We investigate how to obtain a non-trivial geometric phase gate for a two-qubit spin chain, with Ising interaction in different magnetic fields. Indeed, one of the spins is driven by a time-varying rotating magnetic field, and the other is coupled with a static magnetic field in the direction of the rotation axis. This is an interesting problem both for the purpose of measuring the geometric phases and in quantum computing applications. It is shown that the static magnetic field does not change the adiabatic states of the system, and it does not affect the geometric phases, whereas it may be used to control the dynamic phases. In addition, by considering the exact two-spin adiabatic geometric phases, we find that a non-trivial two-spin unitary transformation, purely based on Berry phases, can be obtained by using two consecutive cycles with opposite directions of the magnetic fields, opposite signs of the interaction constant and the phase shift of the rotating magnetic field. In addition, in the non-adiabatic case, starting with a certain initial state, a cycle can be achieved and thus the Aharonov–Anandan phase is calculated.

1. Introduction

In some circumstances, when a Hamiltonian undergoes a cyclic evolution, the quantum state picks up a geometric phase in addition to the dynamical phase. The adiabatic geometric phase, called the Berry phase, is accumulated in an instantaneous eigenstate of an adiabatically varying Hamiltonian that is cyclic in the parameter space [1]. Since Berry’s initial discovery, the geometric phase has been found to occur in more general circumstances: non-adiabatic [2] and non-cyclic evolutions [3], non-Abelian forms [4] and for mixed states [5–7]. The non-adiabatic geometric phase,
called the Aharonov–Anandan (AA) phase, is the geometric part of the phase accumulated in a state that is cyclic in the projected space of rays [2].

The geometric phases for spin systems, first discussed by Berry [1], have been investigated in many references [8–19]. Recently, the study of geometric phases has generalized to the case of a pair of spins coupling with a magnetic field [14–19]. Notably, excellent pioneering discussions and a nuclear magnetic resonance (NMR) experimental demonstration have been made on two-qubit geometric phases, with one of the two qubits regarded as decoupled with the rotating field [20]. For example, Yi et al. [14] studied the effect of interSubsystem coupling on the adiabaticity of the composite system and that of its subsystems; Tong et al. [15] calculated the geometric phases for states of two non-interacting spin-$\frac{1}{2}$ particles in a rotating magnetic field; Zhou & Zhang [18] examined the Berry phase of a bipartite system described by the antiferromagnetic Heisenberg XXZ model and driven by a one-site magnetic field; and Shi [21] examined the implementation of geometric quantum computation based on Zeeman coupling with a rotating magnetic field and isotropic Heisenberg interaction. Here, we investigate the effects of an extra static magnetic field coupled with the other spin. This extra magnetic field may be used to control the dynamic phase, particularly, it can be used to generate dark states [22]: dark states have a zero energy eigenvalue, and thus the dynamical phase will always be zero during the evolution. The analysis differs from previous work on geometric phases of spin-pair systems in that the two spins are affected by different magnetic fields: while one of the spins is exposed to a time-independent magnetic field, the other spin is exposed to a rotating magnetic field. This allows for a systematic cancellation of the dynamical phases acquired during the evolution, which, in turn, provides means to perform geometric two-qubit gates in this system.

In addition to theoretical interest, applications of geometric phases can be found in various physical fields. Recently, geometric quantum computation, expected as an intrinsical fault-tolerant scheme, has become one of the most important applications [22–27]. It was proposed by using NMR [28], superconducting nanocircuits [29], trapped ions [22] or semiconducting nanostructures [30]. As is well known, for an adiabatic (or non-adiabatic) cyclic evolution, the associated total phase shift consists of both dynamic and geometric components, where the geometric phase is interpreted as a holonomy of the Hermitian fibre bundle over the parameter (projective Hilbert) space. Indeed, the geometric phase has received renewed interest owing to several proposals for their use in the implementation of quantum logical gates [31]. Because the geometric phase depends only on the global geometry of the path executed in the evolution, a set of quantum logical gates related only to the pure geometric phase shifts in the gate operations is likely to have an advantage that is insensitive to stochastic operation errors [32,33]. One approach to realize geometric quantum gates uses optical control in the computational Hilbert space, together with ancillary states [22]. Here, we examine another approach, namely the direct use of an interacting two-qubit Hamiltonian to construct non-trivial two-qubit unitary transformation purely based on geometric phases. In this approach, the Hamiltonian involves only the computational Hilbert space, without ancillary space. We consider the Ising interaction, which is an important form of the interaction that often appears in quantum computing implementations [34]. It is an approximation to the isotropic Heisenberg interaction in liquid NMR [28], and can also be realized in cold atoms in optical lattices [35], Josephson junction circuits [36] and so on. The Ising interaction was used in a pioneering construction of conditional phase gates based on geometric phases [20], which pointed out that a crucial issue is to cancel the dynamical phases. On the other hand, in the proposed schemes, the systems generally consist of two spins, one of which is coupled with a time-varying magnetic field [18].

