Nonlinear analysis of thermally and electrically actuated functionally graded material microbeam

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In this paper, we provide a unified and self-consistent treatment of a functionally graded material (FGM) microbeam with varying thermal conductivity subjected to non-uniform or uniform temperature field. Specifically, it is our objective to determine the effect of the microscopic size of the beam, the electrostatic gap, the temperature field and material property on the pull-in voltage of the microbeam under different boundary conditions. The non-uniform temperature field is obtained by integrating the steady-state heat conduction equation. The governing equations account for the microbeam size by introducing an internal material length-scale parameter that is based on the modified couple stress theory. Furthermore, it takes into account Casimir and van der Waals forces, and the associated electrostatic force with the first-order fringing field effects. The resulting nonlinear differential equations were converted to a coupled system of algebraic equations using the differential quadrature method. The outcome of our work shows the dramatic effect and dependence of the pull-in voltage of the FGM microbeam upon the temperature field, its gradient for a given boundary condition. Specifically, both uniform and non-uniform thermal loading can actuate the FGM microbeam even without an applied voltage. Our work also reveals that the non-uniform temperature field is more effective than the uniform temperature field in actuating a FGM cantilever-type microbeam. For the clamped-clamped case, care must be taken to account for the effective use of thermal loading in the
design of microbeams. It is also observed that uniform thermal loading will lead to a reduction in the pull-in voltage of a FGM microbeam for all the three boundary conditions considered.

1. Introduction

Devices in microelectromechanical systems (MEMS) are typically designed to operate in one or more energy domains owing to their unique characteristics, such as small size, lower power consumption, lower operating costs, increased reliability and higher precision. One of those appealing MEMS applications are radio frequency (RF) switches [1], which hold promise for replacing conventional solid-state switches for RF and microwave applications. Several hurdles remain unresolved and require the attention of the research community so as to ensure the reliability of RF–MEMS switches and turn them into commercially viable products. Reducing the pull-in voltage requirement is one of the major concerns in the design of RF–MEMS switches.

Some work has been done regarding the single-phase material microbeam. Tilmans & Legtenberg [2] studied the static and dynamic characteristics of electrostatically driven vacuum-encapsulated polysilicon resonators using the Rayleigh–Ritz method. Abdel-Rahman et al. [3] and Kuang & Chen [4] presented the nonlinear model for electrostatically actuated microbeams with both ends fixed. A distributed parameter model was used by Ramezani et al. [5] to study the pull-in instability of cantilever nanomechanical switches subjected to intermolecular and electrostatic forces.

It is known that for a conventional MEMS device made of a single layer material, it is almost impossible to simultaneously meet all material and operational requirements posed by MEMS structural layers. The motivation for using functionally graded materials (FGMs) manifests in their superior stress relaxation and capabilities of withstanding high temperatures and large thermal gradients, high fracture toughness and improved stress distribution. Typically, the FGM microbeam is formed using bimaterials, such as graded ceramic diffused in metal. The ceramic constituent of the material provides the high temperature resistance owing to its low thermal conductivity, while the ductile metal constituent accommodates the high thermal stresses resulting from the high thermal transients [6–9].

Recently, FGMs have been considered for use in micro- and nanostructures, such as thin films in the form of shape memory alloys [10,11], micro- and nanoelectromechanical systems (MEMS and NEMS) [12,13] and atomic force microscopy (AFM) [14]. Witvrouw and Mehta proposed the use of a non-homogeneous FGM polycrystalline-SiGe to achieve certain desired electrical and mechanical properties [13,15]. They also conducted an experimental study on the mechanical characterization of poly-SiGe layers for CMOS–MEMS integrated applications [16]. Carbonari et al. [17] studied the multi-physics topology optimization of a piezoactuator, an XY nanopositioner actuated by two graded piezoceramics. Cao et al. [18] presented a new technique for measuring the variation of the material properties along the thickness in a freestanding inhomogeneous thin film. Recently, Jia et al. [19,20] investigated the nonlinear pull-in characteristics of the microswitches made of either homogeneous material or inhomogeneous functionally graded material (FGM) with two material phases under the combined electrostatic and intermolecular force.

In the operating conditions of the microswitch, the temperature would rise owing to the electric current. Thermal loading is also one of the basic actuation parameters that can tune the system directly. Thermal actuation is known for its capability in producing large displacements as a result of heat. In microcapacitive thermal sensors and thermal tuneable capacitors, the use of FGM microbeam is not only subjected to an electrostatic force but also to a thermal bending moment caused by the mismatch in the coefficients of thermal expansion of the constituents. Hasanyan et al. [21] studied the pull-in instabilities in a functionally graded MEMS owing to the heat produced by the electric current. Mohammadi-Alasti et al. [22] investigated the mechanical behaviour of a cantilever FGMs microbeam subjected to a nonlinear electrostatic
pressure and temperature changes. Careful review of the literature indicates that only a limited number of papers exist that address the effect of temperature changes upon the effectiveness of electrostatically actuated microstructures, especially FGM microbeams under a non-uniform temperature field.

In this work, we develop a comprehensive and unified model that covers the important aspects pertaining to the mechanical behaviour of an FGM microswitch as represented by a microbeam. The length scale is considered by introducing a length-scale parameter which is based on the modified couple stress theory. The material length-scale parameter can be viewed as an internal state variable of the microbeam. The non-uniform temperature field is solved using the steady-state heat conduction equation. A mathematical model involving thermal loading, van der Waals force and Casimir force is developed to study the pull-in behaviour of the FGM microbeam. The effect of the microscopic size of the beam, the electrostatic gap, the temperature field and material property on the pull-in voltage of the microbeam under different boundary conditions is discussed.

