On a bifurcation structure mimicking period adding

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In this work, we investigate a piecewise-linear discontinuous scalar map defined on three partitions. This map is specifically constructed in such a way that it shows a recently discovered bifurcation scenario in its pure form. Owing to its structure on the one hand and the similarities to the nested period-adding scenario on the other hand, we denoted the new bifurcation scenario as nested period-incrementing bifurcation scenario. The new bifurcation scenario occurs in several physical and electronical systems but usually not isolated, which makes the description complicated. By isolating the scenario and using a suitable symbolic description for the asymptotically stable periodic orbits, we derive a set of rules in the space of symbolic sequences that explain the structure of the stable periodic domain in the parameter space entirely. Hence, the presented work is a necessary step for the understanding of the more complicated bifurcation scenarios mentioned above.

Keywords: piecewise smooth systems; border collision bifurcations; simple limiter control; nested period-incrementing bifurcation scenario

1. Introduction

There are a large number of physical, biological and economic systems which, under certain conditions, can be mathematically described by one-dimensional maps defined on three partitions, whereby the system function is constant in one of the partitions. For example, maps with a horizontal part occur frequently in power electronics as models for DC–DC converters (Kabe et al. 2007; Saito et al. 2007). Maps of this kind have also been proposed as a simple method for controlling cardiac chaos (Glass & Zeng 1994) or, in general, controlling chaotic behaviour using the simple limiter control method (Corron et al. 2000; Murali & Sinha 2003; Stoop & Wagner 2003; Zhou & Yu 2005). They occur also as local maps in coupled map lattices used in biological and economic models (Sinha 1994). Furthermore, it has been shown in Amann et al. (2003) that the dynamics of the electron accumulation and depletion fronts in semiconductor superlattices can be approximately described by a one-dimensional map with a horizontal part. Thus, the results reported in this paper are relevant in understanding the dynamics of systems of much diversity.

In dynamical systems, as a parameter is varied, one often observes bifurcation scenarios that share some common and characteristic properties. The period-doubling scenario, the nested period-adding scenario and the period-incrementing

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scenario are three such bifurcation scenarios reported in the literature. In a period-doubling scenario (first discovered by Myrberg (1963) and later independently rediscovered and investigated in more detail by Grossmann & Thomae (1977) and Feigenbaum (1978)), a period-$n$ orbit directly bifurcates to a period-$2n$ orbit causing the periods of consecutive periodic orbits to form a geometric sequence: $p_n = p_02^n$. In contrast to this, the main characteristics of the period-incrementing and the nested period-adding scenarios are that the periods of consecutive basic periodic orbits form an arithmetic sequence: $p_n = p_0 + n\Delta p$. It is known that three different situations are possible. In the first case, the existence regions of basic orbits with periods $p_n$ and $p_{n+1}$ overlap and we denote this bifurcation structure as period-incrementing scenario with coexistence of attractors (Avrutin & Schanz 2006). In the second case, the existence regions of basic orbits with periods $p_n$ and $p_{n+1}$ are adjacent without overlapping and without a gap between them. Accordingly, we denote this situation as pure period-incrementing scenario (Avrutin & Schanz 2006). In the last case, the situation is more complex. There is a gap between the existence regions of two consecutive basic periodic orbits and the behaviour in these gaps can be either periodic or chaotic. In the first case, one well-known situation is given by the self-similar Farey tree-like bifurcation structure where between the existence regions of two consecutive periodic orbits with periods $p_n$ and $p_{n+1}$, there exists a region with period $p_n + p_{n+1}$ and so on ad infinitum. We denote this scenario as nested period-adding scenario.

The question may arise why we use the notation nested period-adding scenario. This is due to a principal difference between this and the period-doubling and the period-incrementing scenarios. Recall that the period-doubling scenario is formed by one sequence of periodic orbits whose stability regions are converging to one point in the parameter space (where one observes a so-called Feigenbaum attractor). Similarly, a period-incrementing scenario is formed by one family of periodic orbits and also has one accumulation point. In contrast to this, a period-adding structure contains an infinite number of periodic orbits (for a detailed description of these families and their recursive definition we refer to Avrutin et al. (2010)). Consequently, between the existence regions of two consecutive periodic orbits with periods $p_n$ and $p_{n+1}$, there exists an infinite (uncountable) set of accumulation points of the nested period-adding scenario. Owing to the self-similarity of the scenario, an arbitrary small vicinity of each of these accumulation points is similar (both in the state and in the parameter space) to the complete scenario, so we call it nested. Until now the nested period-adding scenario was the only one with an infinite number of accumulation points. However, below we report a new scenario (denoted as nested period-incrementing scenario), and therefore we prefer to emphasize a difference between both nested scenarios from the situations with only one accumulation point like the period-doubling and period-incrementing scenarios.

