting and squeezing. The theory is also applicable to macroscopic quantum mechanics described by the nonlinear Schrödinger equation.

Keywords: atom manipulation; mechanical control of atoms; quantum transport

1. Introduction

An adiabatic process occurs when the external parameter of the Hamiltonian of the system is adiabatically changed. The quantum adiabatic theorem (Born & Fock 1928; Kato 1950; Messiah 1962), which states that, if the system is initially in an eigenstate of the instantaneous Hamiltonian, it remains so during the process, has been studied in various contexts (Berry 1984; Aharonov & Anandan 1987; Samuel & Bhandari 1988; Shapere & Wilczek 1989; Nakamura & Rice 1994; Bouwmeester et al. 1996; Farhi et al. 2001; Roland & Cerf 2002; Sarandy & Lidar 2005; Du et al. 2008). The rate of change in the parameter of the Hamiltonian is infinitesimal, so that it takes an infinite time to achieve the final result in the process.

In our previous paper (Masuda & Nakamura 2008), we investigated a method to accelerate quantum dynamics using an additional phase of a wave function (WF). We can accelerate a given quantum dynamics to obtain a target state in any desired short time. This kind of acceleration is called the fast-forward of adiabatic dynamics.
quantum dynamics. (Readers are recommended to refer to the previous paper for an explanation of the terminologies used in this communication.) Nowadays, we have noble technologies to control nano-scale objects as well as macroscopic objects. As shown by Eigler & Schweizer (1990), we can manipulate even single atoms. As another example, Bose–Einstein condensates, which are composed of many Bose particles and can be described by a single macroscopic WF, have been studied from various aspects. If we are going to apply such technologies for manufacturing purposes, speedup of the manipulation of quantum states will be necessary.

The idea of the adiabatic process seems to be incompatible with that of fast-forward. But here we combine these two ideas, i.e. we propose a theory to accelerate the adiabatic dynamics in quantum mechanics and obtain, in any desired short time, the target state originally accessible after an infinite time through the adiabatic dynamics. By using this theory, we can find a driving potential to generate the target state exactly, except for a spatially uniform time-dependent phase such as dynamical and adiabatic phases (Berry 1984). The fast-forward of the adiabatic dynamics makes an ideal transport of quantum states possible; a stationary wave packet (WP) is moved to an aimed position without leaving any disturbance at the end of the transport. After the transport, the WP becomes stationary again and is in the same energy level as the initial one.

Before embarking upon the main part of the text, we briefly summarize the previous theory of the fast-forward of quantum dynamics (Masuda & Nakamura 2008). In the Schrödinger equation with a given potential $V_0 = V_0(x, t)$ and nonlinearity constant $c_0$ (appearing in macroscopic quantum dynamics),

$$i\hbar \frac{d\Psi_0}{dt} = -\frac{\hbar^2}{2m_0} \nabla^2 \Psi_0 + V_0(x, t) \Psi_0 - c_0 |\Psi_0|^2 \Psi_0,$$

where $\Psi_0(x, t)$ is supposed to be a known function of space ($x$) and time ($t$) and is called a standard state. For any long time $T$, called a standard final time, we choose $\Psi_0(t = T)$ as a target state that we are going to generate. Let $\Psi_\alpha(x, t)$ be a virtually fast-forwarded state of $\Psi_0(x, t)$ defined by

$$|\Psi_\alpha(t)\rangle = |\Psi_0(\alpha t)\rangle,$$

where $\alpha$ is a time-independent magnification factor of the fast-forward. The time evolution of the WF is speeded up for $\alpha > 1$ and slowed down for $0 < \alpha < 1$ like a slow motion. A rewind can occur for $\alpha < 0$, and the WF pauses when $\alpha = 0$.

In general, the magnification factor can be time-dependent, $\alpha = \alpha(t)$. The time evolution of the WF is accelerated and decelerated when $\alpha(t)$ is increasing and decreasing, respectively. In this case, the virtually fast-forwarded state is defined as

$$|\Psi_\alpha(t)\rangle = |\Psi_0(\Lambda(t))\rangle,$$

where

$$\Lambda(t) = \int_0^t \alpha(t') dt'.$$

Since the generation of $\Psi_\alpha$ requires an anomalous mass reduction, we cannot generate $\Psi_\alpha$ (Masuda & Nakamura 2008). But we can, obtain the target state by considering a fast-forwarded state $\Psi_{FF} = \Psi_{FF}(x, t)$, which differs from $\Psi_\alpha$ by an
additional space-dependent phase, \( f = f(x, t) \), as

\[
\Psi_{\text{FF}}(t) = e^{if} \Psi_\alpha(t) = e^{if} \Psi_0(\Lambda(t)).
\]  

(1.5)

The Schrödinger equation for \( \Psi_{\text{FF}} \) is given by

\[
i\hbar \frac{d\Psi_{\text{FF}}}{dt} = -\frac{\hbar^2}{2m_0} \nabla^2 \Psi_{\text{FF}} + V_{\text{FF}}(x, t) \Psi_{\text{FF}} - c_0 |\Psi_{\text{FF}}|^2 \Psi_{\text{FF}},
\]  

(1.6)

where \( V_{\text{FF}}(x, t) \) is called a driving potential. If we appropriately tune the initial and final behaviours of the time dependence of \( \alpha \) (the detail is shown later), the additional phase can vanish at the final time of the fast-forward \( T_F \), and we can obtain the exact target state, i.e.

\[
\Psi_{\text{FF}}(T_F) = \Psi_0(T),
\]  

(1.7)

where \( T_F \) is the final time of the fast-forward defined by

\[
T = \int_0^{T_F} \alpha(t) \, dt.
\]  

(1.8)

(In the case of a constant \( \alpha \), \( T_F = T/\alpha \).)

