We present natural discrete analogues of two integrable classes of shell membranes. By construction, these discrete shell membranes are in equilibrium with respect to suitably chosen internal stresses and external forces. The integrability of the underlying equilibrium equations is proved by relating the geometry of the discrete shell membranes to discrete \( \Omega \) surface theory. We establish connections with generalized barycentric coordinates and nine-point centres and identify a discrete version of the classical Gauss equation of surface theory.

1. Introduction

Thin shell theory has its origin in the nineteenth century and, by now, constitutes a well-established branch of structural mechanics [1]. However, the rich integrable structure residing in the equilibrium equations of thin shell theory in the membrane limit has been uncovered only recently. Thus, in [2], it has been established that the governing equations of shell membrane theory subjected to constant normal loading are amenable to the powerful techniques of soliton theory [3] if the lines of principal stress and curvature coincide. Moreover, the ‘complementary’ case of pure shear stress without external forces has likewise been shown to be integrable [4]. In both cases, the determination of these classes of shell membranes may be formulated as a purely geometric problem. In fact, the (mid-surfaces of the) shell membranes may be located within a large class of the so-called \( \Omega \) surfaces [5] which are integrable and canonically parametrized in terms of curvature coordinates.

Remarkably, a discrete analogue of \( \Omega \) surface theory is readily obtained [6] on use of a discretization of lines of curvature which is standard in discrete differential
geometry [7] and preserves integrability. Accordingly, from a geometric point of view, it should be feasible to construct discrete models (‘plated’ membranes) of the above-mentioned integrable classes of shell membranes, and indeed we here demonstrate that, in the case of pure shear stress, the standard discretization of isothermic surfaces [8] is physically meaningful in the sense that one may regard these discrete shell membranes as being in equilibrium with respect to canonically defined internal shear stresses. However, in the case of the discrete O surfaces which discretize the shell membranes with vanishing shear stress along the lines of curvature, a meaningful physical interpretation is not available. Hence, it is the aim of this paper to demonstrate that, in general, it is necessary to take into account simultaneously the physical and geometric properties in order to obtain discrete shell membranes which are in equilibrium, that is, which are such that the conditions guaranteeing the balance of forces and moments may be satisfied. This is in agreement with the analysis presented in Rogers & Schief [2] in the case of vanishing external force.

In the main part of this paper, we use first principles to obtain a well-defined system of difference equations governing discrete shell membranes in equilibrium which sustain shear-free stresses and constant normal loading. Moreover, if the points on the quadrilateral plates at which the external force acts are chosen carefully then integrability is inherited from the classical case. Hence, there exists a privileged way of discretizing not only the geometry of this class of shell membranes but also the internal and external forces acting on them. In this connection, it is noted that ‘integrable’ quadrilateral meshes have been used in the architectural design of freeform structures [9] and finite-element modelling of plates and shells based on ‘discrete Kirchhoff techniques’ [10] has been a subject of extensive research.

It turns out that each of the above-mentioned points of application may be identified as the canonical analogue for a quadrilateral of the classical nine-point centre of a triangle proposed recently in Myakishev [11]. In order to prove this remarkable fact, we use particular generalized barycentric coordinates [12] which lead to a canonical area-weighted decomposition of the external force. Furthermore, a ‘non-standard’ connection with discrete O surface theory is established. This confirms, in particular, the integrability in the absence of external forces, which has been proved directly in [13] by means of the ‘multi-dimensional consistency’ approach [7]. In the context of special classes of discrete shell membranes which adopt the shape of discrete surfaces of constant Gaussian or mean curvature or, more generally, discrete linear Weingarten surfaces, we retrieve the classical theory in the continuum limit and identify a discrete version of the important Gauss equation of classical differential geometry.