It is worth noticing that the situations with the nested period-adding and the period-incrementing scenarios may be misleading, especially because the notation on this field is not unique and also because both scenarios sometimes occur in the same system in adjacent regions of the parameter space. As far as we know, these scenarios were first observed by van der Pol & van der Mark (1927) but of course they did not denote them as nested period-adding or period-incrementing scenarios. Because they focused on the frequency and not the
period, they introduced the notion of demultiplication, which is another wording for the division of a basis frequency by subsequent integer numbers. Later on many authors did not distinguish between both scenarios as we do and denoted both of them as period-adding (see Bernardo et al. (2008) for references).

Now let us consider the situation from a slightly different perspective. In the history of nonlinear dynamics, the main objective was given for a long time by smooth systems. As these systems demonstrate the period-doubling scenario, this is one of the reasons why this scenario is so well known. However, later it turned out that many applications both in engineering sciences and in nature are required to deal with piecewise smooth systems. Naturally, in the beginning, the research on this field was mainly concentrated on the most simple case, given by systems defined on two partitions. These systems are known to demonstrate the scenarios mentioned above (especially, period-adding and -incrementing). Nowadays, in some sense the situation repeats itself. Several applications mentioned above can be described more adequately by piecewise smooth models defined on three partitions and it is not surprising that these models demonstrate bifurcation scenarios not investigated until now.

Recently, such a bifurcation scenario was observed experimentally (Kabe et al. 2007) in an electronic circuit that can be modelled by a piecewise smooth map defined on three partitions. In this new scenario, the gaps between the existence regions of two consecutive periodic orbits $p_n$ and $p_{n+1}$ are filled with a self-similar structure that superficially resembles or ‘mimics’ period-adding. However, when investigated in more detail, it becomes clear that the periods of the orbits forming this scenario are organized by a rule different from the rule underlying the well-known Farey tree-like structure and hence the scenario cannot be the usual nested period-adding. This scenario was not investigated by Kabe et al. (2007) nor in any other work to date. The aim of this paper is to provide a detailed description of this new scenario.

In order to do so, we have to construct a map with a bifurcation structure that is organized by the new bifurcation scenario in its pure form. Unfortunately, in the map used by Kabe et al. (2007), this scenario occurs not only isolated but also combined with other bifurcation phenomena, which makes the investigation complicated. However, it provided us with hints about the characteristic properties a map must possess for the new scenario to appear in its pure form. The map of Kabe et al. (2007) is piecewise-linear, defined on three partitions and has a zero slope in one of the partitions. Using this information, we constructed the following map to explore the new bifurcation scenario:

$$
\begin{align*}
    x_{n+1} &= \begin{cases} 
        f_L(x_n) = \beta & \text{if } x_n \leq 0 \\
        f_r(x_n) = a \left( 1 - 2 \left| x - \frac{1}{2} \right| \right) & \text{if } x_n > 0 
    \end{cases} \\
    &= \begin{cases} 
        f_L(x_n) = \beta & \text{if } x_n \leq 0 \\
        f_m(x_n) = 2\alpha x_n & \text{if } 0 < x_n \leq \frac{1}{2} \\
        f_r'(x_n) = 2\alpha (1 - x_n) & \text{if } x_n > \frac{1}{2}
    \end{cases} \tag{1.1}
\end{align*}
$$
Since our constructed map (1.1) resembles the tent map, with an additional linear part with zero slope on the left side, we denote it in the following as the extended tent map. Note that since our map (1.1) is defined on three partitions, its dynamic behaviour can be significantly more complicated than the dynamics of piecewise-linear maps defined on only two partitions.

This paper is structured as follows. In §2, we give a broad description of the complete parameter space of map (1.1) and define the region that contains the new bifurcation scenario. This section also contains a short overview of this scenario. Section 3 explains which stable periodic orbits form the new bifurcation scenario and also the border collision bifurcations causing these orbits to appear and to disappear. Section 4 provides then a detailed description of the bifurcation structure. A set of rules is defined, which can be used to derive both the symbolic sequence of every stable periodic orbit in the new scenario and their order in the parameter space. Section 5 contains a summary of our results.

2. Overall scenario description

Before turning to the parameter range of specific interest, let us describe the overall dynamic behaviour of system (1.1) in the parameter plane \((\alpha, \beta)\) with \(\alpha > 0\) (figure 1).