From equations (1.1), (1.4), (1.5) and (1.6), we obtain an equation for the additional phase \( f \) (Masuda & Nakamura 2008),

\[
|\Psi_\alpha|^2 \nabla^2 f + 2 \text{Re}[\Psi_\alpha \nabla \Psi_\alpha^*] \cdot \nabla f + (\alpha - 1) \text{Im}[\Psi_\alpha \nabla^2 \Psi_\alpha^*] = 0,
\]  

(1.9)

and the driving potential of the fast-forward \( V_{\text{FF}} \),

\[
V_{\text{FF}} = \alpha V_0 - \hbar \frac{df}{dt} - \frac{\hbar^2}{2m_0} (\nabla f)^2 + \text{Re} \left[ -(\alpha - 1) \frac{\hbar^2}{2m_0} \nabla^2 \Psi_\alpha \frac{\Psi_\alpha}{\Psi_\alpha} + i \frac{\hbar^2}{m_0} \nabla f \cdot \nabla \Psi_\alpha \frac{\Psi_\alpha}{\Psi_\alpha} \right] - (\alpha - 1) c_0 |\Psi_\alpha|^2.
\]  

(1.10)

In equations (1.9) and (1.10), \( \Psi_\alpha(x, t), f(x, t), \alpha(t), V_{\text{FF}}(x, t) \) and \( V_0(x, \Lambda(t)) \) are abbreviated to \( \Psi_\alpha, f, \alpha, V_{\text{FF}} \) and \( V_0 \), respectively. Using the phase \( \eta = \eta(x, t) \) of the standard state \( \Psi_0 \), \( f \) is given by (Masuda & Nakamura 2008)

\[
f(x, t) = (\alpha(t) - 1) \eta(x, \Lambda(t)),
\]  

(1.11)

which satisfies equation (1.9) and determines \( V_{\text{FF}} \) in equation (1.10). We impose the initial and final conditions for \( \alpha(t) \): \( \alpha \) must start from 1, increase for a while and decrease back to 1 with \( d\alpha/dt = 0 \) at the final time of the fast-forward. Then, the additional phase \( f \) vanishes in the initial and final time of the fast-forward. Once we have a standard state, we can obtain a target state in any desired short time by applying the driving potential with suitably tuned \( \alpha(t) \).

However, in the fast-forward of the adiabatic dynamics, we shall use infinitely large \( \alpha \). Then, the expressions of \( V_{\text{FF}} \) in equation (1.10) and \( f \) in equation (1.11) should diverge. This difficulty will be overcome by regularization of the standard potential and states that will be described in §2. In §3, we apply the theory to the ideal transport by accelerating the adiabatic transport of a WF in a moving confining potential. Ideal manipulations, such as splitting and squeezing of WPs, are shown as other examples of the fast-forward of adiabatic dynamics. Section 4 is devoted to a summary and discussions.
2. Regularization of the standard state and the driving potential for fast-forward

(a) Difficulties involved in standard adiabatic dynamics

We consider the dynamics of a WF, $\Psi_0$, under the potential $V_0 = V_0(x, R(t))$ that varies adiabatically, where $R = R(t)$ is a parameter in the potential that is adiabatically changed from a constant $R_0$ as

$$R(t) = R_0 + \varepsilon t.$$  (2.1)

The constant value $\varepsilon$ is the rate of adiabatic change of $R(t)$ with respect to time and is infinitesimal, i.e.

$$\frac{dR(t)}{dt} = \varepsilon$$  (2.2)

and

$$\varepsilon \ll 1.$$  (2.3)

The Hamiltonian is represented as

$$H_0 = \frac{p^2}{2m_0} + V_0(x, R(t)),$$  (2.4)

and the Schrödinger equation for $\Psi_0$ is given as

$$i\hbar \frac{d\Psi_0}{dt} = -\frac{\hbar^2}{2m_0} \nabla^2 \Psi_0 + V_0(x, R(t)) \Psi_0 - c_0|\Psi_0|^2 \Psi_0,$$  (2.5)

where $c_0$ is a nonlinearity constant. If a system is in the $n$th energy eigenstate at the initial time, the adiabatic theorem guarantees that, in the limit $\varepsilon \to 0$, $\Psi_0$ remains in the $n$th energy eigenstate of the instantaneous Hamiltonian. Then, $\Psi_0$ is represented as

$$\Psi_0(x, t, R(t)) = \phi_n(x, R(t)) e^{-(i/\hbar) \int_0^t E_n(R(t')) dt'} e^{i\Gamma(t)},$$  (2.6)

where $E_n = E_n(R)$ and $\phi_n = \phi_n(x, R)$ are the $n$th energy eigenvalue and eigenstate corresponding to the parameter $R$, respectively, and $\Gamma = \Gamma(t)$ is the adiabatic phase defined by

$$\Gamma(t) = i \int_0^t \int_{-\infty}^{\infty} dx \, dt \phi_n^* \frac{d}{dt} \phi_n,$$  (2.7)

which is independent of space coordinates. $\phi_n$ fulfills

$$\frac{\partial \phi_n}{\partial t} = 0$$  (2.8)

and

$$-\frac{\hbar^2}{2m_0} \nabla^2 \phi_n + V_0(x, R) \phi_n - c_0|\phi_n|^2 \phi_n = E_n(R) \phi_n.$$  (2.9)
The second factor in the right-hand side of equation (2.6) is called a dynamical phase factor, which is also space independent. Such ideal adiabatic dynamics with $\varepsilon \to 0$ takes infinite time until we have an aimed adiabatic state (target state).

Our aim is to accelerate the adiabatic dynamics $\phi_n(R(0)) \to \phi_n(R(T))$ aside from the dynamical and adiabatic phases by applying the theory of fast-forward with an infinitely large magnification factor $\alpha$ and to obtain a target state $\phi_n(R(T))$ in any desired short time, where $T$ is a standard final time that is taken to be $O(1/\varepsilon)$. During the fast-forward, we will take the limit $\varepsilon \to 0$, $\alpha \to \infty$, $T \to \infty$ and $\alpha \varepsilon \sim 1$. In this acceleration, we do not care about spatially uniform phase. (The spatially uniform phase can be controlled by a spatially uniform potential, if necessary.) In applying the fast-forward to the adiabatic dynamics, we have to choose a standard state and a Hamiltonian. But there is some ambiguity in the choice of the standard state and the Hamiltonian because the state that we want to accelerate is represented in the limit $\varepsilon \to 0$ and we will fast-forward it with an infinitely large magnification factor. One might think that we can take $\Psi_0$ and $H_0$ in equations (2.6) and (2.4) as a standard and a Hamiltonian state, respectively. However, such an idea is not adequate because the state in equation (2.6) is an expression of a WF in the limit $\varepsilon \to 0$ and does not satisfy the Schrödinger equation up to $O(\varepsilon)$ with small but finite $\varepsilon$ (Kato 1950; Wu & Yang 2005). In other words, quantum dynamics in equation (1.1) with finite $\varepsilon$ inevitably induces a non-adiabatic transition, but $\Psi_0$ in equation (2.6) ignores such transitions. Then, the original theory of fast-forward (Masuda & Nakamura 2008) is not applicable as it stands. To overcome this difficulty, we shall regularize the standard state and the Hamiltonian corresponding to the adiabatic dynamics.