2. The equilibrium equations of shell membrane theory

We are concerned with the equilibrium equations of classical thin shell theory as first set down by Love [14]. These are obtained by replacing the three-dimensional stress tensor $\sigma_{ik}$ defined throughout the shell by statically equivalent forces and moments acting on the mid-surface $\Sigma$ of the shell and evaluating the conditions of vanishing total force and moment. In modern tensor notation, that is, in terms of tensors $T_{ab}, N^a$ and $M_{ab}$, which encode the forces and moments respectively, the equilibrium equations adopt the compact form [15]

$$
\begin{align*}
T_{ab}^{\alpha} &= h_{\alpha} N^\alpha, & N_{\alpha}^\alpha + h_{\alpha \beta} T_{ab}^{\beta \alpha} + \tilde{p} = 0, \\
M_{ab}^{\alpha} &= N_\alpha, & T_{(ab)} = h_{[\alpha} M_{\beta]}^\alpha,
\end{align*}
$$

(2.1)

where $\tilde{p}$ constitutes the component of a purely normal external force per unit area which is assumed to be constant and the first and second fundamental forms of $\Sigma$ are given by

$$
I = g_{ab} \, dx^a \, dx^b \quad \text{and} \quad II = h_{ab} \, dx^a \, dx^b.
$$

(2.2)

By definition, shell membrane theory is represented by vanishing moments $M_{ab}$. In this case, the equilibrium equations (2.1) imply that the normal components $N^a$ of the internal forces vanish and that the ‘tangential stress tensor’ $T_{ab}$ is symmetric.
(a) Curvature coordinates

Love’s form of the equilibrium equations may be retrieved by choosing curvature coordinates \((x, y)\) on the mid-surface \(\Sigma\) with position vector \(r\) so that the fundamental forms become purely diagonal, that is

\[
\begin{align*}
I &= dr^2 = H^2 dx^2 + K^2 dy^2 \\
\text{and} \\
II &= -dr \cdot dN = \kappa_1 H^2 dx^2 + \kappa_2 K^2 dy^2,
\end{align*}
\]

(2.3)

where \(\kappa_1\) and \(\kappa_2\) designate the principal curvatures of \(\Sigma\) and \(N\) is the unit normal. Thus, if we introduce the parametrization

\[
\begin{align*}
T_1 &= T_1^1, & T_1^2 &= H/K T_1^2, & N_1 &= H N^1 \\
T_2 &= T_2^2, & T_2^1 &= K/H T_2^1, & N_2 &= K N^2
\end{align*}
\]

(2.4)

then, in the case of shell membranes, the moment balance equations (2.1) reduce to

\[
N_1 = N_2 = 0 \text{ and } T_{12} = T_{21} = S
\]

(2.5)

as indicated above, while the remaining force balance equations (2.1) become

\[
\begin{align*}
(K T_1)_x + (HS)_y + H_y S - K_x T_2 &= 0, \\
(H T_2)_y + (KS)_x + K_x S - H_y T_1 &= 0 \\
\kappa_1 T_1 + \kappa_2 T_2 + \bar{p} &= 0.
\end{align*}
\]

(2.6)

The equilibrium equations are coupled to the Gauss–Mainardi–Codazzi equations [16]

\[
\begin{align*}
\kappa_1 y + (\ln H)_y (\kappa_1 - \kappa_2) &= 0, \\
\kappa_2 x + (\ln K)_x (\kappa_2 - \kappa_1) &= 0 \\
\left(\frac{K_x}{H}\right)_x + \left(\frac{H_y}{K}\right)_y + HK\kappa_1\kappa_2 &= 0,
\end{align*}
\]

(2.7)

which guarantee that the quadratic forms (2.3) may indeed be associated with a surface (membrane) \(\Sigma\). It is the aim of this paper to demonstrate that the physical notion of balance of forces and moments gives rise to a (geometric) discretization of canonical classes of membranes which preserves the integrable structure residing in the underlying classical system (2.6) and (2.7).

(b) Integrable structure

In order to establish the connection with integrable systems, it is necessary to recall how the surfaces \(\Sigma\) are reconstructed from the solutions of the Gauss–Mainardi–Codazzi equations (2.7). To this end, it is convenient to reformulate the latter as the first-order system

\[
H_{xy} = p K_x, \quad K_{xy} = q H_x \quad \text{and} \quad q_x + p_y + H_x K_y = 0
\]

(2.8)

together with

\[
H_y = p K \quad \text{and} \quad K_x = q H,
\]

(2.9)
where \( H_\circ = -\kappa_1 H \) and \( K_\circ = -\kappa_2 K \). The orthonormal frame \((X, Y, N)\) of \( \Sigma \) is then uniquely determined (up to rotations) by the linear system

\[
\begin{pmatrix}
X \\
Y \\
N
\end{pmatrix}_x = \begin{pmatrix}
0 & -p & -H_\circ \\
p & 0 & 0 \\
H_\circ & 0 & 0
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
N
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
X \\
Y \\
N
\end{pmatrix}_y = \begin{pmatrix}
0 & q & 0 \\
-q & 0 & -K_\circ \\
0 & K_\circ & 0
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
N
\end{pmatrix}.
\]

