Lévy information and the aggregation of risk aversion

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When investors have heterogeneous attitudes towards risk, it is reasonable to assume that each investor has a pricing kernel, and that these individual pricing kernels are aggregated to form a market pricing kernel. The various investors are then buyers or sellers depending on how their individual pricing kernels compare with that of the market. In Brownian-based models, we can represent such heterogeneous attitudes by letting the market price of risk be a random variable, the distribution of which corresponds to the variability of attitude across the market. If the flow of market information is determined by the movements of prices, then neither the Brownian driver nor the market price of risk are directly visible: the filtration is generated by an ‘information process’ given by a combination of the two. We show that the market pricing kernel is then given by the harmonic mean of the individual pricing kernels associated with the various market participants. Remarkably, with an appropriate definition of Lévy information one draws the same conclusion in the case when asset prices can jump. As a consequence, we are led to a rather general scheme for the management of investments in heterogeneous markets subject to jump risk.

1. Introduction

The importance of the flow of information in financial markets is clearly evident. As market participants we are all ‘signal processors’. It is logical therefore to base our financial models as much as reasonably possible on the information available to market participants, and to try to understand the way in which market signals are processed. One has to admit from the outset that markets are complicated, not only in terms of their structure, but also in terms of the investor psychology. Nevertheless,
we shall show that there is scope for building relatively simple but intuitively natural mathematical models that capture the effects of information flows in heterogeneous markets, thus allowing one to address a variety of practical issues arising in the general area of investment management that might otherwise seem unapproachable.

In a typical financial model, it is usual to begin with a probability space \((\Omega, \mathcal{F}, P)\), together with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\). The so-called ‘physical’ probability measure \(P\) is meant to summarize the system of market probability assignments to various possible events, and the ‘market filtration’ \(\{\mathcal{F}_t\}\) is meant to summarize, for each time \(t \geq 0\), the totality of information available to market participants up to time \(t\). Clearly, both of these ideas involve a good deal of idealization, and it may be taking the notion of ‘market efficiency’ too far to suppose that such a characterization of the market is realistic. It makes better sense perhaps to suggest that each market participant in some way implicitly builds their own version of the basic model, and then by some process all these different versions of the basic model are amalgamated to produce an overall effective model that represents the market. The individual investor then trades in a way that is consistent with the relation of their ‘private’ model to that of the market as a whole.

Our purpose here is to examine a particular example of such a scheme, arising in connection with the apparent variability of opinion one observes concerning the expected rates of returns on financial assets. It is plainly obvious to the investment management community that well-informed, intelligent market participants will often have significantly differing opinions on rates of return—indeed, not only with respect to their views on the expected returns associated with individual assets, but also the expected returns associated with the market as a whole. Clearly, some sort of hypothesis of ‘natural variation’ is needed to model such a situation. The question thus arising is the following: how one can reconcile such a view with those well-established techniques of financial engineering that entail some sort of equilibrium, or, at least, the absence of arbitrage, as part of the very basis of the modelling framework, usually coupled implicitly with the assumption of a high degree of homogeneity and uniformity across the market—some version of the law of one price?

Our approach will be to model the excess rate of return (above the interest rate) as a random variable, the interpretation of which reflects the spread of opinion in the market about the rate of return to be demanded in exchange for the assumption of a given level of risk. We shall assume that the investors can be modelled as having a degree of rationality, in the sense that they recognize that other investors have differing opinions, and that the collective effect of these differing opinions will have an effect on market movements.

In the context of financial modelling, we take the accepted ‘modern’ view that financial models are by their nature ephemeral. By that, we mean that models are always used ‘in the present’ for a specific purpose—pricing, hedging, asset allocation, decision-making. Once the model has been used, then it is (so to speak) thrown away, and another model is constructed for the next task. The ‘new’ model may be identical in structure to its predecessor—perhaps differing only slightly in the assignment of some parameter values. Nevertheless, it is different: one starts each day (or minute, or microsecond) with a fresh model. In practice, the term ‘new model’ is usually applied only if there is some significant structural difference involved—for example, in the sense that the Vasicek and Cox–Ingersoll–Ross interest rate models are structurally different. In casual discussion, one would not normally say that two different versions of the Vasicek model with different values of the mean reversion parameter were distinct models. But it is useful to maintain the idea that one really is in fact working with different models—perhaps a better choice of words would be to say that one is working with a parametric family of models. Then the periodic adjustment of the parameters is the ‘calibration’ of the model. Indeed, the pervasive need for a regular regimen of robust model recalibration is a dominant feature of much of modern banking.

How does one use a financial model? That depends on the particular type of problem one is trying to solve (pricing, hedging, asset allocation, and so on), but typically one uses the freshly calibrated model to generate (by simulation, or numerical integration, or exact solution) the trajectories of the asset prices under different outcomes of chance; and then certain functions
of the trajectories are averaged, with appropriate weightings, to provide the figures needed for the particular application.

The point is that when ‘averaging’ is carried out in a financial model, one is typically averaging over the outcome of chance in two different senses simultaneously—the first being the usual sense of the development of the trajectory as time goes by (for example, one averages over a multitude of distinct random walks); and the second being the sense that one averages over different views or characteristics of a multiplicity of market participants. Thus, if one models the excess rate of return as a random variable, the ‘randomness’ of the excess rate of return is not necessarily to be interpreted—in the model—exclusively in the sense that one particular value turns out to be the ‘correct’ one selected by chance (as if by coin flip), but rather in the sense that if one were to select an investor at random then one could say with what probability that investor will have a view or characteristic that lies in a certain range. Thus in the model, it is the collective effect (via the weighted average) of variation in the future trajectory that determines the solution to a problem posed in the present.

In this paper, we make use of pricing kernel methods, which turn out to be particularly useful, allowing one to distinguish between pricing issues and hedging issues. See Cochrane [1] for an informal but comprehensive introduction to the application of pricing kernels in finance. The study of heterogeneous markets is still, one could probably say, in its infancy, and in a state of active development. Indeed, one can be overwhelmed by trying to contend with all the different types of heterogeneity that can arise in financial markets—heterogeneity in risk attitude, in impatience, in probability assignment, in transmission of information, in network connectivity, in information processing speed, and so on. See Brown & Rogers [2], Duffie [3], Ziegler [4], and references cited therein, for overviews of some of the issues connected to heterogeneity in financial markets currently being pursued. There is also a large literature devoted to portfolio management under partial information (see, for example, Björk et al. [5] and references cited therein). Our approach in what follows is novel inasmuch as it combines pricing-kernel methods with information-based pricing and elements of behavioural finance in an intuitive yet mathematically rigorous treatment aimed at problems of asset allocation and investment decision, with a view particularly on how to manage such problems in the face of issues involving jumps in asset prices, in situations where the ‘hedging paradigm’ for derivative pricing breaks down, and in the context of a post-crisis world-view where buy-side concerns are taken as seriously as sell-side concerns.

The structure of the paper is as follows. In §2, we consider the problem of a heterogeneous market in which investors have variegated attitudes towards risk. We specialize to the case in which asset prices are driven by Brownian motion, and we model the variation in attitude towards risk by taking the excess rate of return to be a random variable. We introduce the idea of an ‘information process’ as the generator of the market filtration, and derive the conditional distribution of the market price of risk. In §3, we work out the form that the pricing kernel takes in such a model and derive the remarkable result that the market pricing kernel is given by the harmonic mean of the pricing kernels attributable to the various market participants based on their attitudes towards risk. In §4, we work out the dynamics of a typical financial asset under the assumptions that we have made, and show how the dynamics can be represented in a way that is explicitly consistent with the absence of arbitrage. In particular, we are able to establish the existence of a Brownian motion adapted to the filtration generated by the information process and such that the dynamical equations of both the asset and the pricing kernel are driven by this ‘market’ Brownian motion. Finally, in §5 and §6, we show how the general framework that we have developed in the case of Brownian motion based models can be extended very naturally to a wide family of models admitting jumps.