In the fast-forward of the adiabatic dynamics in the limit $\varepsilon \to 0$, $\alpha \to \infty$ and $\alpha \varepsilon \sim 1$, the standard final time $T$ is chosen as $T = O(1/\varepsilon)$. In this case, a regularized standard state and a Hamiltonian should fulfil the following conditions:

(i) a regularized standard Hamiltonian and a state of the fast-forward should agree with $H_0$ and $\Psi_0$, except for the space-independent phase, respectively, in the limit $\varepsilon \to 0$ and
(ii) the regularized standard state should satisfy the Schrödinger equation corresponding to the regularized standard Hamiltonian up to $O(\varepsilon)$ with finite $\varepsilon$.

Hereafter, $\Psi_0^{\text{(reg)}}$ and $H_0^{\text{(reg)}}$ are prescribed to a regularized standard state and a Hamiltonian, respectively, which fulfil the conditions (i) and (ii).

(b) **Regularized standard state**

Here, we derive expressions for $\Psi_0^{\text{(reg)}}$ and $H_0^{\text{(reg)}}$. Let us consider a regularized Hamiltonian $H_0^{\text{(reg)}}$,

$$H_0^{\text{(reg)}} = \frac{\mathbf{p}^2}{2m_0} + V_0^{\text{(reg)}},$$

where the potential $V_0^{\text{(reg)}}$ in the regularized Hamiltonian is given as

$$V_0^{\text{(reg)}}(\mathbf{x}, t) = V_0(\mathbf{x}, R(t)) + \varepsilon \bar{V}(\mathbf{x}, t).$$

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Here, $\tilde{V}$ is a real function of $x$ and $t$ to be determined \textit{a posteriori}, which is introduced to incorporate the effect of non-adiabatic transitions. It is obvious that $H_0^{(\text{reg})}$ agrees with $H_0$ in the limit $\varepsilon \to 0$, i.e.

$$\lim_{\varepsilon \to 0} H_0^{(\text{reg})}(x, t) = H_0(x, R(t)).$$

(2.12)

Suppose the regularized standard state is represented as

$$\Psi_0^{(\text{reg})}(x, t, R(t)) = \phi_n(x, R(t))e^{-(i/\hbar)\int_0^t E_n(R(t'))dt'} e^{i\varepsilon\theta(x, t)},$$

where $\theta$ is a real function of $x$ and $t$. $\phi_n$ and $E_n$ are the same as those used in equation (2.6). $\Psi_0^{(\text{reg})}$ coincides with $\Psi_0$, except for the spatially uniform phase in the limit $\varepsilon \to 0$.

Thus, $\Psi_0^{(\text{reg})}$ in equation (2.13) and $H_0^{(\text{reg})}$ in equation (2.10) fulfil condition (i) given in the previous subsection. $\theta$ and $\tilde{V}$ are chosen so that $\Psi_0^{(\text{reg})}$ satisfies the Schrödinger equation corresponding to $H_0^{(\text{reg})}$ up to $O(\varepsilon)$ to fulfil condition (ii).

Here, we shall obtain $\theta$ and $\tilde{V}$. The Schrödinger equation for $\Psi_0^{(\text{reg})}$ is represented as

$$i\hbar \frac{d\Psi_0^{(\text{reg})}}{dt} = -\frac{\hbar^2}{2m_0} \nabla^2 \Psi_0^{(\text{reg})} + V_0^{(\text{reg})}(x, R(t)) \Psi_0^{(\text{reg})} - c_0|\Psi_0^{(\text{reg})}|^2 \Psi_0^{(\text{reg})}.$$  

(2.14)

Substituting equations (2.11) and (2.13) into equation (2.14), we find

$$i\hbar \frac{\partial \phi_n}{\partial R} + E_n \phi_n - \hbar \frac{d\theta}{dt} \varepsilon \phi_n = -\frac{\hbar^2}{2m_0} [\nabla^2 \phi_n + 2i\varepsilon \nabla \theta \cdot \nabla \phi_n - \varepsilon^2 (\nabla^2 \theta)^2 \phi_n + i\varepsilon (\nabla^2 \theta) \phi_n] + V \phi_n + \varepsilon \tilde{V} \phi_n - c_0|\phi_n|^2 \phi_n.$$  

(2.15)

Substituting equation (2.9) into equation (2.15), we have

$$i\hbar \frac{\partial \phi_n}{\partial R} - \hbar \frac{d\theta}{dt} \phi_n = -\frac{\hbar^2}{2m_0} [2i\nabla \theta \cdot \nabla \phi_n + i(\nabla^2 \theta) \phi_n] + \tilde{V} \phi_n,$$  

(2.16)

where we eliminated the second-order term in $\varepsilon$. Multiplying both sides of equation (2.16) by $(i/\hbar)\phi^*_n$, we have

$$-\phi^*_n \frac{\partial \phi_n}{\partial R} - i\hbar \frac{d\theta}{dt} |\phi_n|^2 = \frac{\hbar}{2m_0} [2\phi^*_n \nabla \phi_n \cdot \nabla \theta + |\phi_n|^2 \nabla^2 \theta] + i\tilde{V} |\phi_n|^2.$$  

(2.17)

Equation (2.17) is equivalently represented by decomposing it into real and imaginary parts as

$$|\phi_n|^2 \nabla^2 \theta + 2 \text{Re}[\phi_n \nabla \phi^*_n] \cdot \nabla \theta + \frac{2m_0}{\hbar} \text{Re} \left[ \phi_n \frac{\partial \phi^*_n}{\partial R} \right] = 0$$  