For \(\alpha \in (0, 1)\) and \(\beta \in (0, 1)\), the interval \(I = (0, 1)\) is globally attracting. Hereby, for \(\alpha \in (0, \frac{1}{2})\), all orbits converge to the boundary \(x = 0\), which is by definition of the system function an attracting non-invariant set, whereas for \(\alpha \in (\frac{1}{2}, 1)\), the asymptotic dynamics on the interval \(I\) is given by a chaotic attractor as in the case of the usual tent map. The case \(\alpha = \frac{1}{2}\) is a degenerated situation where the slope of the system function becomes one and each point \(x \in (0, \frac{1}{2}]\) represents a fixed point. For \(\alpha \in (0, 1)\) and \(\beta \not\in (0, 1)\), as the interval \(I\) is no longer globally attracting, there is a coexisting super-stable fixed point \(x^L = \beta \in (-\infty, 0]\) for \(\beta < 0\) and a coexisting stable period-2 orbit \(\{x^{LR}_0 = 2\alpha(1 - \beta), x^{LR}_1 = \beta\}\) for \(\beta > 1\).

For \(\alpha \geq 1\), the region of coexistence ends and all typical initial conditions converge in a finite number of steps either to the stable fixed point (for \(\beta \leq 0\)) or to the stable period-2 orbit (for \(\beta \geq 1\)) or to a periodic orbit with a period \(p \geq 3\) containing points located in all three partitions (for \(0 < \beta < 1\)). In the first two cases, the dynamics for each specific initial condition is restricted on at most two partitions. Therefore, we consider only the parameter region in the following:

\[
P = \{ (\alpha, \beta) \mid \alpha \geq 1, 0 < \beta < 1 \},
\]

where the stable orbits enter all three partitions. In this region, map (1.1) has an unstable fixed point

\[
x^R = \frac{2\alpha}{2\alpha + 1} \in \left[\frac{2}{3}, 1\right).
\]

Remarkably, the bifurcation structure of the extended tent map in the parameter region \(P\) (figure 2b) looks very similar to the nested period-adding
structure (for example, Bai-Lin (1989), Avrutin & Schanz (2006)) shown in figure 2a for the map
\[ x_{n+1} = ax_n + \mu - \text{sign}(x_n), \] (2.3)
which represents a well-known simplified model of an electronic circuit (Σ/Δ-modulator with a 1-bit quantizer) investigated, for example, in Gray (1987), Feely & Chua (1991), Feely (1992) and Avrutin & Schanz (2008). However, when comparing the corresponding period diagrams (figure 2c,d), it becomes clear that both bifurcation structures are entirely different. Note again that this bifurcation structure was already presented in Kabe et al. (2007) but it is neither described nor explained there.

Recall that a very condensed description of the nested period-adding scenario is the following: between the parameter intervals corresponding to the period-\( n \) orbit \( O_\sigma \) and the period-\( m \) orbit \( O_\rho \) there is a parameter interval where the period is \( n + m \), and the symbolic sequence of the periodic orbit is \( \sigma\rho \), which means the concatenation of \( \sigma \) and \( \rho \). As these intervals are not adjacent, the same rule applies again between the intervals corresponding to \( O_\sigma \) and \( O_{\sigma\rho} \) as well as \( O_{\sigma\rho} \) and \( O_\rho \). This process continues recursively ad infinitum and leads to the well-known self-similar nested period-adding structure.

On the same level of description, the scenario occurring in system (1.1) can be summarized as follows: both to the left and to the right of the parameter interval corresponding to a period-\( n \) orbit there are parameter intervals corresponding to period-(\( n + 1 \)) orbits. Also in this case the rule applies again, so there are intervals corresponding to period-(\( n + 2 \)) orbits on both sides of each interval corresponding to period-(\( n + 1 \)) orbits. Again, this process continues recursively ad infinitum and leads to the self-similar bifurcation structure which looks in the two-dimensional parameter space amazingly similar to the nested period-adding structure, although the period diagrams in both cases are completely different.

For example, the stable periodic orbit with the lowest period is the period-3 orbit, which can be seen in the middle of figure 3. To the left and to the right of it there are two period-4 orbits. Each of those two period-4 orbits has a period-5 orbit to the left of it, and a period-5 orbit to the right of it and so on. This means...
that for a given value of $\alpha$, the number of stable orbits with period $n+1$ in the complete interval $\beta \in (0,1)$ is always two times the number of stable orbits of period $n$. Since the lowest stable period that can be found is three, there are two different stable orbits with period-4, four different stable orbits with period-5 and,
3. Stable periodic orbits

It is well known that an adequate description of periodic orbits in piecewise smooth systems can be achieved based on symbolic sequences. In the case of map (1.1), the following partitioning of the state space turns out to be useful for the symbolic description:

$$\text{symbol}(x) = \begin{cases} 
\mathcal{L} & \text{if } x \leq 0 \\
\mathcal{M} & \text{if } 0 < x \leq \frac{1}{2} \\
r & \text{if } \frac{1}{2} < x \leq x^\mathcal{R} \\
\mathcal{R} & \text{if } x > x^\mathcal{R}.
\end{cases} \quad (3.1)
$$

It is worth noticing that although map (1.1) is defined on three partitions, we use four symbols in the symbolic representation. As one can see, the left and the middle partition correspond to the symbols $\mathcal{L}$ and $\mathcal{M}$, respectively, whereas in the right partition we distinguish between points to the left and to the right of the unstable fixed point $x^\mathcal{R}$. As we will see below, this is necessary for an adequate description of the stable periodic orbits.