2. Random risk aversion and market heterogeneity

It will be useful if one regards the market filtration \( \{ \mathcal{F}_t \} \) as being generated by a set of one or more ‘information processes’. By an information process, we mean a process that carries noisy or
imperfect information about some quantity that is of relevance to market participants. The notion is a quite general one, and a number of different situations arise in which one can model the flow of information relevant to the formation of prices. Examples include information flows concerning the market factors that determine dividends on stocks, defaults on bonds, or claims on insurance contracts. The approach that we are adopting is that of ‘information-based asset pricing’, as represented in Brody et al. [6–10], Brody & Friedman [11], Filipovic et al. [12], Hoyle [13], Hoyle et al. [14], Hughston & Macrina [15,16], Macrina [17], Macrina & Parbhoo [18], and Rutkowski & Yu [19]. An important advantage of thinking of the filtration as being generated by information processes is that the treatment of informationally heterogeneous markets can then be pursued in a relatively straightforward way. We consider a set of information processes, some of which are accessible to investor A, some to investor B, some to investor C, and so on, generally with some overlap. If the overlap is substantial for a relatively large number of investors, then we can for some purposes call this the ‘market filtration’. Generally speaking, the system of filtrations has a kind of hierarchical structure that takes the form of a lattice. In what follows, we shall work with a single market filtration, since our main concern is with heterogeneous attitudes towards risk rather than heterogeneous information flows, but the set-up will be structured in such a way that the consideration of heterogeneous information flows is also feasible.

An interesting and important example of an information process arises rather naturally in the context of geometric Brownian motion (GBM) models when we try to generalize such models to situations where the rate of return on the stock is not known exactly. The GBM models are, needless to say, a little too simple as such to be taken seriously as real-world models for asset price dynamics. Nevertheless, they do capture rather succinctly certain key elements of the relation between risk and return, and it is in that context that we confine the discussion initially to the GBM class. The idea is that once one obtains some understanding of how to deal with random rates of return in the case of constant-parameter GBM models, then one might focus on how to generalize the modelling framework to incorporate more realistic features, such as stochastic volatility or the inclusion of jumps.

We begin by consideration of the case of a single risky asset in the standard GBM family of models. The discussion that follows readily generalizes to the situation where a number of risky assets are traded. For simplicity, we present the case of a single such asset, and we assume that no dividends are paid over the time horizon considered. For the price we write

\[ S_t = S_0 e^{(r+\sigma \lambda)t} e^{\sigma B_t - \frac{1}{2} \sigma^2 t}, \tag{2.1} \]

where \( S_0 \) is the initial price, \( \{B_t\}_{t \geq 0} \) is a standard Brownian motion, \( r > 0 \) is the interest rate, \( \sigma > 0 \) is the volatility and \( \lambda > 0 \) is the risk aversion factor. The term \( \sigma \lambda \) is called the ‘risk premium’ or ‘excess rate of return’. For a fixed level of risk aversion, the risk premium increases if one increases the level of riskiness (as represented by the volatility), and for a fixed level of riskiness, the risk premium increases if one increases the level of risk aversion. Since \( \lambda \sigma \) is linear in each factor, it follows that \( \lambda \) has the interpretation of being the ‘excess rate of return per unit of risk’, or ‘market price of risk’ in the GBM model. It should be evident, on the other hand, that there is no \textit{a priori} reason why the excess rate of return should be bilinear. In fact, the case of a bilinear risk premium is quite special. For example, in a general Lévy model the excess rate of return is a nonlinear function of the risk aversion and the volatility (Brody et al. [20], Mackie [21]), and the notion of ‘market price of risk’ is inappropriate.

To complete the specification of the model, we need a pricing kernel \( \{\pi_t\}_{t > 0} \), which in the standard GBM model takes the form

\[ \pi_t = e^{-rt} e^{-\lambda B_t - \frac{1}{2} \lambda^2 t}. \tag{2.2} \]

The pricing kernel in an arbitrage-free model has the property that its product with the price of any non-dividend-paying asset gives a martingale under the physical measure \( \mathbb{P} \). In the present situation we have

\[ \pi_t S_t = S_0 e^{(\sigma - \lambda) B_t - \frac{1}{2} (\sigma - \lambda)^2 t}, \tag{2.3} \]
and one sees that the martingale condition is indeed satisfied. It is important to observe that for the expression of the principle of no arbitrage one needs to specify both the asset price and the pricing kernel.

Thus, in summary, in the case of a single risky asset (and under the assumption that no dividends are paid over the time horizon considered), the model is given by the price process (2.1) and the pricing kernel (2.2). These processes are defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with respect to which \(\{B_t\}_{t \geq 0}\) is a standard Brownian motion, and we can take the market filtration \(\mathcal{F}_t\) as being the standard augmented filtration generated by \(\{B_t\}\). The parameters of the model are \(S_0, r, \lambda\) and \(\sigma\). Once specified, the model can be used at time 0 to value and risk-manage certain other classes of asset. For example, if \(H_T\) represents a random cash flow (perhaps the payoff of an investment strategy) at time \(T\) determined by the trajectory of \(\{S_t\}_{0 \leq t \leq T}\) over the time interval \([0, T]\), then in the model constructed at time 0, the random value \(H_t\), at any time \(t \geq 0\), of the asset that delivers \(H_T\) at time \(T\) is represented by

\[
H_t = \frac{1}{\pi_1} \mathbb{I}(t < T) \mathbb{E}_t[\pi_T H_T],
\]

(2.4)

where \(\mathbb{I}(\cdot)\) denotes the indicator function and \(\mathbb{E}_t[\cdot]\) denotes conditional expectation with respect to \(\mathcal{F}_t\). For example, if \(H_T = \max(S_T - K, 0)\), then a calculation shows that \(H_t\) is given by the familiar Black–Scholes formula for the value at time \(t\) of a call option with strike \(K\) and maturity \(T\). Note that the pricing kernel methodology gives this result rather directly, without the involvement of hedging arguments, replication portfolios, market completeness, risk neutrality, change of measure or the solution of partial differential equations—all of which, important and useful as they are in various specific contexts, are ultimately irrelevant to the determination of the price of an option once the pricing kernel has been specified.

As another example, consider the optimal investment problem for an investor with utility \(U(x)\) and initial endowment \(H_0\), who wishes to invest in such a way as to maximize the expected utility of a contract that pays \(H_T\) at time \(T\). Assuming that \(U(x), x > 0\), is a standard utility function satisfying \(U'(x) > 0\) and \(U''(x) < 0\), and writing \(I(y), y > 0\), for the inverse marginal utility satisfying \(I(U'(x)) = x\) for all \(x > 0\), a variational argument shows that the optimal investment is in a contract that at time \(T\) pays \(H_T = I(\beta I_T)\), where the parameter \(\beta\) is the (unique) solution to the budget constraint \(H_0 = \mathbb{E}[\pi_T I(\beta I_T)]\). In the case of logarithmic utility \(U(x) = \ln x\), for instance, one finds that \(H_T = H_0/\pi_T\). Such results follow more or less directly from the pricing kernel methodology, without the need for consideration, for example, of the optimal portfolio strategy (if such exists) that will generate \(H_T\). In principle the investor simply pays \(H_0\) and buys the contract that delivers \(H_T\), and it is up to the seller whether they prefer (i) to accept the unhedged risk of delivering the contracted payment \(H_T\) at \(T\), or (ii) to hedge the risk by using \(H_0\) to construct a portfolio that is then managed in such a way as to produce the required \(H_T\) at \(T\).

One sees that derivative contracts and investment management contracts are much the same thing from the point of view of the investor, at least if we add the further provision that the derivatives should have non-negative payoffs. Perhaps the investment management paradigm has the moral advantage that at least in some sense the investor is clearly being sold a product that is optimal. Whether such well-defined optimization criteria enter into the actual decision-making processes involved in the marketing of investment opportunities and the targeting of clients is another matter—but clearly they should, to the extent that this is practically possible, if we may speak normatively, and the same goes for the marketing of investment-grade derivatives. A key point is that the optimal investment plan typically involves characteristics of the investor (as modelled, for example, with the specification of a utility function), together with the pricing kernel—but the microstructure of the market, as represented by the various stocks that are traded, and so forth, does not come into play.