(2.18)

and

$$\frac{\hbar}{m_0} \text{Im}[\phi^*_n \nabla \phi_n] \cdot \nabla \theta + \frac{i\tilde{V}}{\hbar} |\phi_n|^2 + \text{Im} \left[ \phi^*_n \frac{\partial \phi^*_n}{\partial R} \right] + i\hbar \frac{d\theta}{dt} |\phi_n|^2 = 0.$$  

(2.19)
\( \theta(x, t) \) should satisfy equation (2.18). Equation (2.18) can also be derived from the continuity equation for \( \Psi_0^{(\text{reg})} \) (see appendix A). \( \tilde{V} \) is then given in terms of \( \theta \) and \( \phi_n \), i.e.

\[
\tilde{V} = -\hbar \frac{d\theta}{dt} - \hbar \text{Im} \left[ \frac{\partial \phi_n / \partial R}{\phi_n} \right] - \frac{\hbar^2}{m_0} \text{Im} \left[ \frac{\nabla \phi_n}{\phi_n} \right] \cdot \nabla \theta. \tag{2.20}
\]

Equation (2.18) indicates that \( \theta \) is a function that is not dependent on \( t \) explicitly, leading to

\[
\frac{d\theta}{dt} = \frac{\partial R}{\partial t} \frac{\partial \theta}{\partial R} = \varepsilon \frac{\partial \theta}{\partial R}. \tag{2.21}
\]

Therefore, in our approximation to suppress the terms of \( O(\varepsilon^2) \), equation (2.20) is reduced to

\[
\tilde{V} = -\hbar \text{Im} \left[ \frac{\partial \phi_n / \partial R}{\phi_n} \right] - \frac{\hbar^2}{m_0} \text{Im} \left[ \frac{\nabla \phi_n}{\phi_n} \right] \cdot \nabla \theta, \tag{2.22}
\]

where \( \tilde{V}(x, t) \), \( \phi_n(x, R) \) and \( \theta(x, t) \) are abbreviated to \( \tilde{V} \), \( \phi_n \) and \( \theta \), respectively. Thus, we have obtained equations that \( \theta \) and \( \tilde{V} \) should satisfy so that the conditions of the regularized standard state and Hamiltonian are fulfilled.

\[ (c) \text{ Driving potential for fast-forward} \]

Here, we obtain the driving potential \( V_{\text{FF}} \) to fast-forward the regularized standard state \( \Psi_0^{(\text{reg})} \). Such limits are taken as \( \varepsilon \to 0, \alpha \to \infty \) and \( \alpha \varepsilon \sim 1 \) in the fast-forward. The final time of the standard dynamics \( T \) is taken to be \( O(1/\varepsilon) \). The driving potential \( V_{\text{FF}} = V_{\text{FF}}(x, t) \) in equation (1.10) is explicitly represented as

\[
V_{\text{FF}}(x, t) = \alpha(t) V_0^{(\text{reg})}(x, R(\Lambda(t))) - \hbar \frac{df}{dt} - \frac{\hbar^2}{2m_0} (\nabla f)^2 + \text{Re} \left[ -\frac{(\alpha(t) - 1)(\hbar^2/2m_0) \nabla^2 \Psi_0^{(\text{reg})}(x, \Lambda(t))}{\Psi_0^{(\text{reg})}(x, \Lambda(t))} \right. \\
+ \left. i \frac{\hbar^2}{m_0} \nabla f \cdot \frac{\nabla \Psi_0^{(\text{reg})}(x, \Lambda(t))}{\Psi_0^{(\text{reg})}(x, \Lambda(t))} \right] - (\alpha(t) - 1)c_0|\Psi_0^{(\text{reg})}(x, \Lambda(t))|^2, \tag{2.23}
\]

where \( f = f(x, t) \) is the additional phase of the fast-forwarded state \( \Psi_{\text{FF}} \),

\[
\Psi_{\text{FF}}(x, t) = e^{i f(x, t)} \Psi_0^{(\text{reg})}(x, \Lambda(t)), \tag{2.24}
\]

with \( \Lambda(t) \) defined by equation (1.4).

As described in §1, the additional phase \( f \) given by equation (1.11) is not convenient for the fast-forward of adiabatic dynamics because \( V_{\text{FF}} \) in equation (2.23) would diverge owing to the infinitely large \( \alpha \). Instead, we can
take $\nabla f$ as

$$\nabla f = (\alpha - 1) \varepsilon \nabla \theta,$$  \hspace{1cm} (2.25)

which also satisfies equation (1.9) (see appendix B). Therefore, we can determine $f$ such that

$$f(x, t) = (\alpha(t) - 1) \varepsilon \theta(x, \Lambda(t)).$$  \hspace{1cm} (2.26)

With the use of $f$ in equation (2.26), we can avoid divergence of the driving potential. By using equations (2.9), (2.11), (2.13) and (2.26) in equation (2.23), the driving potential $V_{FF}$ is written as

$$V_{FF} = \alpha \varepsilon \tilde{V} + V_0 - (\alpha - 1) \frac{\hbar^2}{2m_0} \varepsilon^2 (\nabla \theta)^2 - \hbar \frac{d\alpha}{dt} \varepsilon \theta - \hbar(\alpha - 1) \varepsilon \frac{d\theta}{dt} - \frac{\hbar^2}{2m_0} (\alpha - 1)^2 \varepsilon^2 (\nabla \theta)^2 + (\alpha - 1) E_n(R(\Lambda)), \hspace{1cm} (2.27)$$

where $V_{FF}(x, t), \alpha(t), \tilde{V}(x, \Lambda(t)), \theta(x, \Lambda(t))$ and $V_0(x, \Lambda(t))$ are abbreviated to $V_{FF}, \alpha, \tilde{V}, \theta$ and $V_0$, respectively, and the same abbreviations will be used in equation (2.28). Noting that $d\theta(R(\Lambda(t)))/dt = \alpha(t) \varepsilon (\partial \theta/\partial R)$ in the limit $\varepsilon \to 0, \alpha \to \infty$ and $\varepsilon \alpha \sim 1$, we can suppress the terms of $O(\alpha^p \varepsilon^q)$ with $q > p \geq 0$ in equation (2.27). While the last term in equation (2.27) would diverge, we omit it because it only concerns the space-independent phase of $\Psi_{FF}$ and has no effect on the dynamics governed by $V_{FF}$. Then, the driving potential reduces to

$$V_{FF} = \alpha \varepsilon \tilde{V} + V_0 - \frac{\hbar}{2m_0} \alpha^2 \varepsilon^2 (\nabla \theta)^2. \hspace{1cm} (2.28)$$