Using the introduced symbolic description, let us consider the mechanism causing stable periodic orbits of system (1.1) to emerge in the parameter region $\mathcal{P}$. Without loss of generality, let us start at some point $x_0 \leq 0$. As $\beta \in (0, 1)$, the
next point $x_1$ belongs to the interval $I = (0, 1)$. On this interval, the stable orbit will perform some number of iteration steps until it reaches the sub-interval $J$ given by

$$J = \left[ \frac{1}{2\alpha}, 1 - \frac{1}{2\alpha} \right] = [x_J^1, x_J^2] \subset I.$$  

In this interval, the value of the system function is larger than one, so if a point $x_{n-2}$ is located in this interval, $x_{n-2} \in J$, then $x_{n-1} \geq 1$ and the point $x_n \leq 0$ is located in the left partition. As the slope of the left part $f_l(x)$ of the system function is zero, the orbit will be mapped to the same starting point $x_1$ and hence it becomes periodic. This implies the following:

**Proposition 3.1.** Let $O_\sigma$ be a periodic orbit of system (1.1) for $(\alpha, \beta) \in \mathcal{P}$. If $O_\sigma$ is visiting the left partition, then it is stable and its symbolic sequence $\sigma$ contains exactly one letter $L$.

Note that system (1.1) has for $(\alpha, \beta) \in \mathcal{P}$ also unstable periodic orbits but all of them are located in $I \setminus J$.

As the symbolic sequences of stable periodic orbits are shift-invariant, for any periodic orbits $O_\sigma$ we can choose $x_0 \leq 0$ as the first point of the orbit and hence $\sigma_0 = L$ as the first symbol of the sequence. Then, the following can be easily shown:

**Proposition 3.2.** If $O_\sigma$ is a stable period-$n$ orbit of system (1.1) and $(\alpha, \beta) \in \mathcal{P}$, then $\sigma_{n-1} = R$.

**Proof.** As $x_0 = x_1 \leq 0$, we must have $x_{n-1} \geq 1$ and hence $\sigma_{n-1} = R$. ■

**Proposition 3.3.** If $O_\sigma$ is a stable period-$n$ orbit of system (1.1) and $(\alpha, \beta) \in \mathcal{P}$, then $\sigma_{n-2} \in \{M, r\}$.

**Proof.** As stated above, $x_{n-1} \geq 1$ implies $x_{n-2} \in J$ and hence $\sigma_{n-2} \in \{M, r\}$. ■

To summarize, we have the following

**Corollary 3.4.** If $O_\sigma$ is a stable period-$n$ orbit of system (1.1) and $(\alpha, \beta) \in \mathcal{P}$, then the symbolic sequence $\sigma = \sigma_0, \ldots, \sigma_{n-1}$ consists of at least three letters and has the following form:

$$\sigma_0 = L, \quad \sigma_1, \ldots, \sigma_{n-3} \in \{M, r, R\}$$

and

$$\sigma_{n-2} \in \{M, r\}, \quad \sigma_{n-1} = R.$$  

Regarding the letters $\sigma_1, \ldots, \sigma_{n-3}$, we can state additionally the following:

**Proposition 3.5.** If $O_\sigma$ is a stable period-$n$ orbit of system (1.1) and $(\alpha, \beta) \in \mathcal{P}$, then the sequence $\sigma$ cannot contain sub-sequences $RR, rR$ or $Rr$.

**Proof.** Assume, $\sigma_i = R$. Then $x_i > x^R$ and hence $x_{i+1} < x^R$, so $\sigma_{i+1} \neq R$. Similarly, if $\sigma_i = r$, then $x_i \in (\frac{1}{2}, x^R)$ and hence $x_{i+1} > x^R$, so $\sigma_{i+1} = R$. ■

It is worth noticing that there are two mechanisms involved in the generation of stable periodic orbits in system (1.1). The first one is active in the interval $x \in (0, \frac{1}{2})$. Here, the system function and the bisecting line form a kind of channel, where an orbit may perform some $k$ number of iteration steps. In the corresponding symbolic sequence, this will be reflected by the sub-sequence $M^k$. 