Now suppose that the risk aversion factor (or excess rate of return per unit of risk) is not directly observable, and that there is uncertainty in the market as to its value. This state of affairs, as we have argued earlier, is in many ways representative of reality, and suggests a simple generalization of the GBM model. Let us therefore write \(X\) for the unknown value of the risk.
aversion factor, which we shall treat as a random variable. Then in our model for the typical asset price (assuming, for simplicity, that $X$ and $\{B_t\}$ are independent, that $X$ is positive, and that the other model parameters are constants), we have

$$S_t = S_0 e^{(r+\sigma X)t} e^{\sigma B_t - \frac{1}{2} \sigma^2 t}. \quad (2.5)$$

Thus if we introduce a so-called ‘information process’ $\{\xi_t\}_{t \geq 0}$ defined by

$$\xi_t = B_t + X_t, \quad (2.6)$$

we can write the price in the form

$$S_t = S_0 e^{rt} e^{\sigma \xi_t - \frac{1}{2} \sigma^2 t}. \quad (2.7)$$

Note that the asset price is monotonic in the information. It follows that the filtration generated by the asset price is the same as the filtration generated by the information process. Therefore, in our model it is rather natural to let this be the market filtration $\{\mathcal{F}_t\}$. Then the ‘true’ value of the market risk aversion factor $X$ remains hidden, and at best can only be estimated by observations of the asset price (or, equivalently, the information). This is a rather satisfactory way of viewing the market, since it conforms to intuition, and allows for an embodiment of the idea that past performance is not necessarily a reliable guide to future performance. The information process has the property that for large $t$ the value of $X$ is revealed. In particular, we have

$$\lim_{t \to \infty} \frac{1}{t} \xi_t = X. \quad (2.8)$$

This relation follows from the fact that Brownian motion grows in magnitude like the square root of $t$. Thus investors do not know in advance the excess rate of return on an asset, but in the long run this is revealed.

By use of the Bayes law, taking advantage of the fact that the random variable $\xi_t$ is conditionally Gaussian given $X$, one can work out the conditional distribution of the market factor $X$ given the relevant market information up to time $t$. The details of the calculation leading to this result are shown in appendix A. One finds that

$$p_t(dx) = \frac{\exp[x \xi_t - \frac{1}{2} x^2 t] p(dx)}{\int_0^\infty \exp[z \xi_t - \frac{1}{2} z^2 t] p(dz)}, \quad (2.9)$$

where the measure $p(dx)$ determines the unconditional distribution of $X$. The conditional distribution of the risk aversion factor is then given by

$$\mathbb{P}(X < x \mid \mathcal{F}_t) = \int_0^x p_t(dz), \quad (2.10)$$

with $p_t(dx)$ as in (2.9), and it follows in particular that its conditional mean is

$$\mathbb{E}[X \mid \mathcal{F}_t] = \int_0^\infty x p_t(dx). \quad (2.11)$$

The statement ‘$X$ is unknown’ can be interpreted in several ways. One is that there is a ‘secret’ value of $X$ which none of the market participants know but the asset somehow ‘knows’, and that over time this secret value of $X$ works its way through the dynamics of the asset price to contribute to the eventual return displayed by the asset. Many people like to think in this way, even if they do not actually believe the asset ‘knows’ anything. It is as though the market somehow ‘knows’. One sees this manner of thinking in the use of animistic language, in phrases like ‘the market is always right’, and also in the language of technical analysis. Another interpretation of ‘$X$ is unknown’ is that there is variability of opinion in the market about the rate of return that ought to be expected for a given level of risk, and that the distribution of $X$ represents this spread of opinion. Such variation might well be elemental, in the sense that each participant has their own private level of risk aversion, and that the distribution of $X$ reflects this. Indeed, it is a matter of human nature that equally intelligent and well-informed individuals can and will, by choice or disposition, exhibit markedly differing levels of risk tolerance and risk aversion. This is a practical
fact of life that one encounters constantly in day-to-day interactions with other people (or for that matter animals). We know from experience that even a single individual can, depending on mood and circumstance, exhibit significantly variable attitudes towards risk. It seems therefore both necessary and reasonable to suppose that an equilibrium can be established in a market where investors have widely differing attitudes towards risk, and that market prices are obtained by averaging in some sense over all these different attitudes.

Is it possible to reconcile the ‘X is secret’ point of view with the ‘X represents variation’ idea? From a modelling perspective, it would appear so. In particular, to calculate prices, one needs to form weighted averages over a large number of trajectories. One can imagine that each trajectory invoked in the averaging procedure involves some specific ‘secret’ value of $X$; or alternatively, one can think of averaging over the whole market, taking into account all of the various risk preferences: the result is the same.

3. Modelling the pricing kernel

With these thoughts in mind, we need to consider how to model the pricing kernel in a situation where heterogeneous attitudes towards risk prevail. One might be inclined simply to replace the parameter $\lambda$ in the GBM pricing kernel (2.2) with the random variable $X$ to give a tentative expression of the form

$$\pi_t \overset{?}{=} e^{-rt}e^{-X_t^2 t}$$

(3.1)

as a candidate for the pricing kernel. Unfortunately, this will not quite work, since once we introduce $\xi_t$ we obtain

$$\pi_t \overset{?}{=} e^{-rt}e^{-X_t^2 t}$$

(3.2)

which is clearly not $\mathcal{F}_t$-measurable, on account of the explicit appearance of $X$, and as a consequence the associated process $\{\pi_t\}$ is not adapted to the filtration $\{\mathcal{F}_t\}$ generated by $\{\xi_t\}$. However, if we take the conditional expectation of the expression above with respect to $\mathcal{F}_t$, this gives a better candidate for the pricing kernel, namely

$$\pi_t = E_t \left[ \exp \left( -rt - X_t^2 t \right) \right] = \int_0^\infty \exp \left( -rt - X_t^2 t + \frac{1}{2} X_t^2 t \right) p_t(dx),$$

(3.3)

which has the virtue of being $\mathcal{F}_t$-measurable. Then after insertion of expression (2.9) for $p_t(dx)$ we obtain

$$\pi_t = \frac{1}{\int_0^\infty \exp(rt + x\xi_t - \frac{1}{2} x^2 t) p(dx)}.$$  

(3.4)

This formula is perhaps most easily understood as follows. The process $\{n_t\}$ defined in terms of the pricing kernel by $n_t = 1/\pi_t$ is the so-called ‘natural numeraire’ or ‘benchmark process’ (see, for example, Flesaker & Hughston [22], Long [23]). The price process of any non-dividend-paying asset, when expressed in units of the natural numeraire, is a martingale in the market filtration. Thus in the present context we have

$$n_t = \int_0^\infty \exp \left( rt + x\xi_t - \frac{1}{2} x^2 t \right) p(dx).$$

(3.5)

We note that for each value of $x$ the integrand corresponds to an asset with unit initial price and of the form (2.7) with volatility $x$. Therefore, in the case of an unknown risk aversion factor, we form a weighted portfolio of the numeraire assets obtained for various specific values of the risk aversion, and then invert this to obtain the pricing kernel $\pi_t = 1/n_t$. This leads us to the following important conclusion: the pricing kernel associated with random risk aversion is given by the harmonic mean of the pricing kernels arising for various specific values of the risk aversion.

We have thus obtained a nice example of the use of information processes in shedding light on a problem of considerable interest in the investment community. Indeed, in the behavioural finance literature (see [24,25], and works cited therein), a good deal of evidence has been gathered to the effect that the correct way of amalgamating risk aversion in a heterogeneous market is
not by simply averaging the risk aversion parameter over the market, but rather by taking a suitable average (typically a Hölder mean) of the associated stochastic discount factors. Thus, if agent A says that the market price of risk should be \( x \), and agent B says with equal conviction that the market price of risk should be \( y \), then instead of naively averaging these numbers to obtain \( \frac{1}{2}(x + y) \) and inserting this figure into the stochastic discount factor to express the aggregate view, the behaviouralists propose first to work out the stochastic discount factors corresponding separately to the views of A and B, and then to take a suitable average. As we have seen above, our calculations support this general line of argument, and indeed we are able to go further by deducing from first principles a specific rule for the aggregation of risk aversion.