From equation (2.28), the driving potential coincides with the standard potential $V_0$ when $\alpha \varepsilon = 0$ and $d\alpha/dt = 0$. Thus, $\alpha \varepsilon$ should grow from 0 and come back to 0 at $t = T_F$, where $T_F$ is the final time of the fast-forward defined by equation (1.8). On the other hand, $\alpha \varepsilon = O(1)$ during the fast-forward. The change in parameter $R$ in equation (2.1), during $t = 0$ and $T_F$, is given by

$$\Delta R \equiv R(\Lambda(T_F)) - R(0) = \varepsilon \int_0^{T_F} \alpha(t) \, dt. \hspace{1cm} (2.29)$$

What we have to do for the fast-forward of an adiabatic dynamics is to obtain $\theta$ from a given adiabatic dynamics $\phi_n(R(t))$ and then apply the driving potential $V_{FF}$ given in terms of $\theta$ and $\phi_n$ in equation (2.28).

### 3. Examples

We show some examples of the fast-forward of adiabatic processes in the linear regime ($c_0 = 0$), by numerically iterating equation (1.6) with $V_{FF}(x, t)$ in equation (2.28). (Examples in the nonlinear regime ($c_0 \neq 0$) will be reported elsewhere.) We can obtain the target state in the adiabatic process in any desired short time by applying the theory in §2. In the following examples, the
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\[ \alpha(t) \varepsilon = \bar{v} \cos \left( \frac{2\pi}{T_F} t + \pi \right) + \bar{v}, \]  

where \( \bar{v} \) is the time average of \( \alpha(t) \varepsilon \) during the fast-forwarding, and the final time of the fast-forward \( T_F \) is related to the standard final time \( T \) as \( T_F = \varepsilon T / \bar{v} \) (see equation (1.8)). We take \( \varepsilon T \) and \( T_F \) as finite, where \( \varepsilon \) is infinitesimal and \( T \) is infinitely large. Namely, we aim to obtain the target state in finite time, while the state is supposed to be obtained after infinitely long time \( T \) in the original adiabatic dynamics. The time dependence of \( \alpha(t) \varepsilon \) is shown in figure 1.

(a) Fast-forward of adiabatic transport

Suppose we have a stationary WP in a confining potential in two dimensions and we want to transport the WP without leaving any disturbance on the WF after the transport. If we move the confining potential rapidly to transport the WP fast, the WF is radically affected and it will oscillate in the confining potential after the transport. However, we can realize an ideal transport without leaving any disturbance at the end of the transport by applying the present theory of fast-forward of the adiabatic transport. Suppose there is a stationary WF, \( \psi(x) e^{-(i/h)E_n t} \), in the confining potential \( U = U(x, y) \). Then, let the potential adiabatically slide with infinitesimal velocity \( \varepsilon \) in the \( x \)-direction. The sliding potential \( V_0 = V_0(x, y, t) \) is given by

\[ V_0 = U(x - \varepsilon t, y). \]  

The regularized WF in the potential is supposed to be given as

\[ \Psi^{(reg)}_0 = \phi_n e^{-(i/h)E_n t} e^{i\varepsilon \theta}, \]

where

\[ \phi_n = \psi(x - \varepsilon t, y) = \psi(x - R(t), y) \]

and

\[ R(t) = \varepsilon t. \]

From equation (3.4), we have

\[ \frac{\partial \phi_n}{\partial R} = -\frac{\partial \phi_n}{\partial x}. \]
Substituting equation (3.6) into equation (2.18) or equation (A5), we have
\[
\frac{\hbar}{m_0} \nabla \cdot [\phi_n^2 \nabla \theta] = 2 \text{Re} \left[ \frac{\partial \phi_n^*}{\partial x} \phi_n \right].
\] (3.7)

It can be easily confirmed that
\[
\theta = \frac{m_0}{\hbar} x
\] (3.8)
fulfils equation (3.7) for any function \( \phi_n \). By substituting equations (3.6) and (3.8) into equation (2.22), we find
\[
\tilde{V} = 0, \quad (3.9)
\]
and the potential \( V^{(\text{reg})}_0(x, y, t) \) in equation (2.11) in the regularized Hamiltonian \( H^{(\text{reg})}_0 \) is given as
\[
V^{(\text{reg})}_0 = V_0 = U(x - \epsilon t, y). \quad (3.10)
\]

It should be noted that we can also derive equations (3.8) and (3.9) from the Galilean transformation of the coordinates and omitting phase of the WF in \( O(\epsilon^2) \), which does not affect the fast-forward.

Using equations (3.8) and (3.10) in equation (2.28), we have
\[
V_{\text{FF}}(x, y, t) = U(x - \epsilon \Lambda(t), y) - \frac{\partial \alpha}{\partial t} m_0 \epsilon x - \frac{m_0}{2} \alpha^2 \epsilon^2. \quad (3.11)
\]

Since we have derived \( \theta \) without giving any specific form of \( \phi_n \), the formula for the driving potential in equation (3.11) is independent of the profile of the WF that we are going to transport.