2. Basic equations

Let us consider the electrostatically actuated FGM microbeam depicted in figure 1. A voltage drop $V$ is applied across the microbeam and the fixed electrode underneath it. Assume a microbeam with length $L$, thickness $h$ and width $b$. Let $x$ be the coordinate along the microbeam with its origin at the left end and $z$ be the coordinate along the cross section with its origin at the mid-plane of the microbeam.

It is assumed that the material properties of FGM microbeam are varying along its thickness [23], and the top surface is made of pure metal and the bottom surface from a diffusion of ceramic into the metal. Subscripts ‘c’ for ceramic and ‘m’ for metal are used. The material properties with respect to the $z$ coordinate can be expressed as

$$P_k(z) = P_{mc}e^{\beta_k(z+h/2)}, \quad (2.1)$$

where $P_k$ denotes the material properties, such as Young’s modulus $E$, Poisson’s ratio $\nu$, mass density $\rho$, coefficient of thermal expansion $\alpha$ and thermal conductivity $K$. It is clear that at the top surface, $P_k(−h/2) = P_m$, and the constants $\beta_k$ can be obtained by

$$\beta_k = \frac{1}{h} \ln \left( \frac{P_h}{P_m} \right), \quad (2.2)$$

where $P_h$ are the material properties at the bottom surface of the microbeam.

Let us now consider the effect of thermal effects upon the pull-in and actuation voltage. Two situations may arise. In the first, the operating conditions of the microswitch would lead to temperature rise. The second, thermal loading is deliberately applied to effectively actuate/tune the microswitch. In order to evaluate these thermal stresses, we should first obtain the temperature distribution in the microbeam. In this study, the temperature field is considered to be uniform over the microbeam length, but vary along its thickness. In this case, it is assumed that no source of heat generation exists within the microbeam system. Hence, the temperature distribution along the thickness direction can be obtained by solving the following
one-dimensional steady-state heat conduction equation through the thickness of the beam [24]

\[ K(z) \frac{d^2T}{dz^2} + \frac{dK(z)}{dz} \frac{dT}{dz} = 0, \]

(2.3)

where \( K(z) \) is the thermal conductivity, which is also varying along the thickness of the FGM microbeam. The following types of thermal boundary conditions are considered:

\[ z = -\frac{h}{2} : T = T_1 \quad \text{and} \quad z = \frac{h}{2} : T = T_b. \]

(2.4)

Here, \( T_1 \) and \( T_b \) are the respective temperatures at the top and bottom surfaces of the FGM microbeam. Assuming \( K(z) \) has the form \( K(z) = K_m e^{\nu(z+h/2)} \) \((\nu = 1/h \ln(K_h/K_m))\) in (2.2), the solution of (2.3) subjected to the boundary condition described in (2.4) will lead to

\[ T(z) = C_0 - C_1 e^{-\nu(z+h/2)}, \]

(2.5)

where \( C_0 = (e^{h\nu} T_b - T_1)/(e^{h\nu} - 1) \) and \( C_1 = e^{h\nu} (T_b - T_1)/(e^{h\nu} - 1) \).

Based on the modified couple stress theory [25], the elastic strain energy, taking into account length-scale effects, can be written as

\[ U = \frac{1}{2} \int_\Omega \left( \sigma_{ij} \varepsilon_{ij} + m_{ij} \chi_{ij} \right) d\Omega, \]

(2.6)

where \( \sigma_{ij} \) is the stress tensor, \( \varepsilon_{ij} \) is the strain tensor, \( m_{ij} \) is the deviatoric part of the couple stress tensor and \( \chi_{ij} \) is the symmetric curvature tensor. The displacement field \((\bar{u}, \bar{w})\) of an arbitrary point on the microbeam can be expressed as

\[ \begin{align*}
\bar{u} &= u - zw_x \\
\bar{w} &= w
\end{align*} \]

(2.7)

where \((u, w)\) are the axial and transverse displacements of a point on the mid-plane (i.e. \( z = 0 \). The strain and curvature tensor associated with the above displacement field are

\[ \varepsilon_x = \varepsilon_0 + zk = u_x + \frac{1}{2} w_x^2 - zw_{xx} \]

and

\[ \chi_{xy} = \frac{-w_{xx}}{2}. \]

(2.8)

The constitutive relations can be written as

\[ \begin{align*}
\sigma_x &= E(\varepsilon_x - \alpha \theta) \\
m_{xxy} &= 2^\kappa \mu_0 \chi_{xy}
\end{align*} \]

(2.9)

where \( \theta = (T(z) - T_0) \) is the temperature change, measured with respect to the initial temperature \( T_0 \). It is considered that there exists a plane stress condition. For a wide beam \((b \geq 5h)\), plane strain condition prevails and \( E \) must be replaced by \( E/(1 - \nu^2) \) and \( \alpha \) by \( \alpha/(1 - \nu) \). Let \( \mu_0 = E/2(1 + \nu) \) be the Lame’s coefficient and \( l \) is a material length-scale parameter. Therefore, the following stress resultants per unit width can be defined as:

\[ \begin{align*}
N &= \int_A \sigma_x dA = A_1 \left( u_x + \frac{1}{2} w_x^2 \right) - B_1 w_{xx} - N_T, \\
M &= \int_A z\sigma_x dA = B_1 \left( u_x + \frac{1}{2} w_x^2 \right) - D_1 w_{xx} - M_T
\end{align*} \]

and

\[ \begin{align*}
Y &= \int_{-h/2}^{h/2} m_{xxy} dz = -\frac{1}{2} G_1 w_{xx},
\end{align*} \]