(2.10)

It is noted that the above system of Gauss–Weingarten type is compatible modulo (2.8) and does not depend on the metric coefficients \( H \) and \( K \). It therefore determines \( \Sigma \) up to Combescure transformations [17] which, by definition, preserve the coordinate tangent vectors. Thus, for any given \( p \) and \( q \) obeying the Gauss–Mainardi–Codazzi equations (2.8), any solution of the linear system (2.9) gives rise to the position vector \( r \) of a unique surface \( \Sigma \) via the compatible system

\[
r_x = HX \quad \text{and} \quad r_y = KY.
\]

(2.11)

A privileged Combescure transform \( \Sigma_\circ \) within the class of Combescure-related surfaces \( \Sigma \) is obtained by choosing \((H_\circ, K_\circ)\) as a solution of the linear system (2.9) which then coincides with (2.8). Indeed, this choice gives rise to the spherical representation \( N \) of \( \Sigma \) via (2.10), that is,

\[
N_x = H_\circ X \quad \text{and} \quad N_y = K_\circ Y.
\]

(2.12)

(i) Vanishing shear stress

In the case of vanishing shear stress \( S \), that is, if the lines of principal stress and curvature coincide, then the equilibrium equations (2.6) reduce to

\[
T_{1x} + (\ln K)_x(T_1 - T_2) = 0 \quad \text{and} \quad T_{2y} + (\ln H)_y(T_2 - T_1) = 0
\]

and coincide algebraically with the Mainardi–Codazzi equations (2.7) via the correspondence \((\kappa_1, \kappa_2) \leftrightarrow (T_2, T_1)\). Hence, the reduction

\[
T_1 = \frac{\lambda \kappa_2 + \mu}{2} \quad \text{and} \quad T_2 = \frac{\lambda \kappa_1 + \mu}{2},
\]

(2.14)

where \( \lambda \) and \( \mu \) are constants, is admissible and the remaining equilibrium equation (2.6) reduces to the purely geometric condition

\[
\lambda \kappa + \mu \mathcal{H} + \bar{p} = 0
\]

(2.15)

on the membranes \( \Sigma \). The above constant linear combination of the Gaussian and mean curvatures

\[
\kappa = \kappa_1 \kappa_2 \quad \text{and} \quad \mathcal{H} = \frac{\kappa_1 + \kappa_2}{2}
\]

(2.16)

is well known in classical differential geometry and is equivalent to stating that the membranes constitute ‘linear Weingarten’ surfaces [16]. In the case of a ‘homogeneous’ stress distribution \( T_1 = T_2 = \mu/2 \) corresponding to \( \lambda = 0 \), the classical Young–Laplace relation [18,19]

\[
\mathcal{H} = -\frac{\bar{p}}{\mu}
\]

(2.17)

modelling thin films (‘soap bubbles’) is retrieved. If \( \mu = 0 \) so that the stress components \( T_1 \) and \( T_2 \) are proportional to the principal curvatures \( \kappa_2 \) and \( \kappa_1 \), respectively, the membranes are of constant Gaussian curvature

\[
\kappa = -\frac{\bar{p}}{\lambda}.
\]

(2.18)

Linear Weingarten surfaces and, in particular, surfaces of constant mean or Gaussian curvature are known to be integrable in that the underlying Gauss–Mainardi–Codazzi equations are amenable to the techniques of soliton theory [3,20]. In fact, in [2], it has been shown that,
remarkably, the generic nonlinear system (2.6) and (2.7) is integrable in the shear-free case. This has been achieved by observing that the algebraic structure of the first two equilibrium equations implies that one may interpret these as the Mainardi–Codazzi equations associated with a fictitious surface, where $T_1$ and $T_2$ play the role of principal curvatures. Indeed, if we set

$$H = HT_2 \quad \text{and} \quad \tilde{K} = KT_1$$

(2.19)

then the pair (2.13) turns out to be yet another copy of the linear system (2.9), that is,

$$\tilde{H}_y = p\tilde{K} \quad \text{and} \quad \tilde{K}_x = q\tilde{H}.$$  

(2.20)