In this paper, we examine the general situation of two Ising-interacting spin-$\frac{1}{2}$s coupled with different magnetic fields. An explicit physical scenario for this model could be an NMR experiment where $^{13}$C-labelled chloroform in $d_6$ acetone may be used as the sample (for a review, see Laflamme et al. [37]), in which the single $^{13}$C nucleus and the $^1$H nucleus play the role of the two spin-$\frac{1}{2}$ particles. The constant of spin–spin coupling $J_{\sigma_1^z\sigma_2^z}$ in this case is $J \simeq (2\pi)214.5$ Hz [14]. Individual addressing can be realized in NMR by using spins of nuclei of different isotopes, such as those of different species of atoms, as the subsystems. These spins usually have precession
Finally, in §5, the main results are summarized. We set $J$ and ferromagnetic for $B$ and antiferromagnetic for $-B$. Using the static magnetic field for cancelling, the total phase through a cycle is described in §4 for initial states that are prepared to be eigenstates of a certain time-independent Hamiltonian. Indeed, we discuss a two-cycle scheme using two cycles with opposite directions of the magnetic fields, opposite signs of the interaction constants and a phase shift of the rotating magnetic field. At the end of these two cycles, the dynamical phases are always cancelled, so that the phases become purely geometric. Consequently, one obtains a unitary transformation purely based on Berry phases, which can act on any unknown initial state.

The paper is organized as follows. The model is introduced in §2. The adiabatic evolution is discussed in §3. In this section, after the calculation of Berry phases in instantaneous eigenstates in §3a, a unitary transformation purely based on Berry phases is constructed by using a two-cycle scheme in §3b. In §4, we study the exact evolution without the adiabatic condition. Following the exact solution of the Schrödinger equation in §4a, the cycling and AA phases are discussed in §4b, for initial states that are prepared to be eigenstates of a certain time-independent Hamiltonian. Using the static magnetic field for cancelling, the total phase through a cycle is described in §4c. Finally, in §5, the main results are summarized. We set $\hbar = 1$ throughout this paper.

2. The model

The Hamiltonian, describing a system consisting of two Ising-interacting spin-$\frac{1}{2}$s in the presence of external magnetic fields, takes the form

$$H = \frac{1}{2} [J(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \sigma_1^z \sigma_2^z) + B_1 \cdot \sigma_1 + B_2 \cdot \sigma_2],$$

(2.1)

where $J$ is the real spin–spin coupling coefficient. The model is called antiferromagnetic for $J > 0$ and ferromagnetic for $J < 0$. $\sigma_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$, where $\sigma_j^i$ are Pauli operators of subsystem $j$ ($j = 1, 2$). In addition, $B_j$ is the magnetic field on site $j$. We will choose $B_1 = B_1 \hat{z}$ and $B_2 = B_2 \hat{n}$ with the unit vector $\hat{n} = (\sin \theta \cos \varphi(t), \sin \theta \sin \varphi(t), \cos \theta)$, where the time dependence of the angle $\varphi(t)$ is arbitrary. Therefore, spin 1 is coupled with the static magnetic field, whereas spin 2 is driven by the rotating magnetic field.

The variation period of the Hamiltonian is $T = 2\pi/\omega$. In a space spanned by $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle$, and $|\uparrow\downarrow\rangle$, the Hamiltonian equation (2.1) can be written as

$$H = \frac{1}{2} \begin{pmatrix} J + B_1 + B_2 \cos(\theta) & B_2 \sin(\theta) e^{-i\varphi(t)} & 0 & 0 \\ B_2 \sin(\theta) e^{i\varphi(t)} & -J + B_1 - B_2 \cos(\theta) & 0 & 0 \\ 0 & 0 & -J - B_1 + B_2 \cos(\theta) & B_2 \sin(\theta) e^{-i\varphi(t)} \\ 0 & 0 & B_2 \sin(\theta) e^{i\varphi(t)} & J - B_1 - B_2 \cos(\theta) \end{pmatrix}.$$

(2.2)