As an example of the fast-forward of the adiabatic transport, we choose a ground state in a harmonic potential \( V^{(\text{reg})}_0 = (m_0 \omega^2 / 2) \{(x - x_0(t))^2 + (y - y_0)^2\} \). The centre of potential is adiabatically moved in the \( x \)-direction as
\[
x_0(t) = x_0(0) + \epsilon t. \quad (3.12)
\]
where \( x_0 \) corresponds to \( R \) in §2. The regularized standard state \( \Psi^{(\text{reg})}_0 \) with \( E_{n=0} = \hbar \omega \) is
\[
\Psi^{(\text{reg})}_0(x, t) = \phi_n e^{i \epsilon} e^{-i / \hbar} \int_0^t E_n(R) \, dt
\]
\[
\equiv \left( \frac{m_0 \omega}{\pi \hbar} \right)^{1/2} \exp \left[ - \frac{m_0 \omega}{2 \hbar} \{(x - x_0(t))^2 + (y - y_0)^2\} - i \omega t \right] e^{i \epsilon} e^{-i \omega t}, \quad (3.13)
\]
with \( \theta \) given by equation (3.8). While any positive values are allowed for \( \epsilon T \) and \( T_F \), we put \( \epsilon T = 20 \) and \( T_F = 6.25 \), for example, where \( \epsilon \) is infinitesimal and \( T \) is infinitely large. The centre of mass moves from \((x, y) = (x_0(0), y_0)\) to \((x_0(0) + \epsilon T, y_0)\) during the fast-forward, where \( y_0 \) is a constant. Figure 2 is the spatio-temporal dependence of the driving potential (dashed line) and the amplitude of the fast-forwarded state (solid line) at \( y = y_0 \). Figure 3 shows the spatio-temporal dependence of the real part of the WF of the fast-forwarded state at \( y = y_0 \). Owing to the additional phase \( f \), the real part of the WF shows spatial oscillations during
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Figure 2. Spatio-temporal dependence of the driving potential (dashed lines) and the amplitude of the fast-forwarded state $|\psi_{FF}|^2$ (solid lines) at $y = y_0$. The parameters are taken as $\epsilon T = 20$, $T_F = 6.25$, $\hbar/m_0 = 1$, $\omega = 1$, $x_0(0) = 6$ and $y_0 = 16$.

Figure 3. Spatio-temporal dependence of the real part of the WF of the fast-forwarded state at $y = y_0$. The parameters are the same as in figure 2.

the fast-forward, but at the final time of the fast-forward, such an oscillation disappears and $\psi_{FF}$ agrees with the target state, except for the spatially uniform phase. We can also fast-forward the adiabatic transport of the WP in any excited state and obtain the target state in finite time. This is the ideal transport of WPs because we can obtain the target state without any disturbances at the final time of the fast-forward. In addition, the WP becomes stationary again after the transport when $\alpha \epsilon = 0$ and $d\alpha/dt = 0$.

To numerically check the validity of the fast-forward, we calculate the fidelity defined as

$$F = |\langle \psi_{FF}(t)|\psi^{(\text{reg})}_0(A(t))\rangle|,$$

i.e. the overlap between the fast-forwarded state $\psi_{FF}(t)$ and the corresponding standard one $\psi^{(\text{reg})}_0(A(t))$, where $\psi^{(\text{reg})}_0(A(T_F)) \equiv \psi^{(\text{reg})}_0(T)$. The time dependence of the fidelity defined by equation (3.14) is shown in figure 4. We find that the fidelity first decreases from unity owing to the additional phase $f$ of the
fast-forwarded state, but at the final time, it becomes unity with high precision ($|1 - F| \leq 10^{-5}$) again. We have thus obtained the adiabatically accessible target state with a good accuracy in finite time $T_F = 6.25$.

(b) Fast-forward of wave packet splitting

We then show the fast-forward of WP splitting in one dimension as an example of the manipulation of a macroscopic WF. Here, we calculate $\theta$ by a numerical integration of equation (2.18) or equation (A5). Let us consider the dynamics of a Gaussian WP to be split into a pair of separated WPs. The regularized standard WF with $E_{n=0} = 0$ is chosen as

$$
\psi_0^{(\text{reg})} = \phi_n e^{i \theta} e^{-i(t/\hbar) \int_0^t E_n(R) \, dt} \equiv h(R) \left\{ (1 - R) \exp \left[ -\frac{a}{2} x^2 \right] + R x^2 \exp \left[ -\frac{a}{2} x^2 \right] \right\} e^{i \theta},
$$

(3.15)

where $R$ is defined by equation (2.1) with $R_0 = 0$ and $h(R)$ is a normalization constant represented as

$$
h(R) = \left[ (1 - R)^2 \frac{\sqrt{\pi}}{a} + (1 - R) R \frac{\sqrt{\pi}}{a^{3/2}} + \frac{3}{4} R^2 \frac{\sqrt{\pi}}{a^{5/2}} \right]^{-1/2}.
$$

(3.16)

The potential in the regularized standard Hamiltonian is represented as

$$
V_0^{(\text{reg})} = \frac{\hbar^2}{2m_0} \frac{Ra^2 x^4 + [(1 - R)a^2 - 5Ra]x^2 + 2R - (1 - R)a}{Rx^2 + (1 - R)}.
$$

(3.17)

Note that $\tilde{V}$ in equation (2.22) vanishes owing to the absence of the space-dependent phase of $O(1)$ (Im($\partial \phi_n / \partial R$)/$\phi_n$) = Im($\partial \phi_n / \partial x$)/$\phi_n$ = 0) in the regularized standard state in equation (3.15). It is easily confirmed that for constant $R$, $\psi_0^{(\text{reg})}$ in equation (3.15) with $\theta = 0$ is a zero-energy eigenstate under the potential $V_0^{(\text{reg})}$ in equation (3.17). While for $R = 0$, the WF is a simple Gaussian, for $R = 1$, it becomes spatially separated double Gaussians. The spatial distribution of $|\psi_0^{(\text{reg})}|^2$ for various $R$ from 0 to 1 is shown in figure 5. The standard final time $T$ is taken as $T = 1/\varepsilon$ and $R(0) = 0$. As $R$ is gradually changed from 0 to 1, $\psi_0$ changes from a single-peaked WP at $t = 0$ to a double-peaked one at $t = T$, namely the splitting of a WP occurs. The final time of
the fast-forward $T_F$ is related to $T$ as $T_F = \varepsilon T/\bar{v}$. $\theta$ is obtained by numerical integration of equation (2.18) under the boundary conditions $(\partial \theta/\partial x)(x = 0) = 0$ and $\theta(x = 0) = 0$. The spatio-temporal dependence of $\theta$ is shown in figure 6.