The second mechanism influences the behaviour in the interval $x \in (\frac{1}{2}, 1)$. Here, the orbits form a swirl around the unstable fixed point $x^R$ leading to the subsequences $(Rr)^j$ or $(rR)^j$ in the symbolic representation, where $j$ refers to the number of rotations around the unstable fixed point $x^R$. Remarkably, both mechanisms may work in the same stable periodic orbits: after some number of iteration steps in the channel, the orbit may be mapped to the swirl, then after some number of rotations it may be mapped back to the channel, and so on, until one of the mechanisms (either the channel or the swirl) maps the orbit to the interval $\mathcal{J}$. This interplay of both mechanisms is the reason for the complexity of the observed bifurcation structure.

Now we can identify the border collision bifurcations bounding the region of existence of any stable period-$n$ orbit of system (1.1). As one can see, this orbit exists as long as the point $x_{n-2}$ is located within the interval $\mathcal{J}$.

If a parameter variation leads the point $x_{n-2}$ to collide with the point $x_{\mathcal{J}}^{1}$ from the right, the stable orbit will be destroyed, since in this case we have $x_{n-1} = 1$ and $x_n = 0$ (figure 4). After the bifurcation, the point $x_{n-2}$ will be mapped to a value $x_{n-1}$ close to 1, which means $x_{n-1} \lessapprox 1$. The next iteration will map the orbit close to 0, which means $x_n \gtrapprox 0$, so that subsequent iterations will lead to many steps in the channel. Consequently, the orbits directly after the border collision bifurcation will have a period much larger than $n$. The situation where a parameter variation leads the point $x_{n-2}$ to collide with the point $x_{\mathcal{J}}^{r}$ from the left is analogous.

If the point $x_{n-2}$ moves towards the partition boundary $x = \frac{1}{2}$ from the left (or from the right, respectively) side another bifurcation takes place. As this point crosses the boundary, the period of the orbits does not change. However, the symbolic sequence of the orbit changes: the symbol $\sigma_{n-2}$ which was $\mathcal{M}$ (or $r$)
before the boundary crossing becomes \( r \) (or \( M \)). As an example, one can consider the situation shown in figure 3a at \( \beta = \frac{1}{2} \), where the period-3 orbit changes from \( O_{LMR} \) to \( O_{L^+R} \). Similar transitions take place in the middle of each interval corresponding to a stable periodic orbit, for example, at \( \beta \approx 0.2083 \), the stable period-4 orbit \( O_{LM,LMR} \) becomes \( O_{LM,M^+R} \), at \( \beta \approx 0.7917 \), the stable period-4 orbit \( O_{LM^+,LM^+R} \) becomes \( O_{LM^+,M^+,R^+} \) and so on. Note that for reasons of simplicity, we refer throughout this paper to such pairs of period-\( n \) orbits as a single period-\( n \) orbit as their existence intervals are always adjacent.

Finally, note that all border collision bifurcation curves located at \( \beta > \frac{1}{2} \) converge to a single point \( B_1 = (\infty, 1) \). Similarly, all border collision bifurcation curves located at \( \beta < \frac{1}{2} \) converge to a single point \( B_2 = (\infty, 0) \). These two points represent codimension-2 big-bang bifurcation points of the nested period-incrementing type and are the organizing centres of the extended tent map.

### 4. Admissible sequences and transition rules

As described in the last section, the stable periodic orbits in system (1.1) undergo border collision bifurcations. Unfortunately, the bifurcation structure of system (1.1) cannot be described adequately by detailing the situation before and after a border collision bifurcation. Instead, we describe the changes in the stable periodic behaviour which occur when we change the parameter values corresponding to a period-\( n \) orbit to values corresponding to a period-\((n+1)\) orbit.

The values of \( \beta \) at which an orbit of a specific period exists, correspond directly to the pre-images of the interval \( J \). Recall that \( x_1 \in J \) implies \( x_2 > 1 \) and \( x_1 = x_0 \). So every value of \( \beta \) at which \( x_0 \) is mapped into \( J \) leads to a period-3 orbit. To get a period-4 orbit, \( x_1 \) must be mapped into \( J \) instead of \( x_0 \), and therefore the point \( x_0 \) must be located in one of the first-order pre-images of \( J \). It can be easily seen from the shape of the system function that as we consider \( \alpha > 1 \) at each point in \( I \), the function \( f \) has two pre-images. So, the left pre-image of \( J \) is given by \( J_{1}^{-1} = [f_{m}^{-1}(x_{J}^{-}), f_{m}^{-1}(x_{J}^{+})] \) and the right pre-image of \( J \) is \( J_{2}^{-1} = [f_{r}^{-1}(x_{J}^{-}), f_{r}^{-1}(x_{J}^{+})] \) (figure 5). Every value of \( \beta \), which leads to \( x_0 \) being mapped into either \( J_{1}^{-1} \) or \( J_{2}^{-1} \), corresponds to a period-4 orbit.