Accordingly, $\tilde{H}$ and $\tilde{K}$ constitute the metric coefficients of a Combescure transform $\tilde{\Sigma}$ of the membrane $\Sigma$ with position vector $\tilde{r}$ defined by

$$\tilde{r}_x = \tilde{H}X \quad \text{and} \quad \tilde{r}_y = \tilde{K}Y.$$  

(2.21)

Hence, the determination of shell membranes subject to vanishing shear stress turns out to be a purely geometric problem with the geometry of the Combescure transforms $\Sigma, \Sigma_0$ and $\tilde{\Sigma}$ being determined by the remaining equilibrium equation (2.6) which may be brought into the compact form

$$(H, K) = 0,$$  

(2.22)

where the vectors $H$ and $K$ are defined by

$$H = \begin{pmatrix} \tilde{H} \\ H \\ H_o \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} \tilde{K} \\ K \\ K_o \end{pmatrix}$$

(2.23)

and the scalar product is taken with respect to the constant matrix

$$\Lambda = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\bar{p} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$  

(2.24)

It is noted that the above-mentioned reduction to linear Weingarten surfaces corresponds to the superposition $2(H, K) = \mu(H, K) - \lambda(H_o, K_o)$ of solutions of the linear system (2.20).

The quadratic constraint (2.22) merely constitutes a special case of a rich class of integrable constraints, which is obtained by choosing an arbitrary number of Combescure transforms of a surface $\Sigma$ and an arbitrary constant symmetric matrix $\Lambda$ and imposing the orthogonality condition (2.22). The corresponding class of ‘O surfaces’ has been delineated in [5] and it has been shown that it admits a natural discrete analogue which is likewise integrable [6]. In Schief [13], in the case of vanishing ‘pressure’ $\bar{p}$, the equilibrium equations of discrete (‘plated’) membranes have been derived by means of first principles and their integrability has been shown directly, using an approach which is known as ‘multi-dimensional consistency’ [7]. However, as alluded to in the introduction, it turns out that a connection with discrete O surface theory exists and this is revealed as a by-product of the analysis presented in this paper.

(ii) Pure shear stress and vanishing pressure

If we now consider the ‘complementary’ case of pure shear stress and vanishing pressure, that is, $T_1 = T_2 = \bar{p} = 0$, then the equilibrium equations (2.6) reduce to

$$(H^2 S)_y = 0 \quad \text{and} \quad (K^2 S)_x = 0$$

(2.25)

so that the Gauss–Mainardi–Codazzi equations (2.7) are constrained by the compatibility condition

$$\left[ \ln \left( \frac{H}{K} \right) \right]_{xy} = 0,$$  

(2.26)
which guarantees the existence of \( S \). We may therefore choose the curvature coordinates \( x \) and \( y \) in such a manner that \( H^2 = K^2 \) and, hence, the first fundamental form \( I \) simplifies to
\[
\text{d}r^2 = H^2(\text{d}x^2 + \text{d}y^2).
\] (2.27)

Thus, without external forces, shell membranes can sustain pure shear stresses along the lines of curvature if and only if there exist conformal curvature coordinates, that is, if, modulo a suitable reparametrization of the lines of curvature, the associated metric is conformally flat. In the classical literature, surfaces of this kind are known as isothermic surfaces [16] and these have been studied in great detail in the geometric theory of integrable systems (see [20] and references therein). The occurrence of isothermic surfaces in the current shell membrane theoretical context has been examined in detail by Smyth [4]. It turns out that the approach proposed here leads to a natural discretization of these membranes which, remarkably, coincides with the standard integrable discretization of isothermic surfaces [8].

### 3. Discrete shell membranes: pure shear stress

Blaschke [21] demonstrated that Maxwell’s theory of reciprocal diagrams of forces [22] may be adapted to determine the stresses in shell membranes which are in equilibrium. The associated formalism was then taken up by Sauer [23] to examine hyperbolic membranes in detail and, in particular, membranes of constant negative Gaussian curvature. Both Blaschke’s and Sauer’s work led to a physical reinterpretation of Bianchi’s earlier work on isometric deformations of surfaces which preserve conjugate nets [24]. Specifically, the reciprocal-parallel nature of Voss surfaces and surfaces of constant negative Gaussian curvature was thereby retrieved [16]. Subsequently, Sauer [25] and Wunderlich [26] demonstrated that this geometric and physical connection between Voss and pseudospherical surfaces also exists if these classical classes of surfaces are replaced by natural discrete models. Remarkably, almost 50 years later, it was shown [27] that the underlying discrete ‘Gauss equation’ coincides with Hirota’s integrable discretization of the classical sine-Gordon equation [28].