4. Information-based estimation of market risk aversion

The following question can be posed. Is it possible to formulate the price dynamics of the risky asset in such a way that the resulting representations for \( \{S_t\} \) and \( \{\pi_t\} \) are expressible entirely in the language of stochastic differential equations, without explicit reference to the ‘hidden’ risk aversion variable \( X \)? By doing so, we would have a model formulated, so to speak, in the spirit of ‘classical mathematical finance’. That is to say, the model would be proposed in the form of a closed system of dynamical equations satisfied by the asset and the pricing kernel, with appropriate initial conditions, and some parametric freedom. Only later one would discover, as it were, that the solution to this system of equations implies and admits the existence of the hidden variables \( X \) and \( \{B_t\} \).

It turns out that such a programme is feasible, and indeed is rather enlightening, since it allows one to put forward a version of the theory described in the previous sections without the introduction of ‘unobservable’ elements, and yet with exactly the same practical conclusions. Furthermore, at the same time we are able to ‘deduce’ the existence of a random variable \( X \) having the characteristics already discussed, along with the associated ‘true’ noise \( \{B_t\} \), thus allowing the theory to admit the interpretation we have given it.

We proceed as follows. As in the previous sections, we fix a probability space \((\Omega, \mathcal{F}, P)\) and introduce a Brownian motion \( \{B_t\} \) and an independent random variable \( X \). We introduce the information process \( \{\xi_t\} \) defined by (2.6), along with the filtration \( \{\mathcal{F}_t\} \) that it generates, and we define the asset price by (2.7) and the pricing kernel by (3.4). By virtue of the relation \( d\xi_t^2 = dt \), the dynamical equation satisfied by the asset price takes the form

\[
dS_t = rS_t \, dt + \sigma S_t \, d\xi_t.
\] (4.1)

Our goal is to write the dynamics in a way that brings out more explicitly the fact that the price movements are being driven by Brownian motion. The only difficulty is that the Brownian motion \( \{B_t\} \), in terms of which the information process \( \{\xi_t\} \) is defined, is not adapted to the market filtration; and thus we cannot quite say that the asset price is ‘driven’ by \( \{B_t\} \) in the usual sense.

To overcome this problem, we make use of an idea from filtering theory—the idea of a so-called innovations process. In particular, we define a process \( \{W_t\} \) by

\[
W_t = \xi_t - \int_0^t \mathbb{E}[X \mid \mathcal{F}_s] \, ds.
\] (4.2)

One can show, for example, by use of the Lévy criterion (see appendix B) that \( \{W_t\} \) is an \( \{\mathcal{F}_t\} \)-Brownian motion. Next we define a process \( \{\lambda_t\} \) by setting

\[
\lambda_t = \mathbb{E}[X \mid \mathcal{F}_t].
\] (4.3)

By virtue of the relations (2.11), (4.2) and (4.3), the information process \( \{\xi_t\} \) evidently satisfies a stochastic differential equation of the form

\[
d\xi_t = \lambda_t \, dt + dW_t,
\] (4.4)

where

\[
\lambda_t = \frac{\int_0^\infty x \exp(x\xi_t - \frac{1}{2}x^2t)p(dx)}{\int_0^\infty \exp(x\xi_t - \frac{1}{2}x^2t)p(dx)},
\] (4.5)
and it follows that the dynamical equation for the price can be put in the desired form

$$\text{d}S_t = (r + \lambda_t \sigma) S_t \text{d}t + \sigma S_t \text{d}W_t.$$  \hspace{1cm} (4.6)

We note that the resulting ‘effective’ market price of risk $\{\lambda_t\}$ is given by the conditional expectation (4.3), and hence can be interpreted as the best estimate, given the information available, of the ‘true’ value of the random variable $X$. A little reflection shows that we can drop the adjective ‘effective’ and simply assert that $\{\lambda_t\}$ is indeed the market price of risk (or market risk aversion level) in this model and that $X$ is the (unknown) actual excess rate of return per unit of risk. Market participants acknowledge that only ‘the gods’ know what the actual excess rate of return will turn out to be, or to have been, but that $\{\lambda_t\}$, which is knowable, represents the market consensus, the best estimate, the weighted opinion of market experts, the vote. The observable drift of an asset is thus determined not by the actual risk premium, but rather by the market best estimate for the risk premium. In particular, given the price $S_t$ of the asset at time $t$, one deduces by use of (2.7) and (4.5) that the best estimate of the market price of risk is given by the following expression:

$$\lambda_t = \frac{\int_0^\infty x(S_t/S_0)^{x/\sigma} \exp[-\frac{1}{2} x^2 + (\frac{1}{2} \sigma - r/\sigma) xt] p(dx)}{\int_0^\infty (S_t/S_0)^{x/\sigma} \exp[-\frac{1}{2} x^2 + (\frac{1}{2} \sigma - r/\sigma) xt] p(dx)}.$$  \hspace{1cm} (4.7)

This formula shows explicitly how the investor is able to update the a priori estimate for the market price of risk given the current price level of the risky asset.

It follows from (4.4) that the information process $\{\xi_t\}$ is a Brownian motion under the risk-neutral measure $\mathbb{Q}$. Furthermore, if we make use of the market price of risk to effect a change of measure, a calculation shows that (i) the random variables $\xi_t$ and $X$ are independent under $\mathbb{Q}$, and (ii) the probability law for $X$ under $\mathbb{Q}$ is given by $p(dx)$; that is to say, it is the same as it is under the physical measure $\mathbb{P}$. Therefore, an ‘observer’ in the risk-neutral frame of reference $(\Omega, \mathcal{F}, \mathbb{Q})$ detects the ‘message’ $\{\xi_t\}$, or equivalently the price $\{S_t\}$, but finds that it contains no information about the level of risk aversion—this is the sense in which the level of risk aversion cannot be inferred a priori from derivative prices in the context of Brownian-motion-based models. If stronger modelling assumptions are made about the structure of the pricing kernel in a Brownian model, then in some contexts it is possible to infer information about the risk aversion level from derivative prices [20, 26–28]. The approach that we are taking is, perhaps, more practically oriented, inasmuch as an explicit estimation formula for the risk premium, such as that given by (4.7), can be obtained in a direct and transparent manner without any reference to the risk-neutral measure.

The $\{\mathcal{F}_t\}$-dynamics of the pricing kernel can be pursued similarly. In fact, it is more convenient first to work out the dynamics of the natural numeraire. Starting with equation (3.5), by a direct application of Itô calculus we obtain

$$\text{d}n_t = r n_t \text{d}t + \int_0^\infty x \exp \left( rt + x \xi_t - \frac{1}{2} x^2 t \right) p(dx) \text{d}\xi_t.$$  \hspace{1cm} (4.8)

It follows then by use of (4.4) and (4.5) that

$$\text{d}n_t = (r + \lambda_t^2) n_t \text{d}t + \lambda_t n_t \text{d}W_t,$$  \hspace{1cm} (4.9)

and therefore that

$$\text{d}\pi_t = -r \pi_t \text{d}t - \lambda_t \pi_t \text{d}W_t.$$  \hspace{1cm} (4.10)

Hence we are led to the conclusion that the volatility of the pricing kernel is (minus) the market price of risk, as it of course should be in the $\{\mathcal{F}_t\}$-dynamics of the pricing kernel, and we can write

$$\pi_t = \exp \left( -rt - \int_0^t \lambda_s \text{d}W_s - \int_0^t \lambda_s^2 \text{d}s \right).$$  \hspace{1cm} (4.11)