The spectrum of $H$ is obtained as

$$H|\xi_j\rangle = \xi_j|\xi_j\rangle,$$

(2.3)

where the eigenstates and the corresponding eigenvalues are, respectively,

$$|\xi_1\rangle = \frac{1}{\sqrt{N_1}} \begin{pmatrix} -B_2 \sin(\theta) e^{-i\varphi(t)} \\ -\sqrt{B_2^2 + J^2 + 2JB_2 \cos(\theta) + J + B_2 \cos(\theta)} |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle \end{pmatrix},$$

$$|\xi_2\rangle = \frac{1}{\sqrt{N_2}} \begin{pmatrix} -B_2 \sin(\theta) e^{-i\varphi(t)} \\ -\sqrt{B_2^2 + J^2 + 2JB_2 \cos(\theta) + J + B_2 \cos(\theta)} |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle \end{pmatrix},$$

$$|\xi_3\rangle = \frac{1}{\sqrt{N_3}} \begin{pmatrix} B_2 \sin(\theta) e^{-i\varphi(t)} \\ \sqrt{B_2^2 + J^2 - 2JB_2 \cos(\theta) + J - B_2 \cos(\theta)} |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle \end{pmatrix},$$

$$|\xi_4\rangle = \frac{1}{\sqrt{N_4}} \begin{pmatrix} B_2 \sin(\theta) e^{-i\varphi(t)} \\ \sqrt{B_2^2 + J^2 - 2JB_2 \cos(\theta) + J - B_2 \cos(\theta)} |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle \end{pmatrix},$$

(2.4)
and

$$
\begin{align*}
\xi_1 &= \frac{1}{2}(B_1 + \sqrt{B_2^2 + J^2 + 2JB_2 \cos(\theta)}), \\
\xi_2 &= \frac{1}{2}(B_1 - \sqrt{B_2^2 + J^2 + 2JB_2 \cos(\theta)}), \\
\xi_3 &= \frac{1}{2}(-B_1 + \sqrt{B_2^2 + J^2 - 2JB_2 \cos(\theta)}), \\
\xi_4 &= \frac{1}{2}(-B_1 - \sqrt{B_2^2 + J^2 - 2JB_2 \cos(\theta)}),
\end{align*}
$$

(2.5)

where $N_j$ ($j = 1, 2, 3, 4$) is the normalization coefficient of the $j$th eigenstate. Although the eigenvalues are time independent, the dynamics is not trivial because the eigenstates are clearly time dependent. Indeed, although the time dependence of the eigenstates is only due to the phases, the relative phases of the expansion coefficients may change with time. We also see that the eigenvalues are symmetric with respect to $B_2$ and $J$. This symmetry implies that the eigenvalues of this system are physically equivalent to those of a similar system, but with exchange of values between $B_2$ and $J$. On the other hand, if the rotating magnetic field is on the $x$–$y$ plane, namely $\theta = \pi/2$, the eigenvalues are even functions of $J$. In this situation, when the static magnetic field $B_1$ is changed, the degeneracy $\xi_1 = \xi_3$ ($\xi_2 = \xi_4$) is removed.

### 3. Berry phases

#### (a) Berry phases in instantaneous eigenstates

Let us consider the adiabatic limit of the Hamiltonian, where $d\phi(t)/dt \approx 0$. Under the usual assumptions of the adiabatic theorem, an approximate solution of the time-dependent Schrödinger equation with an initial condition $|\Psi(0)\rangle = |\xi_j(0)\rangle$ is

$$
|\Psi(t)\rangle = \exp[i(\delta_j + \gamma_j)]|\xi_j(t)\rangle,
$$

(3.1)

where

$$
\delta_j(t) = - \int_0^t \xi_j(t') \, dt'
$$

(3.2)

is the dynamical phase, while

$$
\gamma_j(t) = i \int_0^t dt' (\langle \xi_j(t') | \dot{\xi}_j(t') \rangle)
$$

(3.3)

is the Berry phase. The phases generated after the system undergoing a cyclic evolution are obtained as follows:

$$
\delta_j = - \xi_j T
$$

(3.4)

and

$$
\gamma_j = 2\pi \left(1 - \frac{1}{N_j}\right),
$$

(3.5)

where

$$
\begin{align*}
N_1 &= 1 + \left(\frac{B_2 \sin(\theta)}{-\sqrt{B_2^2 + J^2 + 2JB_2 \cos(\theta)} + J + B_2 \cos(\theta)}\right)^2, \\
N_2 &= 1 + \left(\frac{B_2 \sin(\theta)}{\sqrt{B_2^2 + J^2 + 2JB_2 \cos(\theta)} + J + B_2 \cos(\theta)}\right)^2, \\
N_3 &= 1 + \left(\frac{B_2 \sin(\theta)}{\sqrt{B_2^2 + J^2 - 2JB_2 \cos(\theta)} + J - B_2 \cos(\theta)}\right)^2, \\
N_4 &= 1 + \left(\frac{B_2 \sin(\theta)}{-\sqrt{B_2^2 + J^2 - 2JB_2 \cos(\theta)} + J - B_2 \cos(\theta)}\right)^2.
\end{align*}
$$

(3.6)
Figure 1. (a–d) The Berry phases versus $B_2$ and $\theta$ for $J = 1$. (Online version in colour.)