In the same manner, as the pre-images of \( J \), the pre-images of \( J_{1}^{-1} \) and \( J_{2}^{-1} \) can be determined. This leads to the four intervals \( J_{1}^{-2}, J_{2}^{-2}, J_{3}^{-2} \) and \( J_{4}^{-2} \), which are the second-order pre-images of \( J \). The intervals \( J_{1}^{-2} \) and \( J_{2}^{-2} \) are located to the left and to the right of \( J_{1}^{-1} \) and the intervals \( J_{3}^{-2} \) and \( J_{4}^{-2} \) to the left and to the right of \( J_{2}^{-1} \). Each point mapped into one of these four intervals is mapped in the next step into \( J_{1}^{-1} \) or \( J_{2}^{-1} \) and in the step after that into \( J \). So, every value of \( \beta \), which leads to \( x_0 \) being mapped into one of the intervals \( J_{i}^{-2} \), \( i = 1, \ldots, 4 \), corresponds to a period-5 orbit. This reasoning can be continued, so that all intervals \( J_{1}^{-k}, \ldots, J_{2}^{-k} \) (\( k \)-th order pre-images of \( J \)), corresponding to all period-(\( k + 3 \)) orbits, can be determined for arbitrary \( k \). Note that each interval \( J_{m}^{-k} \) is surrounded by exactly two intervals of the next pre-image step: to the left of \( J_{m}^{-k} \) is \( J_{2m-1}^{-((k+1)} \) and to the right is \( J_{2m}^{-((k+1)} \). This explains the whole bifurcation
structure of system (1.1): for each value of $\alpha > 1$, the interval of $\beta$ leading to a stable orbit of period $n \geq 3$ is surrounded by two intervals leading to stable orbits of period $(n + 1)$, as is clearly visible in figure 3. Note that this reasoning holds for any (arbitrary large) $n$ because the definition of the system function guarantees that there exist two pre-images for all values $x \in (0, 1)$.

However, it is worth noticing that the intervals $J_{2m-1}^{-(k+1)}$ and $J_{2m}^{-(k+1)}$ are not necessarily the pre-images of $J_m^{-(k)}$. As an example, in figure 5, the interval $J_2^{-1}$ is surrounded by $J_3^{-2}$ and $J_4^{-2}$. Only one of them, namely $J_3^{-2}$ is its pre-image, whereas the other one $J_4^{-2}$ is a pre-image of $J_1^{-1}$. Therefore, the symbolic dynamics of periodic orbits within the interval $I$ cannot be explained in the same simple manner as the periods. To explain this let us consider the structure of symbolic sequences corresponding to periodic orbits.

Recall that each stable period-$n$ orbit corresponds to two symbolic sequences, which differ only in the symbol $s_{n-2}$ ($\S 3$). The parameter region occupied by a specific orbit can always be split into a left and a right part, each corresponding to one of these two sequences and containing one of the two boundaries of the regions of existence of the orbit.

To simplify the description, we give each sequence of each stable orbit of each period a specific name. We know, that there are $2^{n-3}$ stable orbits of period $n$. Each of these orbits corresponds to two symbolic sequences. We can index all stable orbits of a given period $n$ uniquely by numbering them from the left to the right with respect to their interval of existence in the parameter interval $\beta \in (0, 1)$. We denote the $i$th stable orbit of period $n$ by $O_i^n$. We can then add another index which denotes the left or the right of the two sequences of $O_i^n$, so that we get $\rho_{i,d}^n$, with $d \in \{\ell, r\}$. For example, the two sequences of the single stable period-3 orbit are called $\rho_{1,1}^3$ and $\rho_{1,r}^3$. In figure 3, the point $b$ corresponds to $\rho_{1,1}^4$, the point $e$ to $\rho_{3,r}^5$ and the point $g$ to $\rho_{2,r}^4$.

Figure 5. The first two sets of pre-image intervals of system (1.1) for $\alpha = 1.2$. 
Based on this, we can describe the change of period $n$ to $(n + 1)$ in a structured way: given $\alpha$ and changing $\beta$, we start either at the left or the right sequence of the $i$th stable period-$n$ orbit, $r_{n,i}^l$ or $r_{n,i}^r$. If we start at $r_{n,i}^l$, we move in the left direction to the right sequence of the nearest stable period-$(n + 1)$ orbit, which is $r_{n+1,2i-1}^r$. Accordingly, if we start at $r_{n,i}^r$, we move in the right direction to the left sequence of the nearest stable period-$(n + 1)$ orbit, which is $r_{n+1,2i}^r$. This way, we can connect each of the sequences of length $n$ to a sequence of length $(n + 1)$ in a unique way.