Such an expression for the pricing kernel or state price density is often used as the starting point of various investigations in the theory of finance—but note that we have not assumed that $\{\pi_t\}$ takes this form, we have deduced it.
As a model constructed in the spirit of classical mathematical finance, without direct mention of the risk aversion variable \( X \), one thus has the following. We begin with a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), on which a standard Brownian motion \((W_t)\) is defined, and let \( \{\mathcal{F}_t\} \) be the associated filtration. The model inputs include the initial price \( S_0 \), the interest rate \( r \), the volatility \( \sigma \), and a measure \( p(dx) \) on \( \mathbb{R}^+ \). With this data at hand, one defines a smooth function of two variables \( \lambda: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+ \) given by \((\xi, t) \to \lambda(\xi, t)\), where

\[
\lambda(\xi, t) = \frac{\int_0^\infty x \exp(x\xi - \frac{1}{2}x^2t) p(dx)}{\int_0^\infty \exp(x\xi - \frac{1}{2}x^2t) p(dx)}.
\]  

(4.12)

One can check that for fixed \( t \) the function \( \lambda(\xi, t) \) is increasing in the variable \( \xi \). In particular, a calculation shows that \( \lambda'(\xi, t) > 0 \), where the prime denotes differentiation with respect to \( \xi \). The process \( \{\xi_t\} \) is then defined as the solution to the stochastic differential equation

\[
d\xi_t = \lambda(\xi_t, t) dt + dW_t,
\]

(4.13)

with the initial condition \( \xi_0 = 0 \). Having obtained \( \{\xi_t\} \), one defines the process \( \{\lambda_t\} \) by setting \( \lambda_t = \lambda(\xi_t, t) \). The SDE for the asset price is taken to be (4.6), with the initial condition \( S_0 \), and the SDE for the pricing kernel is taken to be given by (4.10), with the initial condition unity. That gives a complete characterization of the dynamics of the asset price and the pricing kernel.

Having constructed the model in the filtration \( \{\mathcal{F}_t\} \) without reference to the random variable \( X \), one might ask whether it is possible in some sense to reconstruct \( X \). It turns out that one can. In fact, we can derive expressions for the two ‘hidden’ objects \( X \) and \( \{B_t\} \) appearing in (2.6) from the ingredients arising in the \( \{\mathcal{F}_t\} \) version of the modelling framework. Specifically, let \( \{\xi_t\} \) satisfy the stochastic differential equation (4.13), with \( \xi_0 = 0 \). We can then show (i) that the random variables defined by

\[
X = \lim_{T \to \infty} T^{-1}\xi_T \quad \text{and} \quad B_t = \xi_t - X_t
\]

(4.14)

are independent for all \( t \), (ii) that the distribution of \( X \) is given by \( p(dx) \), and (iii) that the process \( \{B_t\} \) thus arising is a standard \( \mathbb{P} \)-Brownian motion. The details of the arguments involved in establishing these facts are summarized in appendix C.

5. Geometric Lévy models

Remarkably, the considerations that we have presented in connection with GBM models generalize very naturally to the context of geometric Lévy models (GLMs). As a consequence, we are able to construct a large and rich family of financial models for asset prices with jump risk in situations where the market exhibits variation among its participants in the excess rate of return required (above the interest rate) as compensation for the assumption of such risk. We assume familiarity with basics of the theory and application of Lévy processes in what follows, as discussed for example in Anderson & Lipton [29], Cont & Tankov [30], Kyprianou [31], Protter [32] or Schoutens [33]. Numerous investigations have been pursued concerning the development of Lévy-based models in finance, and as a consequence the literature is very extensive. We mention, for example, the works of Madan & Senata [34], Madan & Milne [35], Heston [36], Gerber & Shiu [37], Eberlin & Keller [38], Eberlin & Jacod [39], Madan et al. [40], Chan [41], Carr et al. [42], Kallsen & Shiryaev [43], Hubalek & Sgarra [44], Baxter [45], and Yor [46]. To set the notation, we begin with a few definitions. A Lévy process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is a process \( \{X_t\} \) such that \( X_0 = 0 \), \( X_t - X_s \) is independent of \( \mathcal{F}_s \) for \( t \geq s \) (independent increments), and

\[
\mathbb{P}(X_t - X_s \leq y) = \mathbb{P}(X_{t+h} - X_{s+h} \leq y)
\]

(5.1)

(stationary increments), where \( \{\mathcal{F}_t\} \) denotes the augmented filtration generated by \( \{X_t\} \). In order for \( \{X_t\} \) to give rise to a GLM, we require that it should satisfy

\[
\mathbb{E}[e^{\alpha X_t}] < \infty
\]

(5.2)
for all \( t \geq 0 \), for some connected real interval \( \alpha \in A \) containing the origin. We consider Lévy processes satisfying such a condition. It follows by the stationary and independent increments property that there exists a function \( \psi(\alpha) \), the Lévy exponent, such that

\[
\mathbb{E}[e^{\alpha X_t}] = e^{t \psi(\alpha)} \tag{5.3}
\]

for \( \alpha \in \{ w \in \mathbb{C} : \text{Re}(w) \in A \} \), and one can check that the process defined by

\[
M_t = e^{\alpha X_t - t \psi(\alpha)} \tag{5.4}
\]

is an \( \{F_t\} \)-martingale. We call \( \{M_t\} \) the geometric Lévy martingale (or Esscher martingale) associated with \( \{X_t\} \), with parameter \( \alpha \). For example, in the case of a standard Brownian motion the Lévy exponent is given by

\[
\psi(\alpha) = \frac{1}{2} \alpha^2, \tag{5.5}
\]

which is defined for all real \( \alpha \), and the associated geometric Lévy martingale is a compensated GBM with volatility \( \alpha \). A Lévy process is fully characterized by its exponent. As a consequence, it is useful to define and classify such processes by presentation of their exponents. For example, the Poisson process with rate \( m \) is defined by

\[
\psi(\alpha) = m(e^\alpha - 1), \tag{5.6}
\]

The gamma process with rate \( m \) and scale unity is given by

\[
\psi(\alpha) = -m \ln \left(1 - \frac{\alpha^2}{2m}\right). \tag{5.7}
\]

The variance-gamma (VG) process is given for \( \alpha^2 < 2m \) by

\[
\psi(\alpha) = -m \ln \left(1 - \frac{\alpha^2}{2m}\right). \tag{5.8}
\]

The theory of Esscher transformations plays an important role in the analysis of the relation between risk and return in Lévy models. We say that two Lévy exponents \( \psi(\alpha) \) and \( \tilde{\psi}(\alpha) \) are related by an Esscher transformation with parameter \( \lambda \) if

\[
\tilde{\psi}(\alpha) = \psi(\alpha + \lambda) - \psi(\lambda). \tag{5.9}
\]

The parameter \( \lambda \) must lie in the domain \( A \) of \( \psi(\alpha) \), and the domain of \( \tilde{\psi}(\alpha) \) consists of those values of \( \alpha \) such that \( \alpha + \lambda \in A \). The operation is reversible in the sense that if \( \tilde{\psi}(\alpha) \) is an Esscher transformation of \( \psi(\alpha) \) with parameter \( \lambda \), then \( \psi(\alpha) \) is an Esscher transformation of \( \tilde{\psi}(\alpha) \) with parameter \( -\lambda \). Thus, if \( \psi(\alpha) = \frac{1}{2} \alpha^2 \) represents a Brownian motion, then \( \tilde{\psi}(\alpha) = \frac{1}{2} \alpha^2 + \alpha \lambda \) represents a Brownian motion with drift \( \lambda \). If \( \psi(\alpha) = m(e^\alpha - 1) \) represents a Poisson process with rate \( m \), then \( \tilde{\psi}(\alpha) = me^\alpha(e^\alpha - 1) \) represents a Poisson process with rate \( me^\lambda \), and so on. Each example is rather different in character.