As in the absence of external forces the internal stresses may be shown to be necessarily tangential to the asymptotic lines [23] on hyperbolic surfaces, Sauer’s discrete model of membranes parametrized in terms of asymptotic coordinates consists of a mesh of ‘strings’ which is such that the four strings meeting at any vertex are coplanar and the forces acting along these four strings annihilate each other. Indeed, meshes with coplanar stars have been used extensively as natural discretizations of asymptotic lines on surfaces [7]. By contrast, we here focus on meshes which possess planar faces and the internal forces are assumed to act on the edges rather than the vertices. Thus, throughout this paper, a discrete shell membrane \( \Sigma \) is defined as a map
\[
r : \mathbb{Z}^2 \to \mathbb{R}^3
\]
(3.1)
such that the quadrilaterals \( [r, r_{(1)}, r_{(12)}, r_{(2)}] \) are planar and the internal forces are encoded in a vector field
\[
F : E \to \mathbb{R}^3,
\]
(3.2)
where \( E \) denotes the set of edges of the \( \mathbb{Z}^2 \)-lattice and any index \((i)\) represents a relative shift in the corresponding coordinate direction so that, for instance,
\[
r = r(n_1, n_2), \quad r_{(1)} = r(n_1 + 1, n_2) \quad \text{and} \quad r_{(12)} = r(n_1 + 1, n_2 + 1).
\]
(3.3)
For notational convenience, we decompose \( F \) into two vector fields
\[
F_1 : E_v \to \mathbb{R}^3 \quad \text{and} \quad F_2 : E_h \to \mathbb{R}^3,
\]
(3.4)
where \( E_v \cong \mathbb{Z}^2 \) and \( E_h \cong \mathbb{Z}^2 \) denote the two sets of edges (‘vertical’ and ‘horizontal’), respectively. Furthermore, we assume that the internal forces act at the midpoints of the edges or, equivalently, that the internal forces are homogeneously distributed along the edges. It is noted that
Figure 1. A quadrilateral in equilibrium of Koenigs type (a) and its associated diagram of forces (b). (Online version in colour.)

(multi-dimensional) lattices composed of planar quadrilaterals discretize classical conjugate nets and have been the subject of extensive investigation in the context of the geometric theory of discrete integrable systems [7,29].

In order to recall the method of discretization proposed in [13], we here focus on the aforementioned case of pure shear stress and vanishing external force, which has not been considered before. However, in the first instance, we do not make use of any notion of discrete curvature nets but consider the more general case of discrete conjugate nets as defined above. Thus, let $A$, $B$, $C$ and $D$ be the internal tangential forces acting at the midpoints of the edges of a quadrilateral $[r, r_{(1)}, r_{(12)}, r_{(2)}]$, as depicted in figure 1a. In order for this quadrilateral to be in equilibrium, the total force and moment are required to vanish. Accordingly, the equilibrium equations read

$$A + B + C + D = 0$$

and

$$\begin{align*}
(r_{(1)} + r) \times A + (r_{(12)} + r_{(1)}) \times B + (r_{(12)} + r_{(2)}) \times C + (r_{(2)} + r) \times D &= 0.
\end{align*}$$

(3.5)

The force balance equation (3.5)1 may be interpreted as the local closing condition for the existence of another quadrilateral $[\tilde{r}, \tilde{r}_{(1)}, \tilde{r}_{(12)}, \tilde{r}_{(2)}]$ with oriented edges $A$, $B$, $C$ and $D$, as illustrated in figure 1b. Moreover, since the internal forces acting on the common edge of any two adjacent quadrilaterals must cancel each other, we may make the identification

$$F_{1(1)} = B = -D_{(1)} \quad \text{and} \quad F_{2(2)} = C = -A_{(2)}.$$  (3.6)

Hence, the global condition of the balance of forces is equivalent to the requirement that the internal forces be parametrized in terms of a Combescure transform [30] $\tilde{r}$ of $r$ via

$$F_1 = \tilde{r}_{(2)} - r \quad \text{and} \quad F_2 = -(\tilde{r}_{(1)} - r).$$  (3.7)

By definition, two discrete conjugate nets $r$ and $\tilde{r}$ are related by a Combescure transformation if the corresponding edges of $r$ and $\tilde{r}$ are parallel, that is,

$$\tilde{r}_{(i)} - r \parallel r_{(i)} - r, \quad i = 1, 2.$$  (3.8)

Resolution of the moment balance equation (3.5)2 now leads to the following result (cf. figure 1).