(2.10)

where

\[(A_1, B_1, D_1) = \int_{-h/2}^{h/2} E(1, z, z^2) dz, \quad G_1 = \int_{-h/2}^{h/2} \frac{E l^2}{2(1 + \nu)} dz.\]
and
\[ \begin{align*}
N_T &= \int_{-h/2}^{h/2} E \alpha \theta \, dz, \\
M_T &= \int_{-h/2}^{h/2} E \alpha \theta z \, dz.
\end{align*} \]

\( A_1 \) and \( D_1 \) are the in-surface stiffness and bending stiffness matrices based on the geometric middle surface, respectively. \( B_1 \) is the stretching–bending coupling matrix, which does not equal to zero, owing to the misalignment of the geometric and physical neutral surface of the FGM structures. Our formulae are based on the geometric middle surface. However, they can be easily shown to be equivalent to the formulae provided in [26,27] which are based on the physical neutral plane of the FGM structures. The terms \( N_T \) and \( M_T \) in (2.10) are the thermal stress resultants and moment caused by the temperature field. Based on the modified couple stress theory [25], the variation of the elastic strain energy of unit width can be written as
\[
\delta U = \int_\Omega \left( \sigma_x \delta \varepsilon_x + 2 m_{xy} \delta \chi_{xy} \right) \, dv = \int_0^L \left( N \delta \varepsilon_0 + M \delta \kappa + 2 Y \delta \chi_{xy} \right) \, dx. \tag{2.11}
\]

Similarly, the variation of the work done by the external force can be expressed as
\[
\delta W = \int_0^L q \delta w \, dx, \tag{2.12}
\]
where
\[
q = F_e + F_v + F_c, \tag{2.13}
\]
where \( F_e \) is the electrostatic force and is given by [28]
\[
F_e = \frac{1}{2} \frac{\varepsilon_v b V^2}{(g_0 - w)^2} \left( 1 + 0.65 \frac{(g_0 - w)}{b} \right), \tag{2.14}
\]
with the first-order fringing field effect included. The absolute dielectric constant in vacuum is taken to be \( \varepsilon_v = 8.8542 \times 10^{-12} \) F m\(^{-1}\). van der Waals force \( F_v \) according to Liu et al. [29] is assumed to take the form
\[
F_v = \frac{A_{12} b}{6\pi (g_0 - w)^3}, \tag{2.15}
\]
where the Hamaker constant \( A_{12} = 4 \times 10^{-20} \) J. The Casimir force according to Liu et al. [29] is
\[
F_c = \frac{\pi^2 h c b}{240(g_0 - w)^4}, \tag{2.16}
\]
where the Planck contact \( h = 6.625 \times 10^{-34} J \cdot s \) and the speed of light \( c = 3 \times 10^8 \) m s\(^{-1}\).

Enforcing the theorem of minimum potential energy
\[
\delta U - \delta W = 0 \tag{2.17}
\]
and substituting (2.11) and (2.12) into equation (2.17), we can obtain the nonlinear governing equations for the microbeam as follows:
\[
\begin{align*}
N_{xx} &= 0, \\
M_{xx} + Y_{xx} + Nw_{xx} + q &= 0. \tag{2.18}
\end{align*}
\]
The boundary conditions can be written as
\[
\begin{align*}
x = 0, L : N = 0 \text{ or } u = 0, \\
M_x + Y_x + Nw_x = 0 \text{ or } w = 0 \tag{2.19}
\end{align*}
\]
\[
M + Y = 0 \text{ or } w_x = 0.
\]
Substituting (2.11) to (2.16) into (2.18), we obtain the following equilibrium equations for the microbeam:

\[
\begin{align*}
A_1(u_{xx} + w_x w_{xx}) - B_1 w_{xxx} &= 0 \\
B_1(u_{xxx} + w_x w_{xxx}) - D_1 w_{xxxx} - \frac{1}{2} G_1 w_{xxxx} + \left[ A_1 \left( u_x + \frac{1}{2} w_x^2 \right) - N_T \right] w_{xx} + \\
&+ \frac{A_{12}}{6\pi (g_0 - w)^3} + \frac{\varepsilon_b V^2}{2\pi (g_0 - w)^2} \left[ 1 + 0.65 \frac{(g_0 - w)}{b} \right] + \frac{\pi^2 \hbar c}{240 (g_0 - w)^4} = 0.
\end{align*}
\]

When the scale parameter \( l \) is taken to be zero, that is \( G_1 = 0 \) in equation (2.20), the classical equilibrium equation of the Euler–Bernoulli beam is readily obtained.

Introducing the following dimensionless parameters:

\[
\begin{align*}
\bar{\xi} &= \frac{x}{L}, \quad W = \frac{w}{g_0}, \quad U = \frac{u}{L}, \quad \bar{A}_1 = \frac{A_1}{E_m h'}, \quad \alpha_1 = \frac{g_0}{L}, \\
\bar{B}_1 &= \frac{B_1}{E_m h'^2}, \quad \bar{D}_1 = \frac{D_1}{E_m h'^2}, \quad \bar{G}_1 = \frac{G_1}{E_m h'^2}, \\
\alpha_2 &= \frac{\varepsilon_b L V_0^2}{2E_m h' g_0}, \quad \bar{V} = \frac{V}{V_0}, \quad \alpha_3 = 0.65 \frac{g_0}{b}, \quad \alpha_4 = \frac{A_{12} L}{6E_m h' g_0^2}, \quad \alpha_5 = \frac{\pi^2 \hbar c L}{240E_m h' g_0^2}, \\
\gamma &= \frac{l}{h'}, \quad \bar{N}_T = \frac{N_T}{E_m h'}, \quad \bar{M}_T = \frac{M_T}{E_m h'},
\end{align*}
\]