With these definitions at hand, we can present a development of the theory of GLMs for asset pricing [20] that is particularly well adapted to the analysis of jump risk aversion. The straightforward approach to GLMs is as follows. First we construct the pricing kernel \( \{\pi_t\}_{t \geq 0} \). Let \( \{X_t\} \) be a Lévy process with exponent \( \psi(\alpha) \) where \( \alpha \in A \). Let \( \lambda > 0, -\lambda \in A \), and set

\[
\pi_t = e^{-rt} e^{-\lambda X_t - t \psi(-\lambda)}. \tag{5.10}
\]

We require that the product of the pricing kernel and the asset price should be a martingale, which we assume is of the geometric Lévy form

\[
\pi_t S_t = S_0 e^{\beta X_t - t \psi(\beta)} \tag{5.11}
\]

for some \( \beta \in A \), and we deduce that

\[
S_t = S_0 e^{rt} e^{\sigma X_t + t \psi(-\lambda) - t \psi(\sigma - \lambda)}, \tag{5.12}
\]
where \( \sigma = \beta + \lambda \). We shall assume that \( \sigma > 0 \) and that \( \sigma \in A \). It follows that the price can be expressed by the formula

\[
S_t = S_0 e^{\int R(\lambda, \sigma) dt} e^{\sigma X_t - t\psi(\sigma)},
\]

where

\[
R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda).
\]

It is a remarkable fact that the excess rate of return function \( R(\lambda, \sigma) \) thus arising is positive and is increasing with respect to both of its arguments.

### 6. On the aggregation of jump-risk aversion

In our analysis of jump-risk aversion, it will be useful to cast the foregoing formulation of GLMs into a slightly different form which turns out to be well suited for our purpose. Let \( \{X_t\} \) and \( \psi(\alpha) \) be defined as above, and set

\[
\phi(\alpha) = \psi(\alpha - \lambda) - \psi(-\lambda).
\]

Clearly, we have

\[
\psi(\alpha) = \phi(\alpha + \lambda) - \phi(\lambda),
\]

and we observe that \( \psi(\alpha) \) is given by an Esscher transform of \( \phi(\alpha) \), with parameter \( \lambda \). An exercise then shows that the asset price \( (5.12) \) can be expressed in the form

\[
S_t = S_0 e^{\int \sigma X_t - t\phi(\sigma)}.
\]

Thus, we see that given a ‘fiducial’ Lévy exponent \( \phi(\alpha) \), if we let the exponent of the Lévy process \( \{X_t\} \) be an Esscher transform of \( \phi(\alpha) \), with parameter \( \lambda \), then the process \( S_t \) defined as above by \( (6.3) \) will be a submartingale, which we can take to be the asset price process. The associated pricing kernel is then a supermartingale that takes the form

\[
\pi_t = e^{-\lambda X_t + t\phi(\lambda)}.
\]

The advantage of this representation of the GLM (which is entirely equivalent to that of the previous section) is that it readily generalizes to the situation where the risk aversion parameter is uncertain, thus allowing one to generalize the scheme set out in §§1–4 to models with jumps.

To proceed further, it will be expedient to make use of the general filtering theory associated with Lévy noise developed in Brody et al. [47]. We recall that, when phrased in the language of signal processing, what amounts to the ‘signal’ in the present investigation is the unknown level of risk aversion. In the Brownian context, it is natural for the signal to be obscured by an additive noise. However, in the case of general Lévy noise with jumps, the signal is no longer obscured by noise in an additive manner. In fact, each different type of Lévy process, when viewed as a model for noise, ‘carries’ the signal in its own distinctive manner.

Rather than developing the theory case by case, here we propose to present the theory in such a way that is applicable to the whole category of Lévy models. For this purpose, we introduce the important notion of Lévy information. By a Lévy information process \( \{\xi_t\} \) with signal \( X \) on a probability space \((\Omega, \mathcal{F}, P)\), we mean a process such that conditional on the sigma field \( \mathcal{F}^X \) generated by \( X \), \( \{\xi_t\} \) is a Lévy process with exponent

\[
\psi_X(\alpha) = \phi(\alpha + X) - \phi(X)
\]

for \( \alpha \in \{w \in \mathbb{C} : \text{Re}(w) = 0\} \). That is to say, we have the relation

\[
\mathbb{E}[e^{a\xi_t} | \mathcal{F}^X] = e^{\psi_X(a)}
\]

for imaginary values of \( a \). In terms of the Lévy information process \( \{\xi_t\} \), the asset price can be expressed in the form

\[
S_t = S_0 e^{\int \sigma \xi_t - t\phi(\sigma)},
\]

which can be thought of as the general Lévy-based analogue of the price process \( (2.7) \). One way of looking at the formula above is to view \( \{\xi_t\} \) as the driving Lévy process in the risk-neutral measure.
where $\xi_t$ is given by the formula

$$\pi_t = e^{-rt}e^{-X_{\xi_t}+t\phi(X)}.$$  \hfill (6.8)

This expression suffers from the fact that it is not measurable with respect to the sigma field $F_t$ generated by the trajectory of the information process up to time $t$. It turns out, fortunately, and perhaps surprisingly, that the approach taken in the case of the Brownian example carries through to the general Lévy context, and for the pricing kernel we have

$$\pi_t = \mathbb{E}_t[\exp(-rt - X_{\xi_t} + \phi(X)t)] = \int_0^\infty \exp(-rt - x\xi_t + \phi(x)t) p_t(dx),$$  \hfill (6.9)

where

$$p_t(dx) = \frac{\exp(x\xi_t - \phi(x)t)p(dx)}{\int_0^\infty \exp(z\xi_t - \phi(z)t)p(dz)}$$  \hfill (6.10)

is the $(F_t)$-conditional measure for the distribution of $X$, and $p(dx)$ is the unconditional measure. Substituting (6.10) in (6.9), we thus deduce that the pricing kernel in a GLM with random risk aversion is given by the following formula:

$$\pi_t = \frac{1}{\int_0^\infty \exp(rt + x\xi_t - \phi(x)t)p(dx)}.$$  \hfill (6.11)

Similarly, for the natural numeraire in the case of a Lévy model with random risk aversion we can write

$$n_t = \int_0^\infty \exp(rt + x\xi_t - \phi(x)t)p(dx).$$  \hfill (6.12)

Again, as in the Brownian situation, one observes in the Lévy case the key point that the market numeraire asset can be viewed as a portfolio, the elements of which correspond, with appropriate weights, to the numeraire assets of the various investors, each with a volatility given by the risk aversion factor associated with the particular investor.

With the conditional density (6.10) at hand, we are able to determine the optimal estimate (in the sense of least quadratic error) for the level of jump-risk aversion. This is given by

$$\lambda_t = \frac{\int_0^\infty x \exp(x\xi_t - \phi(x)t)p(dx)}{\int_0^\infty \exp(x\xi_t - \phi(x)t)p(dx)}.$$  \hfill (6.13)

Since the asset price (6.7) is a simple invertible function of $\xi_t$ we are thus in a position to obtain an explicit formula for the jump-risk aversion factor $\lambda_t$ in terms of the price level $S_t$ in the general setting of a geometric Lévy model. Once $\lambda_t$ has been determined, the excess rate of return associated with jump risk is given by $R(\lambda_t, \sigma)$. It is perhaps remarkable that the analysis presented in the case of the geometric Brownian model extends so straightforwardly to the case of the general geometric Lévy model, even though the powerful tools of the traditional Ito calculus are not directly applicable in the general Lévy context. This can be viewed as a vindication of the usefulness of pricing kernel methods. Indeed, to grasp the relation between risk, risk aversion, and return, the pricing kernel is an indispensable tool, and this is especially clear when prices can jump, as is in any event typically the case in real financial markets. In particular, since optimal investment strategies depend solely for their specification on the pricing kernel and the risk profile of the investor (as given, for example, by an appropriate utility function), we are led by this reasoning to be able to present a clear account of such strategies in the case of markets with price jumps.

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Appendix A. Conditional distribution of the risk aversion factor

In this appendix, we present the details of the calculation leading to the general expression given by equation (2.9) for the conditional distribution of the risk aversion factor in the case of a filtration generated by a Brownian information process.

We consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which a Brownian motion \(\{B_t\}_{t \geq 0}\) is defined along with an independent square-integrable random variable \(X\), and we assume that \(X > 0\) almost surely. The filtration \(\{\mathcal{F}_t\}\) is taken to be generated by the information process \(\{\xi_t\}_{t \geq 0}\) defined by \(\xi_t = B_t + Xt\). The conditional distribution of \(X\) can be worked out as follows.