It is important to note that the Berry phases are independent of $B_1$ because eigenstates (2.4) do not depend on this static magnetic field. On the other hand, eigenvalues (2.5) linearly depend on $B_1$. Therefore, this extra magnetic field may be used to control the dynamic phase. In particular, choosing $B_1 = -\sqrt{B_2^2 + J^2 + 2JB_2 \cos(\theta)}$, $B_1 = \sqrt{B_2^2 + J^2 + 2JB_2 \cos(\theta)}$, $B_1 = \sqrt{B_2^2 + J^2 - 2JB_2 \cos(\theta)}$ or $B_1 = -\sqrt{B_2^2 + J^2 - 2JB_2 \cos(\theta)}$, we can generate a dark state (see equation (2.5)). Therefore, geometric quantum computation may be achieved by eliminating the dynamic phase accumulated in the cyclic evolution. The dark state method was proposed for geometric quantum computation with trapped ions [22], and then described in NMR systems [38].

The Berry phases are shown in figures 1 and 2 as functions of $J$, $B_2$ and $\theta$. Figure 2 shows that with $J \to \pm \infty$, Berry phases $\gamma_j \to 0$. Indeed, $\lim_{J \to +\infty} \gamma_2 = \lim_{J \to +\infty} \gamma_3 = 0$ and $\lim_{J \to -\infty} \gamma_1 = \lim_{J \to -\infty} \gamma_4 = 0$ [39,40].

It is worth noting that if $J$ reverses sign, $\theta$ and $\phi$ transform into $\pi - \theta$ and $\phi + \pi$, respectively, and eigenvalues (2.5) remain unchanged,

$$
\begin{align*}
\xi_1[-J, \pi - \theta, \phi + \pi] = \xi_1[J, \theta, \phi], \\
\xi_2[-J, \pi - \theta, \phi + \pi] = \xi_2[J, \theta, \phi], \\
\xi_3[-J, \pi - \theta, \phi + \pi] = \xi_3[J, \theta, \phi], \\
\xi_4[-J, \pi - \theta, \phi + \pi] = \xi_4[J, \theta, \phi].
\end{align*}
$$

(3.7)
Figure 2. The Berry phases versus $J$ with $B_2 = 1$ and $\theta = \pi/4$. (Online version in colour.)

whereas the eigenstates of the transformed Hamiltonian change as follows:

$$
\begin{align*}
|\xi_1[-J, \pi - \theta, \varphi + \pi]\rangle &= |\xi_2[J, \theta, \varphi]\rangle, \\
|\xi_2[-J, \pi - \theta, \varphi + \pi]\rangle &= |\xi_1[J, \theta, \varphi]\rangle, \\
|\xi_3[-J, \pi - \theta, \varphi + \pi]\rangle &= |\xi_4[J, \theta, \varphi]\rangle, \\
|\xi_4[-J, \pi - \theta, \varphi + \pi]\rangle &= |\xi_3[J, \theta, \varphi]\rangle.
\end{align*}
$$

(3.8)

This leads to the following symmetry in the Berry phases:

$$
\begin{align*}
\gamma_1[-J, \pi - \theta, \varphi + \pi] &= \gamma_2[J, \theta, \varphi], \\
\gamma_2[-J, \pi - \theta, \varphi + \pi] &= \gamma_1[J, \theta, \varphi], \\
\gamma_3[-J, \pi - \theta, \varphi + \pi] &= \gamma_4[J, \theta, \varphi], \\
\gamma_4[-J, \pi - \theta, \varphi + \pi] &= \gamma_3[J, \theta, \varphi].
\end{align*}
$$

(3.9)