where \( V_0 \) is a unit voltage, equilibrium equation (2.20) of the microbeam can be rewritten in dimensionless form as

\[
\begin{align*}
\bar{A}_1 (U_{\bar{\xi}} + \alpha_1 W_{\bar{\xi}} W_{\bar{\xi}}) - \bar{B}_1 \alpha_1 W_{\bar{\xi} \bar{\xi} \bar{\xi} \bar{\xi}} &= 0 \\
\bar{B}_1 (U_{\bar{\xi} \bar{\xi} \bar{\xi}} + \alpha_2^2 W_{\bar{\xi}} W_{\bar{\xi} \bar{\xi} \bar{\xi}}) - \bar{D}_1 \alpha_1 W_{\bar{\xi} \bar{\xi} \bar{\xi} \bar{\xi}} - \frac{1}{2} \bar{G}_1 \alpha_1 W_{\bar{\xi} \bar{\xi} \bar{\xi} \bar{\xi}} + \\
&+ \left[ \bar{A}_1 (U_{\bar{\xi}} + \frac{1}{2} \alpha_1^2 W_{\bar{\xi}}^2) - \bar{N}_T \right] \alpha_1 W_{\bar{\xi} \bar{\xi}} + \frac{\alpha_2 V^2}{(1 - W)^2} [1 + \alpha_3 (1 - W)] + \\
&+ \frac{\alpha_4}{(1 - W)^3} + \frac{\alpha_5}{(1 - W)^4} = 0.
\end{align*}
\]

The dimensionless boundary conditions for clamped-clamped (C-C), simply supported pinned (S-S) and cantilever (C-F) beams are

\[
\begin{align*}
\text{C-C:} & \quad \bar{\xi} = 0, 1: U = 0, W = 0, W_{\bar{\xi}} = 0, \\
\text{S-S:} & \quad \bar{\xi} = 0, 1: U = 0, W = 0, \bar{B}_1 (U_{\bar{\xi}} + \frac{1}{2} \alpha_1^2 W_{\bar{\xi}}^2) - (\bar{D}_1 + \frac{1}{2} \bar{G}_1) \alpha_1 W_{\bar{\xi} \bar{\xi}} - \bar{M}_T = 0, \\
\text{C-F:} & \quad \bar{\xi} = 0: U = 0, W = 0, W_{\bar{\xi}} = 0, \\
& \quad \bar{\xi} = 1: \bar{A}_1 (U_{\bar{\xi}} + \frac{1}{2} \alpha_1^2 W_{\bar{\xi}}^2) - \bar{B}_1 \alpha_1 W_{\bar{\xi} \bar{\xi}} - \bar{N}_T = 0, \\
& \quad \bar{B}_1 (U_{\bar{\xi} \bar{\xi}} + \alpha_2^2 W_{\bar{\xi}} W_{\bar{\xi} \bar{\xi}}) - (\bar{D}_1 + \frac{1}{2} \bar{G}_1) \alpha_1 W_{\bar{\xi} \bar{\xi} \bar{\xi} \bar{\xi}} = 0, \\
\text{and} & \quad \bar{B}_1 (U_{\bar{\xi} \bar{\xi}} + \frac{1}{2} \alpha_1^2 W_{\bar{\xi}}^2) - (\bar{D}_1 + \frac{1}{2} \bar{G}_1) \alpha_1 W_{\bar{\xi} \bar{\xi}} - \bar{M}_T = 0.
\end{align*}
\]

### 3. Solution methodology

Governing equations (2.22) and associated boundary conditions (2.23) form a system of nonlinear differential equation whose closed form exact solution is not possible. The differential quadrature
method (DQM) was deployed to solve this differential system numerically. DQM involves the approximation of derivatives by using weighted sums of function values within the physical domain, which converts a differential equation system into an algebraic equation system. According to DQM rule, the derivative of the dimensionless deflection \( W \) at an arbitrary point \( x_i \) is approximated by

\[
\frac{d^n W(x_i)}{dx^n} = \sum_{j=1}^{N} C^{(n)}_{ij} W(x_j), \quad i = 1, 2, \ldots, N-1, N. \tag{3.1}
\]

The derivative of the dimensionless displacement \( U \) has a similar form. The weighting coefficients \( C^{(n)}_{ij} \) are dependent on the distribution of the sampling points only and can be calculated from recursive formulae [30] as follows:

\[
\begin{align*}
C^{(1)}_{ij} &= \frac{\prod_{k=1}^{i-1} (x_i - x_j)}{\prod_{k=1}^{j-1} (x_i - x_j)}, \quad i, j = 1, 2, \ldots, N-1, N; j \neq i, \\
C^{(m)}_{ii} &= -\sum_{j=1,j \neq i}^{N} C^{(m)}_{ij}, \quad i, j = 1, 2, \ldots, N-2, N-1; m \geq 1
\end{align*}
\]

(3.2)

and

\[
C^{(r)}_{ij} = r \left[ C^{(r-1)}_{ii} \cdot C^{(1)}_{ij} - \frac{C^{(r-1)}_{ij}}{x_i - x_j} \right], \quad r \geq 2,
\]

where

\[
\prod_{j=1,j \neq i}^{N} (x_i - x_j),
\]