The driving potential is calculated from equation (2.28) using equations (3.15) and (3.17) and $\theta$. $V_{FF}$ (solid line) and $|\Psi_{FF}|^2$ (dashed line), which is accelerated by $V_{FF}$, are shown in figure 7. The conversion from a hump to a hollow in the central region of the potential in figure 7 is caused by the deceleration of $\alpha(t)\varepsilon$ that suppresses the splitting force. The fidelity defined by equation (3.14) is confirmed to be back to unity with a high numerical precision (0.999) at $T_F$ (figure 8).

(c) Fast-forward of wave packet squeezing: case of $\bar{V} = 0$

The fast-forward of WP squeezing is also an important example of the accelerated WF manipulation. Let us consider a WF with a Gaussian distribution in a harmonic potential in one dimension, which is squeezed adiabatically.
The regularized standard state with $E_{n=0} = 0$ is represented as

$$\psi_0^{\text{(reg)}} = \phi_n e^{i\theta} e^{-\left(i/\hbar\right) \int_0^t E_n(R) dt} \equiv \left(\frac{m_0 \omega}{\pi \hbar}\right)^{1/4} \exp\left[-\frac{m_0 \omega}{2\hbar}x^2\right] e^{i\theta}. \quad (3.18)$$

The corresponding regularized standard potential is

$$V_0^{\text{(reg)}} = -\frac{\hbar \omega}{2} + \frac{m_0 \omega^2}{2} x^2. \quad (3.19)$$

In the adiabatic dynamics, the potential curvature $\omega$ is gradually increased as

$$\omega(t) = \omega_0 + \varepsilon t, \quad (3.20)$$

where $\omega_0$ is a constant. In this case, we can easily confirm that

$$\theta = -\frac{m_0}{4\hbar \omega} x^2 \quad (3.21)$$

satisfies equation (2.18) and that $\tilde{V}$ in the regularized standard potential in equation (2.22) vanishes because $\text{Im}((\partial \phi_n / \partial \omega) / \phi_n) = \text{Im}((\partial \phi_n / \partial x) / \phi_n) = 0$. Using

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Figure 9. Spatio-temporal dependence of the driving potential (solid line) and the amplitude of the fast-forwarded state $|\Psi_{FF}|^2 \times 3 \times 10^4$ (dashed line). The parameters are taken as $\omega_0 = 0.25$, $\epsilon T = 0.75$, $T_F = 0.025$ and $(\hbar/m_0) = 1.0$. The inset represents the potential (dark line) and the WF (light line) distribution for $-3 \leq x \leq 3$ at $t = 0$.

Figure 10. Time dependence of fidelity.

equations (3.18), (3.19) and (3.21) in equation (2.28), we obtain the driving potential as

$$V_{FF} = \left[ m_0 \omega^2(\Lambda(t)) - \frac{d\alpha}{dt} \epsilon \frac{m_0}{4\omega(\Lambda(t))} - \frac{3m_0}{8\omega^2(\Lambda(t))} \alpha^2 \epsilon^2 \right] x^2,$$

(3.22)

where $\Lambda(t)$ is defined by equation (1.4). The spatio-temporal dependence of the driving potential $V_{FF}$ is shown in figure 9. The conversion of the potential curvature from positive to negative in figure 9 is caused by the deceleration of $\alpha(t)\epsilon$, which suppresses the squeezing force. The time dependence of the fidelity is shown in figure 10. The fidelity becomes unity with a numerical precision of 0.999 at the final time of the fast-forward, and the WP squeezing has been carried out in a short time.

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(d) Fast-forward of wave packet squeezing: case of $\tilde{V} \neq 0$

In the previous examples, the right-hand side of equation (2.22) was always vanishing (e.g. because of the absence of a space-dependent phase of $O(1)$ in the standard WF). While, in many practical cases, we do not have to choose a stationary state with a space-dependent phase of $O(1)$ for a standard state, the WF can, in principle, have such a phase. Let us finally show the fast-forward of WP squeezing as an example of the fast-forward with non-zero $\tilde{V}$ by having recourse to a stationary state with a space-dependent phase $\eta$ of $O(1)$. Here, the regularized standard state with $E_{n=0} = 0$ is chosen as

$$\Psi_0^{(\text{reg})} = \phi_n e^{i\theta} e^{-(i/\hbar) \int_0^t E_n(R) dt} \equiv \left( \frac{m_0 \omega}{\pi \hbar} \right)^{1/4} \exp \left[ -\frac{m_0 \omega}{2\hbar} x^2 \right] e^{i\eta} e^{i\theta}, \quad (3.23)$$

with $\eta$ given as

$$\eta = B \int_0^x \exp \left[ \frac{m_0 \omega}{\hbar} x'^2 \right] dx', \quad (3.24)$$

where $B$ is a real constant. The regularized standard potential that guarantees the eigenstate in equation (3.23) is written as

$$V_0^{(\text{reg})} = -\frac{\hbar \omega}{2} + \frac{m_0 \omega^2}{2} x^2 - \frac{\hbar^2 B^2}{2m_0} \exp \left[ \frac{2m_0 \omega}{\hbar} x^2 \right] + \varepsilon \tilde{V}, \quad (3.25)$$

where $\omega$ is a parameter that is adiabatically changed as

$$\omega(t) = \omega_0 + \varepsilon t, \quad (3.26)$$

where $\omega_0$ is the initial value of $\omega$. $\Psi_0^{(\text{reg})}$ with $\theta = 0$ and constant $\omega$ in equation (3.23) stands for the zero-energy eigenstate trapped in the central hollow of the potential barrier $V_0^{(\text{reg})}$ with $\tilde{V} = 0$ (see inset in figure 11). And the WP is squeezed adiabatically.

From equation (2.18), which is not affected by $\eta$, $\theta$ is obtained as

$$\theta = -\frac{m_0}{4\hbar \omega} x^2. \quad (3.27)$$

Substituting equations (3.24) and (3.27) and $\phi_n$ from equation (3.23) into equation (2.22), $\tilde{V}$ is obtained as

$$\tilde{V} = -m_0 B \int_0^x x'^2 \exp \left[ \frac{m_0 \omega}{\hbar} x'^2 \right] dx' + \frac{\hbar}{2\omega} B x \exp \left[ \frac{m_0 \omega}{\hbar} x^2 \right]. \quad (3.28)$$

Substituting equations (3.27) and (3.28) into equation (2.28), we obtain the driving potential $V_{FF}$, whose spatio-temporal dependence is shown in figure 11. The solid and dashed lines represent the driving potential and the corresponding amplitude of the WF, respectively. The conversion of the potential curvature from positive to negative in figure 11 can be explained in the same way as in the case of the example in §3c. The amplitude of the fast-forwarded WF is shown in figure 12. The inset in figure 12 represents the time dependence of the fidelity for $0 \leq t \leq T_F$. We have confirmed that the fidelity comes back to unity with a high precision at $T_F$. 