First, we note (i) that \(\{\xi_t\}\) is a Markov process and (ii) that \(X\) is \(\mathcal{F}_\infty\)-measurable. To establish the Markov property, we note the fact that in the case of Brownian motion the random variables \(B_t\) and \(B_s / B_{s_1} / s_1\) are independent for \(t > s > s_1 > 0\) by virtue of the theory of the Brownian bridge. More generally, if \(s > s_1 > s_2 > s_3 > 0\), we find that \(B_s / B_{s_1} / s_1\) and \(B_{s_2} / B_{s_3} / s_2 - B_{s_3} / s_3\) are independent. We observe that for any \(k \geq 1\) we have

\[
\mathbb{P}(\xi_t \leq x \mid \xi_s, \xi_{s_1}, \ldots, \xi_{s_k}) = \mathbb{P}\left(\xi_t \leq x \mid \frac{\xi_s}{s}, \frac{\xi_{s_1}}{s_1}, \ldots, \frac{\xi_{s_{k-1}}}{s_{k-1}} - \frac{\xi_{s_k}}{s_k}\right)
\]

\[
= \mathbb{P}\left(\xi_t \leq x \mid \frac{\xi_s}{s}, \frac{B_{s_1}}{s_1}, \ldots, \frac{B_{s_{k-1}}}{s_{k-1}} - \frac{B_{s_k}}{s_k}\right).
\]

(A 1)

Since \(\xi_t\) and \(\xi_s\) are independent of \(B_s / B_{s_1} / s_1, \ldots, B_{s_{k-1}} / s_{k-1} - B_{s_k} / s_k\), it follows that

\[
\mathbb{P}(\xi_t \leq x \mid \xi_s, \xi_{s_1}, \ldots, \xi_{s_k}) = \mathbb{P}(\xi_t \leq x \mid \xi_s),
\]

(A 2)

and that gives us the Markov property. As regards the \(\mathcal{F}_\infty\)-measurability, this follows from the fact that \(\lim_{t \to \infty} t^{-1} \xi_t = X\). In the calculation of the conditional distribution of \(X\) given \(\mathcal{F}_t\) it thus suffices to determine the conditional distribution of \(X\) given \(\xi_t\). By virtue of the relevant version of the Bayes formula, we have

\[
p_t(dx) = \frac{\rho(\xi_t \mid X = x)p(dx)}{\int_0^\infty \rho(\xi_t \mid X = x)p(dx)},
\]

(A 3)

where \(p(dx) = \mathbb{P}(X \in dx)\) is the a priori distribution of \(X\), assumed known, and where for each \(x\) the function \(\rho(\xi \mid X = x), \xi \in \mathbb{R}\), is the conditional density for the random variable \(\xi_t\) given that \(X = x\), which in (A 3) is then valued at \(\xi = \xi_t(\omega)\) for each outcome of chance \(\omega \in \Omega\). Since \(B_t\) is a Gaussian random variable with mean 0 and variance \(t\), we deduce that

\[
\rho(\xi \mid X = x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(\xi - tx)^2}{2t}\right).
\]

(A 4)

Inserting this expression into the Bayes formula (A 3), one is then immediately led to (2.9).

Appendix B. Emergence of the Brownian driver

In the conventional modelling framework (and in the absence of jumps apart from those associated directly with dividend payments) it is usually assumed that the market filtration is generated by a Brownian motion of one or more dimensions, and that the associated asset prices are adapted to this filtration. Although well-established and mathematically sound, from a financial perspective this view of the market is unsatisfactory in various respects. One gets a hint of the nature of the problem when on the one hand (a) the Brownian motion \(\{W_t\}\) driving the asset is referred to as ‘noise’, and on the other hand (b) the sigma field \(\mathcal{F}_t = \sigma\{W_s\}_{0 \leq s \leq t}\) is referred to as ‘information’. If one presses a finance theorist on this point, the reply will be a shrug of the shoulders and piece of sophistry of the form, ‘Well, it is true that \(\{W_t\}\) is noise, but \(\mathcal{F}_t\) represents the knowledge of the history of the trajectory of that noise, and therefore carries valuable information’. This point of view, ridiculous as it may seem, permeates the whole subject, and is consequently a source of confusion.
Fortunately, there is a resolution of this seemingly paradoxical issue. Prices are driven by the flow of information, and information is usually communicated along noisy channels. The market accepts this as the normal state of affairs, and prices are based on the best estimates of the relevant factors, given the information available, imperfect as it may be. This is perhaps what Norbert Wiener was getting at when he said that ‘Economics is a science of communication’ [48].

In this appendix, we present details of the calculations in §4 leading to the emergence of the Brownian driver. Starting with the relation \( \xi_t = B_t + X_t \), we define the process \( \{W_t\} \) as in (4.2). To prove that \( \{W_t\} \) is an \( \{F_t\} \)-Brownian motion, it suffices by use of the so-called Lévy criterion to show that \( \{W_t\} \) is an \( \{F_t\} \)-martingale and that \( (dW_t)^2 = dt \). First, we shall demonstrate that \( \{W_t\} \) is an \( \{F_t\} \)-martingale. Letting \( t \leq T \), we deduce that

\[
\mathbb{E}[W_T | F_t] = \mathbb{E}[B_T | F_t] + T \mathbb{E}[X | F_t] - \mathbb{E} \left[ \int_0^T \lambda_s ds \right] F_t \\
= \mathbb{E}[B_T | F_t] + T \mathbb{E}[X | F_t] - \int_0^T \mathbb{E}[\lambda_s | F_t] ds,
\]

by use of Fubini’s theorem. Next, we note that

\[
\int_0^T \mathbb{E}[\lambda_s | F_t] ds = \int_0^t \mathbb{E}[\lambda_s | F_t] ds + \int_t^T \mathbb{E}[\lambda_s | F_t] ds = \int_0^t \lambda_s ds + (T - t) \lambda_t. \tag{B1}
\]

Here, we have used the fact that the process \( \{\lambda_t\} \) is by construction an \( \{F_t\} \)-martingale. Substituting (B2) in (B1), we obtain

\[
\mathbb{E}[W_T | F_t] = \mathbb{E}[B_T | F_t] + t \mathbb{E}[X | F_t] - \int_0^t \lambda_s ds. \tag{B3}
\]

Finally, we observe that by the tower property of conditional expectation we have

\[
\mathbb{E}[B_T | F_t] = \mathbb{E}[\mathbb{E}[B_T | F_t^B] , X] | F_t] = \mathbb{E}[B_t | F_t], \tag{B4}
\]

where \( \{F_t^B\} \) denotes the filtration generated by \( \{B_t\} \). Inserting this in (B3), we obtain

\[
\mathbb{E}[W_T | F_t] = \mathbb{E}[B_t | F_t] + t \mathbb{E}[X | F_t] - \int_0^t \lambda_s ds \\
= \mathbb{E}[(B_t + tX) | F_t] - \int_0^t \lambda_s ds \\
= \mathbb{E}[\xi_t | F_t] - \int_0^t \lambda_s ds = W_t, \tag{B5}
\]

and this establishes that \( \{W_t\} \) is an \( \{F_t\} \)-martingale. Next, we observe that since

\[
(dW_t)^2 = (X - \lambda_t)dt + dB_t, \tag{B6}
\]

it follows at once that \( (dW_t)^2 = dt \). Taking this result together with the fact that \( \{W_t\} \) is an \( \{F_t\} \)-martingale, we conclude that \( \{W_t\} \) is an \( \{F_t\} \)-Brownian motion.