(3.3)

and \( N \) (with \( N \) being odd) is the total number of sampling points, with \( x_i \) unevenly distributed over the domain. In the domain \([0, 1]\), the sampling point can be given by Chebyshev polynomial, i.e.

\[
x_i = \frac{1}{2} \left( 1 - \cos \frac{2i - 1}{2N} \pi \right), \quad i = 3, 4, \ldots, N-2
\]

(3.4)

and

\[
x_1 = 0, \ x_2 = \delta, \ x_{N-1} = 1 - \delta, \ x_N = 1 (\delta = 10^{-4}).
\]

Applying DQM approximations to the governing equation (2.22), it yields

\[
\begin{align*}
\bar{A}_1 \sum_{j=1}^{N} C^{(2)}_{ij} U_j + \alpha_1 \bar{A}_1 \left( \sum_{j=1}^{N} C^{(1)}_{ij} W_j \right) - \bar{B}_1 \alpha_1 \sum_{j=1}^{N} C^{(3)}_{ij} W_j &= 0, \\
\bar{B}_1 \sum_{j=1}^{N} C^{(3)}_{ij} U_j + \alpha_1^2 \bar{B}_1 \left( \sum_{j=1}^{N} C^{(1)}_{ij} W_j \right) - (\bar{D}_1 \alpha_1 + \frac{1}{2} \bar{G}_1 \alpha_1) \sum_{j=1}^{N} C^{(4)}_{ij} W_j &+ \left[ \bar{A}_1 \sum_{j=1}^{N} C^{(1)}_{ij} U_j + \frac{1}{2} \alpha_1^2 \bar{A}_1 \left( \sum_{j=1}^{N} C^{(1)}_{ij} W_j \right)^2 - \bar{A}_1 \bar{N}_r \right] \alpha_1 W_{\xi \xi} \\
+ \frac{\alpha_2 V^2}{(1 - W)^2} [1 + \alpha_3 (1 - W)] + \frac{\alpha_4}{(1 - W)^3} + \frac{\alpha_5}{(1 - W)^4} &= 0.
\end{align*}
\]

(3.5)
Accordingly, the boundary conditions become

\[
\begin{align*}
\text{C-C:} \quad U_1 &= 0, U_N = 0, W_1 = 0, \sum_{j=1}^{N} C_{2j}^{(1)} W_j = 0, \sum_{j=1}^{N} C_{(N-1)j}^{(1)} W_j = 0, W_N = 0, \\
\text{S-S:} \quad U_1 &= 0, U_N = 0, \tilde{B}_1 \sum_{j=1}^{N} C_{(N-1)j}^{(1)} U_j + \frac{1}{2} \alpha_1^2 \tilde{B}_1 \left( \sum_{j=1}^{N} C_{2j}^{(1)} W_j \right)^2 \\
&- \left( \tilde{D}_1 + \frac{1}{2} \tilde{G}_1 \right) \alpha_1 \sum_{j=1}^{N} C_{2j}^{(2)} W_j - \tilde{M}_T = 0, \\
\text{C-F:} \quad U_1 &= 0, W_1 = 0, \sum_{j=1}^{N} C_{2j}^{(1)} W_j = 0, \tilde{A}_1 \sum_{j=1}^{N} C_{Nj}^{(1)} U_j + \frac{1}{2} \alpha_1^2 \tilde{A}_1 \left( \sum_{j=1}^{N} C_{Nj}^{(1)} W_j \right)^2 \\
&- \tilde{B}_1 \alpha_1 \sum_{j=1}^{N} C_{Nj}^{(2)} W_j - \tilde{N}_T = 0, \\
&- \left( \tilde{D}_1 + \frac{1}{2} \tilde{G}_1 \right) \alpha_1 \sum_{j=1}^{N} C_{(N-1)j}^{(3)} W_j = 0, \\
&\tilde{B}_1 \sum_{j=1}^{N} C_{Nj}^{(1)} U_j + \frac{1}{2} \alpha_1^2 \tilde{B}_1 \left( \sum_{j=1}^{N} C_{Nj}^{(1)} W_j \right)^2 \\
&- \left( \tilde{D}_1 + \frac{1}{2} \tilde{G}_1 \right) \alpha_1 \sum_{j=1}^{N} C_{Nj}^{(2)} W_j - \tilde{M}_T = 0.
\end{align*}
\]  

(3.6)

The solution of the nonlinear governing equations is obtained by iteration and the nonlinear terms are linearized by iteration as follows:

\[
(X \cdot Y)_m = (X)_m (Y)_{m_p}
\]  

(3.7)

in which \((Y)_{m_p}\) is the value of the former iteration. For the primary iteration, a secondary extrapolation method is introduced to obtain the value of \((Y)_{m_p}\), such that

\[
(Y)_{m_p} = A(Y)_{m-1} + B(Y)_{m-2} + C(Y)_{m-3}.
\]  

(3.8)

The iteration coefficients \(A, B\) and \(C\) at the \(m\)th step are selected as follows:

\[
m = 1: \quad A = 1, B = 0, C = 0, \\
m = 2: \quad A = 2, B = -1, C = 0, \\
m \geq 3: \quad A = 3, B = -3, C = 1.
\]  

(3.9)
Table 1. Material properties of the FGM microbeam.