Note that we can derive not only one but two sequences of length $(n + 1)$ from one sequence of length $n$. This is due to the fact that when we describe a transition from the sequences $r_{n,i}^l$ and $r_{n,i}^r$ to the sequences $r_{n+1,2i-1}^r$ and $r_{n+1,2i}^r$, then we also get the other sequences of the higher period, $r_{2i-1,i}^l$ and $r_{2i,i}^r$, as they differ only in the symbol $\sigma_{n-2}$.

The symbolic sequence-generation scheme of the scenario presented here is given by a binary tree whose first layers are shown in figure 6. Each node of the tree contains two sequences of same length $n$, and is connected with two nodes each containing two sequences of length $n + 1$. Each edge of the binary tree represents the application of one of the rules described below. Note that this infinite binary tree plays the same role for the nested period-incrementing
scenario presented here as the infinite sequence adding scheme for the nested period-adding scenario. A significant difference between both situations is that the infinite sequence adding scheme represents a graph but not a tree. Note also that similar to the nested period-adding structure, the scenario presented here shows in the limiting case an infinite number of stable aperiodic orbits.

On the basis of this binary tree structure, we can describe how the symbolic sequences change when the period is changed. As already mentioned, the rule governing the periods in the presented scenario observed in system (1.1) is easier than the rule that governs the periods in the nested period-adding scenario. By contrast, the rules defining the involved symbolic sequences are much more complicated than the simple concatenation in the period-adding case. The reason for this complexity is that there are two mechanisms leading stable periodic orbits in system (1.1) to emerge, and not only one as in the period-adding case.

These mechanisms result in a set of context-sensitive rules, which are split in three classes of rules denoted as $A$-Rule and $B$-Rules defined by

\[ Rule \, A: \quad \mathcal{MRL} \rightarrow \begin{cases} \mathcal{M}r\mathcal{RL} \\ \mathcal{M}\mathcal{MRL} \end{cases} \]

Rules $B^1_j$: \[ (r\mathcal{R})^j r\mathcal{RL} \rightarrow \begin{cases} \mathcal{R}(r\mathcal{R})^j r\mathcal{RL} \\ \mathcal{R}(r\mathcal{R})^j \mathcal{MRL} \end{cases} \quad \text{with } j \geq 0 \]

Rules $B^2_j$: \[ \mathcal{R}(r\mathcal{R})^j r\mathcal{RL} \rightarrow \begin{cases} (r\mathcal{R})^{j+1} r\mathcal{RL} \\ (r\mathcal{R})^{j+1} \mathcal{MRL} \end{cases} \quad \text{with } j \geq 0. \]

Note that each rule increases the length of the symbolic sequence by one. The $A$-Rule is a single rule, which changes a sub-sequence of length 3 to a sub-sequence of length 4. The $B$ rules are two families of rules of variable length. Each $B^1_j$ rule changes a sub-sequence of length $2j + 3$ to a sub-sequence of length $2j + 4$. Similarly, each $B^2_j$ rule changes a sub-sequence of length $2j + 4$ to one of length $2j + 5$.

Often, more than one of these rules can be applied to a given sequence; however, only one rule leads to the correct sequence. If the $A$-Rule is applicable (that means, the symbolic sequence contains the sub-sequence $\mathcal{MRL}$), then this rule must be applied. Otherwise, the longest of the applicable $B$-Rules must be applied. That means, for example, $B^1_j$ can only be applied if $B^2_j$ is not applicable, and there is no $j' > j$, for which $B^1_{j'}$ is applicable. Note that the left side of each rule ends with $\mathcal{RL}$. This guarantees that the rule will be applied at the end of the symbolic sequence, or, more precisely, that the last three symbols on the left side of each rule are $\sigma_{n-2} \sigma_{n-1} \sigma_0$.

By applying the correct rule, one obtains two new sequences of length $(n + 1)$. These two sequences differ in the symbol $\sigma_{n-2}$, as explained above. One of these sequences always contains the symbol $\sigma_{n-2} = r$ and results directly from the sequence $\rho^n_d$, with $d \in \{l, r\}$ where we started the transition. More precisely, if the transition is $\rho_{i,1}^n \rightarrow \rho_{2i-1,1}^{n+1}$, then the directly resulting sequence $\rho_{2i-1,1}^{n+1}$ contains $\sigma_{n-1} = r$, whereas the remaining sequence $\rho_{2i-1,1}^{n+1}$ contains $\sigma_{n-1} = \mathcal{M}$. Analogously, if the transition is $\rho_{i,1}^n \rightarrow \rho_{2i,1}^{n+1}$, then the directly resulting sequence $\rho_{2i,1}^{n+1}$ contains $\sigma_{n-1} = r$ and the remaining sequence $\rho_{2i,1}^{n+1}$ contains $\sigma_{n-1} = \mathcal{M}$. 