In addition, if $B_1$ also reverses sign, then the eigenstates do not change, but their eigenvalues reverse signs, so

$$
\begin{align*}
| - \xi_1[-J, -B_1, \pi - \theta, \varphi + \pi]\rangle &= |\xi_1[J, B_1, \theta, \varphi]\rangle, \\
| - \xi_2[-J, -B_1, \pi - \theta, \varphi + \pi]\rangle &= |\xi_2[J, B_1, \theta, \varphi]\rangle, \\
| - \xi_3[-J, -B_1, \pi - \theta, \varphi + \pi]\rangle &= |\xi_3[J, B_1, \theta, \varphi]\rangle, \\
| - \xi_4[-J, -B_1, \pi - \theta, \varphi + \pi]\rangle &= |\xi_4[J, B_1, \theta, \varphi]\rangle.
\end{align*}
$$

(3.10)

where the instantaneous eigenstates have been labelled in terms of their eigenvalues.

(b) Design of geometric phase gates

For the purposes of both experimentally detecting geometric phases and application in quantum computing, it is crucial to cancel the dynamic phases, as pioneered by Ekert et al. [20].
Additionally, in order to perform conditional quantum gate operations using geometric phases only, it is necessary to find a way to eliminate the dynamic phase. This cancellation is very important because of the detection intricacy of the Berry phase effect in the presence of the dynamical phase.

Now consider an arbitrary initial unknown state, \(|\psi(0)\rangle\), which evolves under the adiabatically varying Hamiltonian. After \(T\), the state becomes

\[ |\psi(T)\rangle = U(T)|\psi(0)\rangle, \tag{3.11} \]

where

\[ U(T) = \text{diag}(e^{i\eta_1}, e^{i\eta_2}, e^{i\eta_3}, e^{i\eta_4}), \tag{3.12} \]

written on the basis \(\{|\xi_1\rangle, |\xi_2\rangle, |\xi_3\rangle, |\xi_4\rangle\}\), where \(\eta_j\)'s are the total phases. In this scheme, quantum gates can be implemented based on the total phase accumulated in cyclic evolutions.

In the following, we propose a two-cycle method to cancel the dynamical phase \(\delta_j\) for all eigenstates simultaneously, i.e. only the Berry phases \(\gamma_j\)'s appear in the final unitary transformation, consequently,

\[ |\psi(2T)\rangle = U_\delta(2T)|\psi(0)\rangle, \tag{3.13} \]

where

\[ U_\delta(2T) = \text{diag}(e^{2i\gamma_1}, e^{2i\gamma_2}, e^{2i\gamma_3}, e^{2i\gamma_4}), \tag{3.14} \]

with \(\gamma_j\)'s given in equation (3.3).

We make the Hamiltonian undergo two consecutive cycles as, in the second cycle, the parameters \(J\) and \(B_1\) reverse signs, and \(\theta\) and \(\varphi\) transform into \(\pi - \theta\) and \(\varphi + \pi\), respectively. These changes must be carried out at the end of the first cycle instantaneously. Therefore, an instantaneous eigenstate at the end of the first cycle continues to be an eigenstate in the second cycle, but with the sign of the eigenvalue reversed. Consequently, the dynamical phases in the two cycles cancel. Simply speaking, reversing the signs of these three quantities and the phase shift implies reversing the Hamiltonian. Hence, the cancellation of the dynamical phase and doubling of the geometric phase is a generalization of that for a single spin precessing in a rotating magnetic field [21], for which when the magnetic field is reversed in the second cycle, one simply changes \(B\) to \(-B\) while keeping the angular coordinates \((\varphi, \theta)\) unchanged; then each eigenstate does not change, whereas the energy reverses its sign, and consequently the dynamical phase cancels while the Berry phase doubles.

To comprehend the situation clearly and in detail, suppose the state starts as an instantaneous eigenstate \(|\xi_1\rangle\). In the first cycle of the Hamiltonian, it remains as the instantaneous eigenstate, accumulating a phase \(\gamma_1 - \xi_1 T\) at the end of the first cycle. Then, \(J, B_1, \theta\) and \(\varphi\) are suddenly switched to \(-J, -B_1, \pi - \theta\) and \(\varphi + \pi\), respectively, i.e. Hamiltonian (2.2), \(H[J, B_1, \theta, \varphi]\), is switched to \(H[-J, -B_1, \pi - \theta, \varphi + \pi]\), instantaneously. Because \(|\xi_1[J, B_1, \theta, \varphi]\rangle = |\xi_1[-J, -B_1, \pi - \theta, \varphi + \pi]\rangle\), \(e^{i\gamma_1 - i\xi_1 T}|\xi_1[J, B_1, \theta, \varphi]\rangle\) is an eigenstate of the switched Hamiltonian \(H[-J, -B_1, \pi - \theta, \varphi + \pi]\), but with eigenvalue \(-\xi_1\). Therefore, this state continues to be an instantaneous eigenstate in the second cycle, accumulating a phase \(\gamma_1 + \xi_1 T\) in the second cycle. Therefore, the total phase at the end of these two cycles is

\[ \gamma_1 - \xi_1 T + \gamma_1 + \xi_1 T = 2\gamma_1. \]

The situation is similar if the initial state is \(|\xi_n[J, D, B_1, \phi]\rangle\) \((n = 2, 3, 4)\), for which the total phase is \(2\gamma_n\).