<table>
<thead>
<tr>
<th>Material</th>
<th>$E$ (GPa)</th>
<th>$\nu$</th>
<th>$\alpha$ ($10^{-6}$ K$^{-1}$)</th>
<th>$\rho$ (kg m$^{-3}$)</th>
<th>$K$ (W m$^{-1}$ K$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>nickel</td>
<td>204</td>
<td>0.3</td>
<td>13.2</td>
<td>8902</td>
<td>237</td>
</tr>
<tr>
<td>silicon nitride</td>
<td>310</td>
<td>0.27</td>
<td>3.4</td>
<td>3290</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 2. Comparison of pull-in voltage of electrostatically actuated polysilicon microbeam.

<table>
<thead>
<tr>
<th>$L$ (µm)</th>
<th>experimental ref. [2]</th>
<th>present</th>
</tr>
</thead>
<tbody>
<tr>
<td>210</td>
<td>27.95</td>
<td>27.89</td>
</tr>
<tr>
<td>310</td>
<td>13.78</td>
<td>13.73</td>
</tr>
<tr>
<td>410</td>
<td>9.13</td>
<td>9.05</td>
</tr>
<tr>
<td>510</td>
<td>6.57</td>
<td>6.52</td>
</tr>
</tbody>
</table>

For every loading step, we continue the iteration until the difference between two consecutive iterations ($\sum_{j=1}^{N} \{(W_j)_m - (W_j)_{m-1}\}^2 + \{(U_j)_m - (U_j)_{m-1}\}^2$) is less than 0.001%.

4. Numerical results and discussion

(a) Validation of unified model

It is assumed that the investigated FGM microbeam has the following geometrical properties: $L = 500$ µm, $b = 90$ µm, $h = 6$ µm and $g_0 = 2$ µm. It is made of two materials: silicon nitride (Si$_3$N$_4$) and nickel (Ni). The material properties are listed in Table 1. The FGM microbeam was formed from five varied silicon nitride diffused into nickel. These five cases will be identified in our work as Cases 1–5 and relate to 0–100%, in increments of 25%, corresponding to the percentage of diffused Si$_3$N$_4$. The variation in Young’s modulus and coefficient of thermal expansion in the thickness direction are determined by (2.1). In this study, we selected the total number of sampling points $N$ to be 13. This selection was determined as a result of numerous convergence trials. The selection of the step size for the voltage application in each case was also selected by convergence test between successive iterations.

Consider Case 5 FGM microbeam and cantilever-type fixation without accounting for temperature. In this case, the pull-in voltage as obtained from our model is 14.94 compared with 14.85 in the work of Mohammadi-Alasti et al. [22].

To further validate the present analysis, the pull-in voltage results of clamped-clamped microbeam are compared with the experimental results provided in [2]. In this validation example, the microbeam is assumed to be made of pure polysilicon, with Young’s modulus $E = 151$ GPa and the relevant geometric parameters being $b = 100$, $h = 1.5$ and $g_0 = 1.18$ µm. The comparisons which are shown in Table 2 reveal good agreement.

(b) Pull-in voltage of functionally graded material microbeam

In this section, the dependence of the pull-in voltage on the microbeam’s material properties, boundary conditions, uniform and non-uniform temperature fields, Casimir and van der Waals forces and material length-scale parameter are presented and discussed.

Let us first consider the case where there are no thermal effects. Figure 2 shows the variation of the deflection of the FGM microbeam with the applied voltage for the three varied boundary conditions: (a) two ends clamped, (b) two ends simply supported and (c) cantilever microbeam. The abscissa ‘$V$’ denotes the applied voltage and the vertical ordinate ‘$W$’ denotes the mid-point deflection for (a) and (b) and the free-end deflection of cantilever microbeam (c). Comparing figure 2a–c, it can be seen that for the same material properties and geometry, the cantilever microbeam requires the least pull-in voltage, while the clamped-clamped microbeam requires the largest pull-in voltage. As the Si$_3$N$_4$ has a larger Young’s modulus than Ni, the increase
Figure 2. Variation pull-in voltage of a FGM microbeam with type fixations of (a) clamped-clamped, (b) simply supported and (c) cantilever. (Online version in colour.)

in the ceramic content increases the stiffness of the microbeam, which leads to a larger pull-in voltage for Case 5 FGM where Si₃N₄ is 100% at the bottom surface. In the clamped-clamped case, the respective pull-in voltages of Cases 1–5 are 85, 87, 89, 91 and 93. This increase in pull-in voltage is owing to the increased stiffness of the microbeam as a result of the inclusion of Si₃N₄. Effectively, the pull-in voltage requirement increases in the absence of thermal actuation of the considered FGM microbeam. In the following, we will illustrate some superior properties of the FGM microbeam over single-phase microbeam when subjected to both electrostatic and thermal actuation.

Initially, we will consider the microbeam to be subjected to a uniform temperature field such that \( T(z) = T_b = T_0 + \Delta T \) K, where the initial temperature is \( T_0 = 300 \) K with \( \Delta T \) being the temperature change. Unless otherwise specified, the non-uniform temperature field is given by (2.5) with the following boundary conditions: \( T_b = T_0 - \Delta T \) K at the bottom, and \( T_t = T_0 + \Delta T \) K at the top of the microbeam. In the case of a pure nickel microbeam, the non-uniform field reduces to a linear distribution that is governed by the bottom and top temperatures as given by \( T(z) = T_t + (z + h/2)(T_b - T_t)/h \).