4. Conclusion

We have shown a method to accelerate adiabatic dynamics in microscopic and macroscopic quantum mechanics, using an infinitely large magnification factor of fast-forward and the regularization of standard states and a Hamiltonian. One can obtain the target state exactly, except for spatially uniform (dynamical and adiabatic) phases in any desired short time, while in adiabatic dynamics, the target state is accessible after infinite time. A noble feature of the fast-forward of adiabatic dynamics, which is distinct from our previous theory, is that we should regularize in advance the standard state so as to satisfy Schrödinger
equation up to $O(\varepsilon)$ for a small but finite rate $\varepsilon$ of temporal change in the Hamiltonian. To avoid divergence of the driving potential, we used a new form of the additional phase $f$ in equation (2.26) that differs from the one in our previous work (Masuda & Nakamura 2008).

Typical examples of the fast-forward of adiabatic dynamics have been shown, e.g. WF transport and WP splitting and squeezing. The fast-forward of adiabatic transport makes the ideal transport of WFs possible. The WP is rapidly transported to the adiabatically accessible targeted position, leaving neither a disturbance nor an oscillation after the transport; the WF becomes stationary again in the confining potential at the end of the fast-forward. We confirmed that the fidelity first decreases from unity owing to the phase $f$ of the fast-forwarded state, but at the final time, it becomes unity with high precision again. Examples of the WP squeezing and splitting show a way to fast-forward the adiabatic manipulation of WPs without leaving any residual disturbance. The framework of this theory is applicable to the macroscopic quantum mechanics described by the nonlinear Schrödinger equation, and one can expect the high-speed manipulations of WPs in Bose–Einstein condensates, which will be reported elsewhere.

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Appendix A. Derivation of equation (2.18) from the continuity equation

Equation (2.18) is also obtained from the continuity equation. Suppose that the Schrödinger equation for the regularized standard state $\Psi_0^{\text{(reg)}}$ is given by equation (2.14). Suppose also that $\Psi_0^{\text{(reg)}}$ is represented by equation (2.13), where $\phi_n$ satisfies equations (2.8) and (2.9). Then, we have

$$\frac{d}{dt}|\Psi_0^{\text{(reg)}}|^2 = \frac{d\Psi_0^{\text{(reg)*}}}{dt}\Psi_0^{\text{(reg)}} + \Psi_0^{\text{(reg)*}}\frac{d\Psi_0^{\text{(reg)}}}{dt} = 2\varepsilon \text{Re} \left[ \frac{\partial\phi_n^*}{\partial R}\phi_n \right], \quad (A\ 1)$$

where we used equation (2.8). On the other hand, we have the continuity equation

$$\frac{d}{dt}|\Psi_0^{\text{(reg)}}|^2 = -\frac{\hbar}{m_0} \nabla \text{Im}[\Psi_0^{\text{(reg)*}}\nabla \Psi_0^{\text{(reg)}}]. \quad (A\ 2)$$

Substituting equation (2.13) into equation (A2), we have

$$\frac{d}{dt}|\Psi_0^{\text{(reg)}}|^2 = -\frac{\hbar}{m_0} \nabla \text{Im}[\phi_n^*\nabla \phi_n + i\varepsilon|\phi_n|^2\nabla \theta]. \quad (A\ 3)$$

We can obtain

$$\nabla \text{Im}[\phi_n^*\nabla \phi_n] = 0 \quad (A\ 4)$$
by multiplying $\phi_n^*$ on both sides of equation (2.9) and taking the imaginary part. Therefore, by using equations (A1), (A3) and (A4), we obtain

$$\frac{\hbar}{m_0} \nabla \cdot [|\phi_n|^2 \nabla \theta] = 2 \text{Re} \left[ \frac{\partial \phi_n^*}{\partial R} \phi_n \right].$$  \hspace{1cm} (A5)

$\nabla \theta$ must satisfy this equation. From equation (A5), we can reach equation (2.18).

**Appendix B. Additional phase of the fast-forwarded state**

We show that the gradient of the additional phase $f$ in equation (2.25) satisfies equation (1.9). $\Psi_0^{(\text{reg})}$ in equation (2.13) includes the amplitude factor

$$\phi_n(x, R(t)) = |\phi_n|e^{i\eta(x,t)},$$  \hspace{1cm} (B1)

where $\eta = \eta(x, R(t))$ is the space-dependent phase. Using equations (2.9) and (B1), we have

$$\phi_n^* \nabla^2 \phi_n - \phi_n \nabla^2 \phi_n^* = 2i|\phi_n|[2\nabla \eta \cdot \nabla |\phi_n| + (\nabla^2 \eta)|\phi_n|] = 0.$$  \hspace{1cm} (B2)

Thus, $\eta$ satisfies

$$2\nabla |\phi_n| \nabla \eta + |\phi_n| \nabla^2 \eta = 0.$$  \hspace{1cm} (B3)

Noting $\Psi_\alpha$ in equation (1.9) is now given by $\Psi_\alpha(t) = \Psi_0^{(\text{reg})}(\Lambda(t))$ in the regularized case, we substitute equations (2.13) and (B1) into equation (1.9), and obtain

$$|\phi_n|^2 \nabla^2 f + 2|\phi_n| \nabla |\phi_n| \cdot \nabla f + (\alpha - 1)[-2|\phi_n| \nabla (\eta + \varepsilon \theta) \cdot \nabla |\phi_n|$$

$$- |\phi_n|^2 \nabla^2 (\eta + \varepsilon \theta)] = 0.$$  \hspace{1cm} (B4)

By using equation (B3) in equation (B4), we can easily check that $\nabla f$ defined by equation (2.25) satisfies equation (B4), i.e. equation (1.9).

**References**