These four paths, each starting with an instantaneous eigenstate of the Hamiltonian, in the two cycles can be summarized as

\[ |\xi_j\rangle \rightarrow e^{i\gamma_j - i\xi_j T}|\xi_j\rangle \rightarrow e^{2i\gamma_j}|\xi_j\rangle, \tag{3.15} \]

where each arrow represents a cycle.
In this way, we realize a purely geometric unitary transformation, as described in equation (3.14), with

\[ U_g(2T) = U_g(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \text{diag}(e^{2i\gamma_1}, e^{2i\gamma_2}, e^{2i\gamma_3}, e^{2i\gamma_4}). \]  

(3.16)

\( U_g \) may be used as a two-qubit phase gate, if the logic qubit basis states are encoded as the eigenstates |\( \xi_j \rangle \).

Geometric phase gate (3.16) can be manipulated by changing \( B_2, \theta \) and \( J \). According to figure 1, for a given \( \theta \), we can change only one of the Berry phases arbitrarily by varying \( B_2 \). Therefore, this gate is not sufficiently flexible by changing \( B_2 \) only (assuming \( J \) and \( \theta \) are fixed by the system). \( J \) is controllable and can be reversed in some artificial systems such as cold atoms in optical lattices or superconducting qubits in Josephson junction circuits. To give an illustrative example demonstrating the feasibility, consider the superconducting charge qubits, based on Cooper pair boxes. The sign of the coupling constant \( J \) can be reversed in some coupling schemes for superconducting charge qubits. For example, in the scheme of Averin & Bruder [41], two charge qubits are coupled through a variable electrostatic transformer, the coupling constant \( J \) can be reversed by making suitable changes of the gate voltage of the transformer.

In addition to the usual robustness of geometric phases [42], an additional aspect of robustness of the quantum gates constructed above is that for the two consecutive cycles whose dynamical phases cancel each other, only the period \( T \) needs to be the same, whereas the time dependence of \( \phi(t) \) can be completely arbitrary and independent in each cycle. The underlying reason is that the energy for each instantaneous eigenstate is time independent, thus the dynamical phase is simply a product of the energy and the period.

The technique described here constitutes an interesting approach to quantum computation, one that builds conditional phase shift gates entirely based on geometric phases. This technique, as mentioned earlier, is readily implementable with current technology in quantum optics and has already been demonstrated by some of the authors using NMR [20]. It would be interesting to further analyse the robustness of geometric quantum computation to errors.

4. Aharonov–Anandan phases

(a) Exact solution of the Schrödinger equation

In this section, we discuss non-adiabatic geometric phases by exactly solving the Schrödinger equation

\[ i \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle, \]  

(4.1)

where \( H(t) \) is still as given in equation (2.2), but there is no requirement of adiabaticity.

One may make a unitary transformation

\[ |\psi'(t)\rangle = R(t) |\psi(t)\rangle. \]  

(4.2)

Substituting (4.2) into (4.1), the Schrödinger equation in the new gauge becomes

\[ i \frac{\partial}{\partial t} |\psi'(t)\rangle = H'(t) |\psi'(t)\rangle, \]  

(4.3)

where

\[ H' = RHR^\dagger - iR \frac{\partial}{\partial t} R^\dagger. \]  

(4.4)

For a rotating magnetic field with constant angular frequency \( d\varphi/dt = \omega \), the model is exactly solvable. Suitable choice of \( R(t) \) may make \( H'(t) \) time independent and the solutions to (4.3)
where \( \omega t = \varphi(t) \). Therefore,

\[
|\psi'(t)\rangle = e^{i(\omega t)0}\sigma^z_1 + \sigma^z_2)|\psi(t)\rangle.
\]

(4.6)

On the other hand, Hamiltonian (2.2) can be written as

\[
H(t) = e^{-i(\omega t)0}\sigma^z_1 + \sigma^z_2)H(0)e^{i(\omega t)0}\sigma^z_1 + \sigma^z_2).
\]

(4.7)

where

\[
H(0) = \frac{1}{2}(J\sigma^z_1 + B_1\sigma^z_2 + B_2\sin(\theta)\sigma^x_2 + B_2\cos(\theta)\sigma^z_2)
\]

(4.8)

is the Hamiltonian at \( t = 0 \).