**Appendix C. Existence and construction of the hidden variables \( X \) and \( B_t \)**

In this appendix, we present details of the arguments allowing one to establish properties (i), (ii) and (iii) of the constructed versions of the random risk aversion variable \( X \) and the associated ‘pure noise’ process \( \{B_t\} \) stated at the end of §4, starting from the formulation of the theory in which all quantities under consideration at the outset are ‘financial observables’, that is to say, suitably adapted to the market filtration. Let us begin by establishing property (i), the
independence of the random variables $B_t$ and $X$. To this end, it suffices to check that the relation

$$E[e^{aB_t + bX}] = E[e^{aB_t}] E[e^{bX}]$$

(C1)

holds for all $a, b \in \mathbb{C}^1 := \{ w \in \mathbb{C} : \text{Re } w = 0 \}$. Verification that the joint characteristic function factorizes proceeds as follows. By the definitions of $B_t$ and $X$ given at (4.14), we have

$$E[e^{aB_t + bX}] = \lim_{T \to \infty} E[e^{a(\xi_t - t^{-1}\xi_T) + bT^{-1}\xi_T}].$$

(C2)

We shall calculate the expectation in (C2) and show that it factorizes for all $T$. In this connection, it will be useful to construct a solution to the stochastic differential equation (4.13). We begin with a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ on which we introduce a standard Brownian motion $\{\xi_t\}$. Given the measure $p(dx)$, which we assume to admit a second moment, one can check that the function $\Phi(\xi, t)$ defined for $\xi, t \geq 0$ by

$$\Phi(\xi, t) = \int_0^\infty \exp \left( x\xi - \frac{1}{2} x^2 t \right) p(dx)$$

(C3)

is of class $C^2$ in $\xi$ and class $C^1$ in $t$, and has the space–time harmonic property (Yor [49]),

$$\frac{\partial \Phi}{\partial t} = \frac{1}{2} \frac{\partial^2 \Phi}{\partial \xi^2}.$$ 

(C4)

As a consequence, we are able to introduce a process $\{\Phi_t\}$ defined by

$$\Phi_t = \Phi(\xi_t, t) = \int_0^\infty \exp \left( x\xi_t - \frac{1}{2} x^2 t \right) p(dx),$$

(C5)

and it is straightforward to verify that $\{\Phi_t\}$ is a martingale under $\mathbb{Q}$ with respect to the filtration $\{\mathcal{F}_t\}$ generated by $\{\xi_t\}$. Applying Ito’s lemma, and defining the process $\{\lambda_t\}$ as before by $\lambda_t = \lambda(\xi_t, t)$, where the function $\lambda(\xi, t)$ is given by (4.12), one deduces that $d\Phi_t = \lambda_t \Phi_t d\xi_t$, and hence by integration we obtain

$$\int_0^\infty \exp \left( x\xi_t - \frac{1}{2} x^2 t \right) p(dx) = \exp \left( \int_0^t \lambda_s d\xi_s - \frac{1}{2} \int_0^t \lambda_s^2 ds \right),$$

(C6)

which expresses $\{\Phi_t\}$ in the form of an exponential martingale. Since $\{\xi_t\}$ is a $\mathbb{Q}$-Brownian motion, one sees by the use of Girsanov’s theorem that the process $\{W_t\}$ defined by

$$W_t = \xi_t - \int_0^t \lambda_s ds$$

(C7)

is a Brownian motion under the measure $\mathbb{P}$ defined by

$$\frac{d\mathbb{P}}{d\mathbb{Q}}_{|\mathcal{F}_t} = \Phi_t,$$

(C8)

and one concludes from (C7) that $\{\xi_t\}$ satisfies the stochastic differential equation (4.13). We see moreover that $\mathbb{Q}$ is the risk-neutral measure. The conditional expectations in the probability measures $\mathbb{P}$ and $\mathbb{Q}$ are related for $0 \leq t \leq T$ by the scheme

$$E^\mathbb{P}_t [Y_T] = \frac{1}{\Phi_t} E^\mathbb{Q}_t [\Phi_T Y_T] \quad \text{and} \quad E^\mathbb{Q}_t [Y_T] = \Phi_t E^\mathbb{P}_t \left[ \frac{1}{\Phi_T} Y_T \right]$$

(C9)

for any $\mathcal{F}_T$-measurable random variable $Y_T$. 


Equipped with these results, we proceed to work out the expectation (under $\mathbb{P}$) appearing in (C 2). In particular, we need the relation

$$E^\mathbb{P}[Y_T] = E^\mathbb{Q}[\Phi_T Y_T].$$

We thus observe that

$$E[\exp(a(\xi_t-t^{-1}\xi_T)+b T^{-1}\xi_T)] = E^\mathbb{Q}\left[\left(\int_0^\infty \exp\left(-\frac{x^2}{2} T\right) \exp(a(\xi_t-t^{-1}\xi_T)+b T^{-1}\xi_T)\right)\right]$$

$$= \int_0^\infty E^\mathbb{Q}\left[\exp\left(-\frac{x^2}{2} T\right) \exp(a(\xi_t-t^{-1}\xi_T)+b T^{-1}\xi_T)\right] p(dx).$$

But since $\{\xi_t\}$ is a Brownian motion under $\mathbb{Q}$, the inner expectation can be worked out by the use of standard techniques from the theory of Brownian motion. The result is

$$E^\mathbb{Q}\left[\exp\left(-\frac{x^2}{2} T\right) \exp(a(\xi_t-t^{-1}\xi_T)+b T^{-1}\xi_T)\right] = \exp\left(\frac{1}{2} a^2 T(T-t)^{-1} + bx + \frac{1}{2} b^2 T^{-1}\right),$$

from which it follows that

$$E[\exp(a(\xi_t-t^{-1}\xi_T)+b T^{-1}\xi_T)] = \exp\left(\frac{1}{2} a^2 T(T-t)^{-1}\right) \left(\int_0^\infty \exp\left(bx + \frac{1}{2} b^2 T^{-1}\right) p(dx)\right),$$

which exhibits the claimed factorization of the characteristic function for all $T$. In particular, for large $T$ we obtain

$$E[\exp(B_t+bX)] = \exp\left(\frac{1}{2} a^2 T\right) \left(\int_0^\infty \exp\left(bx p(dx)\right)\right),$$

which establishes (C 1), showing that random variables $X$ and $B_t$ defined by (4.14) are independent for all $t$, which is property (i). It follows further that the distribution of $X$ is given by $p(dx)$, which is property (ii), and that $B_t$ is normally distributed with mean zero and variance $t$, which gives us part of property (iii). To complete the proof of property (iii), that $\{B_t\}$ is a Brownian motion, we must verify that $\{B_t\}$ has independent increments. It will suffice to demonstrate that

$$E[\exp(B_t+b(B_u-B_t))] = E[\exp(B_t)] E[\exp(b(B_u-B_t))]$$

holds for $0 \leq t \leq u$ and $a, b \in \mathbb{C}$. Using the definition of $\{B_t\}$, we can write

$$E[\exp(B_t+b(B_u-B_t))] = E^\mathbb{Q}[\Phi_T \exp(a(\xi_t-t^{-1}\xi_T)+b(\xi_u-u\xi_T)-(\xi_t-t\xi_T))],$$

where $\Phi_T = \Phi(\xi_T, T)$, and it follows from the definition of $X$ that

$$E[\exp(B_t+b(B_u-B_t))] = \lim_{T \to \infty} E^\mathbb{Q}[\exp(a(\xi_t-t^{-1}\xi_T)+b(\xi_u-uT^{-1}\xi_T)-(\xi_t-t^{-1}\xi_T)X)].$$

In obtaining (C 17), we have used the theory of the Brownian bridge to deduce that $\xi_T$ (and hence $\Phi_T$) is independent of $\xi_t - t^{-1}\xi_T$ and $\xi_u - uT^{-1}\xi_T$ under $\mathbb{Q}$, and we have used the fact that $E^\mathbb{Q}[\Phi_T] = 1$. One is then left with a calculation involving the expectation of an exponentiated sum of Gaussian random variables, which can be simplified by use of the theory of the Brownian bridge, and for large $T$ we obtain the desired result:

$$E[\exp(B_t+b(B_u-B_t))] = \exp\left(\frac{1}{2} a^2 T + b^2 (u-t)\right).$$

The same line of argument applies for any number of increments. Thus, we conclude that $\{B_t\}$ is normally distributed with zero mean and variance $t$, and has independent increments. Therefore, $\{B_t\}$ is a standard Brownian motion under $\mathbb{P}$.

References