Figure 3 shows the effect of subjecting the microbeam to a uniform temperature field on its deflection for a given applied voltage for Case 5 arrangement for different end fixation. The dashed curve in figure 3a shows that the pull-in voltage of the FGM microbeam is 93 under room temperature. When the microbeam is subjected to a uniform temperature rise of \( \Delta T = 10 \) K, the pull-in voltage is decreased from 93 to 86 (the dash-dotted curve in figure 3a). When the temperature increases, an axial compressive thermal stress is generated, which reduces the microbeam’s resistance to bending loads in the FGM cases considered. By contrast, a decrease in the temperature introduces an axial tensile thermal stress, which increases the pull-in voltage \( (V_p = 100 \) shown by the solid curve, figure 3a).
Figure 3. Effect of uniform temperature variation on pull-in voltage of Case 5 FGM (a) clamped-clamped, (b) simply supported and (c) cantilever microbeam. (Online version in colour.)

Figure 3b depicts the effect of uniform temperature on the pull-in voltage of the simply supported pinned microbeam. It can be observed that even in the absence of the applied voltage, the increase in temperature leads to an initial upward bending ($W < 0$). This is owing to the larger expansion of the metal-rich surface of the FGM microbeam. In this case, the applied voltage has to overcome the resulting upward moment before it proceeds with the pull-in. In spite of this, the FGM microbeam requires the smallest pull-in voltage ($V_p = 40.05$). Specifically stated, this smallest pull-in occurs when the uniform temperature field increases to 310 K, compared with room temperature ($V_p = 41.4$) and when the temperature is decreased to 290 K ($V_p = 43.2$). Again, the reduction in the pull-in voltage in the simply supported pinned microbeam is owing to the reduced resistance to bending under the resulting thermal compressive loads.

Figure 3c shows the effect of temperature change on the pull-in voltage of the FGM cantilever microbeam. The solid and dash-dotted curves depict the initial deflection owing to the thermal strain when $V = 0$. Unlike the simply-simply supported case, when the temperature increases, the top surface stretches more significantly than the bottom, leading to a downward deflection of the free end of the microbeam. This leads to a reduction in the initial electrostatic gap. The downward displacement results in a lower pull-in voltage ($V_p = 12.3$). The pull-in voltages for the room temperature and the temperature drop cases are 14.8 and 16.74, respectively.

Figure 3 leads us to conclude that uniform thermal loading will lead to a reduction in the pull-in voltage of the FGM (Case 5) microbeam under all the three cases of boundary conditions considered.

Let us now consider the case of a thermally actuated microbeam ($V = 0$). Figure 4a,b shows the variation of the pull-in voltage of the FGM cantilever microbeam with different ceramic percentages. In these two figures, the tip deflection of the cantilever microbeam varies linearly with the temperature increase, and thus avoids the sudden increase in the deflection observed
Figure 4. FGM cantilever microbeam deflection versus temperature change ($V = 0$) under (a) uniform temperature field $T(z) = T_b = T_t = T_0 + \Delta T K$ and (b) non-uniform temperature field ($T_b = T_0 - \Delta T K$ and $T_t = T_0 + \Delta T K$). (Online version in colour.)

In figure 3. If we now consider the case of a uniform temperature field, the deflection of the microbeam increases more dramatically with the increase in the ceramic content, as depicted in figure 4a. This is owing to the thermal mismatch in the thermal expansion coefficients within the microbeam material. Moreover, when the microbeam is composed of pure nickel (Case 1), the temperature change does not affect the deflection, as indicated by the solid line shown in figure 4a (overlapping with the abscissa). In this case, the pure nickel microbeam stretches uniformly in the axial direction and does not deform downward. The reason is that there exist uniform thermal strains in the thickness direction without thermal stresses for the homogeneous cantilever microbeam. Uniform thermal loading cannot pull-in this pure nickel microbeam.

The relationship of deflection with temperature change is shown in figure 4b for the considered non-uniform temperature environment. Note that the pure nickel microbeam (Case 1, as shown in the solid line) shows the greatest slope for the assumed non-uniform thermal loading. This is because, in this case, both the bottom and top surface of microbeam have the largest thermal strains compared with the FGM microbeam (Cases 2–5). By comparing figure 4a and b, it can be found that under a non-uniform temperature field a small thermal load can lead to pull-in of the microbeam. In this case, the thermal strain varies across the thickness. Each layer of the beam cannot stretch or shrink freely without affecting the other layers. Owing to the mismatch of the thermal strains in the thickness direction, thermal stresses develop in the FGM beam and noting that the thermal stress resultant through the thickness direction is zero. The varying thermal strain plays a role in changing the pull-in voltage.

Figure 5 shows the pull-in voltage of different cases of FGM cantilever microbeam under thermal actuation with applied actuation voltage ($V = 5$). The applied actuation voltage will introduce an initial deflection when there is no temperature variation. The pure nickel microbeam under uniform temperature environment does not change with the increase in temperature. Comparing figure 5 with figure 4, it can be observed that as the deflection increases, a nonlinear relationship between temperature and microbeam tip-deflection of the FGM cantilever develops. This nonlinearity results in a sudden increase in deflection when thermal actuation is about to pull-in the microbeam for both uniform and non-uniform temperature field. Comparing figures 4 and 5, it can be noted that the pull-in voltage of the microbeam can be reduced by employing the appropriate thermal field.