Substituting (4.5) and (4.7) into (4.4), we obtain

\[
H' = H(0) - \frac{\omega}{2}(\sigma^z_1 + \sigma^z_2).
\]

(4.9)

\( H' \) is just the Hamiltonian in the rotating frame and is time independent. Hence, \( |\psi'(t)\rangle = e^{-iH't}\psi'(0)\). Therefore, the solution of the Schrödinger equation (4.1), in the original frame, is given by

\[
|\psi(t)\rangle = e^{-i(\omega t)0}\sigma^z_1 + \sigma^z_2) e^{-iH't}|\psi(0)\rangle.
\]

(4.10)

(b) Cyclic evolution and Aharonov–Anandan phases

Like the case of a single spin in a magnetic field [2], the state is cyclic in the projected space of rays if starting as an eigenstate of \( H' \). After a cycle, the state picks up a phase factor, which is the sum of a dynamical phase and a geometric phase called the AA phase.

Suppose the initial state, in the original frame, is an eigenstate of \( H' \), written as

\[
|\psi(0)\rangle = |\Phi^+_n\rangle \equiv a'_n|\uparrow\uparrow\rangle + b'_n|\uparrow\downarrow\rangle + c'_n|\downarrow\uparrow\rangle + d'_n|\downarrow\downarrow\rangle,
\]

(4.11)

satisfying

\[
H'|\Phi^+_n\rangle = E'_n|\Phi^+_n\rangle.
\]

(4.12)

Then, according to equation (4.10), the state at time \( t \) becomes

\[
|\psi_n(t)\rangle = e^{-iE'_n t}(a'_n|\uparrow\uparrow\rangle + b'_n|\uparrow\downarrow\rangle + c'_n|\downarrow\uparrow\rangle + d'_n|\downarrow\downarrow\rangle + e^{i\omega t}d'_n|\downarrow\downarrow\rangle),
\]

(4.13)

which is always cyclic in the projected space of rays, with period \( T = 2\pi/\omega \), i.e. the period of the Hamiltonian. Because \( \omega T = 2\pi \),

\[
|\psi_n(T)\rangle = e^{i\eta_n}|\Phi^+_n\rangle,
\]

(4.14)

where

\[
\eta_n = -E'_n T
\]

(4.15)

is the total phase accumulated in each cycling period;

\[
\eta_n = \eta^A_n + \eta^d_n
\]

(4.16)

where the dynamical phase is

\[
\eta^d_n = -\int_0^T \langle \psi_n(t)|H(t)|\psi_n(t)\rangle\,dt = -E'_n T - 2\pi(|a'_n|^2 - |d'_n|^2),
\]

(4.17)

whereas the AA phase \( \eta^A_n = \eta_n - \eta^d_n \)[2] is given by

\[
\eta^A_n = 2\pi(|a'_n|^2 - |d'_n|^2).
\]

(4.18)
We now come to the explicit form of the eigenstate $|\Phi_n'\rangle$ of $H'$. The eigenstates and the corresponding eigenvalues are, respectively,

$$
|\Phi_1'\rangle = \frac{1}{\sqrt{M_1}} \left( -B_2 \sin(\theta) \left| \uparrow\uparrow \right\rangle + \left| \downarrow\downarrow \right\rangle \right),
$$

$$
|\Phi_2'\rangle = \frac{1}{\sqrt{M_2}} \left( -B_2 \sin(\theta) \left| \uparrow\uparrow \right\rangle + \left| \downarrow\downarrow \right\rangle \right),
$$

$$
|\Phi_3'\rangle = \frac{1}{\sqrt{M_3}} \left( \frac{B_2 \sin(\theta)}{2E_{2,1}' - B_1 + J + B_2 \cos(\theta)} \left| \downarrow\uparrow \right\rangle + \left| \down\downarrow \right\rangle \right),
$$

$$
|\Phi_4'\rangle = \frac{1}{\sqrt{M_4}} \left( \frac{B_2 \sin(\theta)}{2E_{3,4}' + B_1 + J - B_2 \cos(\theta)} \left| \down\uparrow \right\rangle + \left| \down\downarrow \right\rangle \right)
$$

and

$$
E_1' = \frac{1}{2} (B_1 - \omega - \sqrt{(J - \omega)(J - \omega + 2B_2 \cos(\theta)) + B_2^2}),
$$