Figure 7. Stable period-4 and -5 orbits of system (1.1). Each diagram is marked with the corresponding symbolic sequence. Below adjacent diagrams of differing periods, the corresponding transition rule is shown. Parameters: \( a = 1.2, \beta \approx 0.1013, 0.1736, 0.2430, 0.3154 \) \((a, b, c, d)\), 0.6846, 0.7570, 0.8264, 0.8987 \((e, f, g, h)\).

Using these rules and starting with the sequences \( \rho_{1,5}^3 = \mathcal{M}\mathcal{R}\mathcal{L} \) and \( \rho_{1,x}^3 = r\mathcal{R}\mathcal{L} \), we can get the sequences corresponding to all the existing stable periodic orbits in system (1.1).

In doing so, we can observe that half of the transitions between periods \( n \) and \( (n + 1) \) are generated using the \( A \)-Rule, while the other half is made with \( B \)-Rules of varying length. Furthermore, each of the rules corresponds to a specific change of behaviour of the corresponding stable periodic orbit. The \( A \)-Rule means that the orbit of the higher period has an additional point in the channel, while a \( B \)-Rule means that the orbit has an additional point in the swirl. Figure 7 illustrates that for the transition from period 4 to 5. Additionally, figure 8\( a, b \) shows the effect of the repeated application of the \( A \)- and \( B \)-Rules, respectively. In the first case (figure 8\( a \)), the orbit performs six iteration steps in the channel before it reaches the interval \( J \). The corresponding symbolic sequence \( \mathcal{M}^6\mathcal{M}\mathcal{R}\mathcal{L} \) is created from the starting sequence \( \mathcal{M}\mathcal{R}\mathcal{L} \) by six applications of the \( A \)-Rule:

\[
\begin{align*}
\mathcal{M}\mathcal{R}\mathcal{L} &\xrightarrow{A} \mathcal{M}\mathcal{M}\mathcal{R}\mathcal{L} \\
&\xrightarrow{A} \mathcal{M}^2\mathcal{M}\mathcal{R}\mathcal{L} \\
&\xrightarrow{A} \mathcal{M}^3\mathcal{M}\mathcal{R}\mathcal{L} \\
&\xrightarrow{A} \mathcal{M}^4\mathcal{M}\mathcal{R}\mathcal{L} \\
&\xrightarrow{A} \mathcal{M}^5\mathcal{M}\mathcal{R}\mathcal{L} \\
&\xrightarrow{A} \mathcal{M}^6\mathcal{M}\mathcal{R}\mathcal{L}.
\end{align*}
\]
Bifurcation structure mimicking period adding

Figure 8. Stable orbits of system (1.1) corresponding to the symbolic sequences $M_6MRL$, resulting from six consecutive applications of the $A$-Rule (a) and $(rR)^3rRL$, resulting from six consecutive applications of $B$-Rules (b). Parameter values: $\alpha = 1.2$, $\beta = 0.002398357905$ (a), $\beta = 0.7050230482$ (b).

Six letters $M$—each resulting from the application of the $A$-Rule—correspond to six steps in the channel, and the remaining suffix $MRL$ corresponds to the last three points of the orbit (in the interval $J$, on the right side outside of $I$ and on the left side). Similarly, in the second case (figure 8b), the orbit performs three rotations around the unstable fixed point. The corresponding symbolic sequence results from the application of $B$-Rules:

$$
\begin{align*}
(rR) & \rightarrow R(rR) \\
& \rightarrow (rR) rRL \\
& \rightarrow (rR)^2 rRL \\
& \rightarrow (rR)^3 rRL.
\end{align*}
$$

As one can see, a consecutive application of a pair of rules $B_1^i$ and $B_2^i$ with $i \geq 0$ adds one rotation around the fixed point.

5. Summary and outlook

In this work, we described a recently discovered but until now not explained bifurcation scenario formed by border collision bifurcations. Because of its internal structure and the striking similarity with the nested period-adding bifurcation scenario, we denoted it as nested period-incrementing scenario. By using a specifically constructed piecewise-linear discontinuous scalar map defined on three partitions, we were able to isolate the bifurcation structure in its pure form. Furthermore, using a suitable symbolic description, we derived a set of rules in the space of symbolic sequences that allowed us to explain the structure of the domain of asymptotically stable periodic orbits in the parameter space. In contrast to the nested period-adding scenario, with its Farey-like self-similar
structure, the self-similar structure of the nested period-incrementing scenario in its pure form follows another simple rule: on each side of a region of a stable period-$n$ orbit there exists a region of a stable period-$n+1$ orbit.

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References