Figure 6 shows the pull-in voltage of the FGM clamped-clamped microbeam under thermal actuation and electrostatic actuation ($V = 50$). It can be seen from figure 6a that the deflection of the microbeam of the pure nickel (Case 1, as shown by the solid line) under a uniform temperature field increases steeply. This phenomenon is different from that observed in the case of a cantilever microbeam (figures 4a and 5a). Let us now demonstrate the role played by thermal actuation on the pull-in voltage. Take for instance Case 5, in the absence of thermal actuation, the electrostatic
pull-in voltage is $V = 86$ (figure 3a). If we now introduce thermal actuation with $\Delta T = 46.5\text{ K}$, the electrostatic actuation reduces to $V = 50$ (figure 6a), leading to a reduction of 42% in the electrostatic voltage. Figure 6b shows the pull-in of the FGM clamped-clamped microbeam under non-uniform temperature ($T_b = T_0 - \Delta T\text{ K}$ and $T_t = T_0 + \Delta T\text{ K}$) field with an electrostatic voltage $V = 50$. In this case, the resultant thermal stress of the pure nickel microbeam (Case 1, in which the temperature field distributed linearly in the thickness direction) equals to 0 and the temperature change does not affect the microbeam deflection. For the FGM microbeam, the thermal expansion coefficient in the top surface is larger than that in the bottom, which leads to a compressive stress in the microbeam. A bigger percentage of diffused ceramic will lead to larger thermal compressive stresses. Compared with other considered cases, the results show that Case 5 FGM microbeam has the largest compressive stress and the lowest pull-in voltage. From figure 6a,b, it can be noted that uniform temperature loading is more effective for the clamped-clamped microbeam than non-uniform thermal loading.

Let us now examine the case presented in figure 7 for different non-uniform temperature distributions. In this case, the temperature distribution of the microbeam is given by $T_t = T_0\text{ K}$, while $T_b = T_0 + 2\Delta T\text{ K}$ for the different cases depicted in figure 7a. By contrast, in figure 7b, the
uppermost temperature is given by $T_t = T_0 + 2\Delta T K$, whereas the bottom fibre temperature is assumed to be constant with $T_b = 0 K$. The results depicted in figure 7a reveal that Case 1 requires less thermal actuation, while Case 5 requires greater thermal actuation than the corresponding cases depicted in figure 6a for uniform temperature to actuate the microbeam investigated. Interestingly, the roles are reversed when considering the second thermal distribution in figure 7b where Case 1 requires greater and Case 5 requires smaller thermal actuation when the results are compared with figure 6a for uniform temperature field.

Let us now examine the effect of Casimir and van der Waals forces which are typically ignored in most analyses. Figure 8 shows the effect of Casimir and van der Waals forces on the pull-in voltage of both the FGM microbeam and the pure nickel microbeam with varying initial gap $a_3 = 0.65g_0/b$ and $\Delta T = 0 K$. The abscissa ‘$V_p$’ denotes the pull-in voltage. For example, when $a_3 = 0.0015$, the pull-in voltage, calculated without considering van der Waals and Casimir forces, is overestimated by 30% when compared with the one that considers the molecular forces. As can be seen from the results, neglecting these forces (the solid curves) may lead to overrating of the pull-in voltage at small initial gaps. As the gap increases, the effect of these intermolecular forces is reduced significantly. It can also be observed that the increase in $a_3$ (increase in gap $g_0$ or decrease in width $b$), leads to an increase in the pull-in voltage of all the considered cases.

Figure 7. Pull-in voltage of FGM clamped-clamped microbeam under different non-uniform temperature field ($V = 50$) (a) $T_t = T_0 K$, $T_b = T_0 + 2\Delta T K$ and (b) $T_t = T_0 + 2\Delta T K$, $T_b = T_0 K$. (Online version in colour.)

Figure 8. Effect of Casimir and van der Waals forces on the pull-in voltage of FGM microbeam. (Online version in colour.)
Another important feature that needs to be accounted for in our work is the effect of the length scale of the microbeam upon the pull-in voltage. Figure 9 shows that the pull-in voltage is underestimated, if the length scale is not taken into account in the model. It should be noted that the material length-scale parameter is obtained as $l = 17.6 \mu m$ for a homogeneous epoxy beam [31]. However, to the authors’ knowledge, there is no available data for the material length-scale parameter of FGM microbeam. As the FGM material length-scale parameter is governed by size and bimaterial properties, in order to study its effect on the pull-in voltage, the material length-scale parameter of FGM microbeam was taken as a variable. It can be observed from the figure that the increase in the material length parameter leads to a significant discrepancy between the results of our work which uses the modified couple stress theory and the classical theory adopted by many in the literature. Ignoring the effect of length scale typically leads to a gross underestimation of the pull-in voltage of the microbeam.

5. Conclusion

In this paper, a unified and comprehensive analytical model is developed to describe the behaviour of FGM microbeam actuated thermally and electrically. The model incorporates the couple stress theory to introduce an intrinsic length scale as a state variable and further considers van der Waals force and Casimir force upon the actuating voltage pull-in and the response of the microbeam switch. Three support cases both for uniform and non-uniform thermal fields are examined and discussed. Our results reveal the following:

(a) Given the same material and geometry, the cantilever microbeam requires the least pull-in voltage, whereas the clamped-clamped microbeam requires the largest.

(b) Uniform temperature rise leads to a reduction in the pull-in voltage of the FGM microbeam of the three different cases of microbeam fixation investigated.

(c) As for the cantilever microbeam, in the uniform thermal actuation case, only the FGM microbeam can be actuated by thermal loading. However, in the non-uniform case, the pure nickel microbeam requires less thermal actuation than the FGM microbeam to pull-in.

(d) Non-uniform thermal actuation is more effective than uniform thermal actuation in achieving the same deflection of the FGM cantilever microbeam in the absence of any actuating voltages.

(e) Length-scale effects must be considered in the formulations of the microbeam. The use of the classical theory, which avoids the deployment of length-scale effects, for example the couple stress theory, can lead to underestimation of the pull-in voltage of the microbeam.
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