$$
E_2' = \frac{1}{2} (B_1 - \omega + \sqrt{(J - \omega)(J - \omega + 2B_2 \cos(\theta)) + B_2^2}),
$$

$$
E_3' = \frac{1}{2} (\omega - B_1 + \sqrt{(J + \omega)(J + \omega - 2B_2 \cos(\theta)) + B_2^2}),
$$

$$
E_4' = \frac{1}{2} (\omega - B_1 - \sqrt{(J + \omega)(J + \omega - 2B_2 \cos(\theta)) + B_2^2}),
$$

with

$$
M_{1,2} = 1 + \left( \frac{B_2 \sin(\theta)}{2E_{2,1}' - B_1 + J + B_2 \cos(\theta)} \right)^2,
$$

$$
M_{3,4} = 1 + \left( \frac{B_2 \sin(\theta)}{2E_{3,4}' + B_1 + J - B_2 \cos(\theta)} \right)^2.
$$

If the initial state is $|\Phi_n'\rangle$, then after $T$, the AA phase is

$$
\eta_n^A = 2\pi \left( 1 - \frac{1}{M_n} \right),
$$

for $n = 1, 2$, whereas

$$
\eta_n^A = \frac{-2\pi}{M_n},
$$

for $n = 3, 4$. Because the terms $2E_n' \pm B_1$ are $B_1$-independent in equations (4.21) (see equations (4.20)), the AA phases are also independent of the static magnetic field coupled with spin 1. Nevertheless, this static magnetic field can be used to control dynamic phases (4.17). An interesting observation is that the AA phase depends on the field parameter $\omega$, and therefore can be controlled by adjusting the driving field. This is of importance in practical applications. The AA phase reduces to the Berry phase of equation (3.5) in the adiabatic limit of the evolution, namely $\omega \to 0$, because in this limit $M_n \to N_n$.

(c) Cancelling phases

In §3b on the adiabatic case, we discussed how to obtain unitary transformations, purely based on Berry phases, by cancelling the dynamic phases through two cycles. Now, the static magnetic field may be used to cancel the total phase through a cycle. For example, choosing $B_1 = \omega + \sqrt{(J - \omega)(J - \omega + 2B_2 \cos(\theta)) + B_2^2}$ with the initial condition $|\psi(0)\rangle = |\Phi_1'\rangle$, after one cycle, we can realize (see equations (4.14), (4.15), (4.20))

$$
|\psi_1(T)\rangle = |\psi(0)\rangle.
$$
Hence, starting as an eigenstate of $H'$, one can realize a cycle with no phase factor, with period $T$, by choosing appropriate values for $B_1$ in the cycle. Furthermore, in this way, one cannot construct a non-trivial two-qubit gate based on AA phases.

5. Summary and conclusions

To summarize, we have investigated the behaviour of geometric phases, including both Berry phases and AA phases, of two Ising-interacting spin-$\frac{1}{2}$s; one of the spins is driven by a time-varying rotating magnetic field, and the other is coupled with a static magnetic field in the direction of the rotation axis. We have calculated the geometric phases of the whole system. It was shown that the geometric phases are independent of the static magnetic field coupled with spin 1. On the other hand, this static magnetic field may be used to control the dynamical phase, in particular, it can be used to generate dark states.

For the adiabatic phases, through two consecutive cycles, the dynamical phases in the four instantaneous eigenstates can be cancelled simultaneously, thus one obtains a phase gate purely based on Berry phases.

We have also studied the exact evolution of the state under the time-dependent Hamiltonian, without the adiabatic condition. If the initial state is an eigenstate of the time-independent Hamiltonian $H'$ given in equation (4.9), then the state is always cyclic with a phase factor that is a sum of the dynamical phase and the AA phase. One can realize a cycle with no phase factor, with period $T$, by choosing appropriate values for $B_1$ in the cycle. In addition, the AA phase recovers the Berry phase in the adiabatic evolution. For two spins with Ising interaction, both the Berry phases and AA phases depend on the interaction constant, $J$, in contrast to the case of isotropic Heisenberg interaction [21], particularly with $J \rightarrow \pm \infty$, Berry phases $\gamma_j \rightarrow 0$.

We would like to address that the interaction between the two spin-$\frac{1}{2}$ particles in our model is not a typical spin–spin coupling as in NMR, but rather a toy model describing a double spin flip. So, we have to make a mapping when we use the presentation in the NMR system and when all subsystems are driven by the classical field.

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References