**Theorem 3.1.** A discrete conjugate net $r$ may be regarded as a discrete membrane in equilibrium with purely tangential internal forces if and only if $r$ constitutes a discrete Koenigs net, that is, if there exists a Combescure transform $\tilde{r}$ such that ‘non-corresponding’ diagonals are parallel:

$$\tilde{r}_{(1)} - \tilde{r}_{(2)} \parallel r_{(12)} - r \quad \text{and} \quad \tilde{r}_{(12)} - \tilde{r} \parallel r_{(1)} - r_{(2)}.$$  (3.9)

The internal forces are then given by (3.7).

**Proof.** If, for any fixed quadrilateral, we choose $r$ to be the origin of the ambient space $\mathbb{R}^3$ then it is readily seen that (3.5)2 reduces to (cf. figure 2)

$$(r_{(12)} - r) \times (B + C) = 0.$$  (3.10)
Alternatively, consideration of the linear combination \((3.5)_2 - [2r \times (3.5)_1]\) leads to the same result so that
\[
(r_{(12)} - r_r(r_{(1)} - r_r(r_{(2)}).
\]
Since the two defining properties \((3.9)_{1,2}\) of Koenigs nets imply each other, the proof is complete.

Discrete Koenigs nets were originally proposed by Sauer [31] and analysed in detail in [32,33] in connection with integrable discrete differential geometry. It is observed that Koenigs nets come in pairs, since the discrete conjugate nets \(r_r\) and \(r_r\) appear on equal footing. Hence, \(r_r\) may also be regarded as a discrete membrane in equilibrium with the internal forces being encapsulated in \(r_r\). Continuous Koenigs nets are classical and governed by linear hyperbolic equations with equal Laplace–Darboux invariants [20]. If the lines of curvature on a surface form a Koenigs net, then the surface is isothermic in the sense of §2 and \(r_r\) is known as the Christoffel transform of \(r_r\). It turns out that this defining property of isothermic surfaces is also valid in the discrete setting. Hence, if one requires that a discrete Koenigs net be a discrete curvature net, then one obtains the discrete isothermic surfaces proposed by Bobenko & Pinkall [8]. Discrete curvature nets are defined by demanding that the quadrilaterals be inscribed in circles, as illustrated in figure 5. Once again, such particular discrete conjugate nets have been used extensively in integrable discrete differential geometry [7].

In conclusion, it is observed that the preceding analysis provides a physical interpretation of discrete isothermic surfaces regarded as particular discrete curvature nets. In fact, the following theorem constitutes the direct analogue of Smyth’ theorem [4] alluded to in §2 and is illustrated in figure 3 for a discrete catenoid and a discrete sphere as its Christoffel dual.

**Theorem 3.2.** A discrete curvature net \(r_r\) may be regarded as a discrete membrane in equilibrium with purely tangential internal forces if and only if \(r_r\) constitutes a discrete isothermic surface. The internal forces are encoded in the discrete Christoffel transform \(r_r\) [8].
4. Discrete shell membranes: vanishing shear stress and constant normal loading

We now focus on the ‘complementary’ case of vanishing shear stress and constant normal loading which turns out to be considerably more involved. This is partially due to the fact that, a priori, it is not evident how one should choose the point \( r_e \) at which the external force \( F_e \) acts on each quadrilateral so that the integrable structure residing in the continuous case is preserved. Thus, we consider discrete curvature nets \( r: \mathbb{Z}^2 \to \mathbb{R}^3 \), internal forces \( F_i \) as defined in (3.4) and an external force \( F_e: \mathbb{R} \to \mathbb{R}^3 \) ‘acting’ at \( r_e: \mathbb{R} \to \mathbb{R}^3 \), where \( \mathbb{F} \equiv \mathbb{Z}^2 \) denotes the set of faces of \( \mathbb{Z}^2 \), with the following properties (cf. figure 4):

- \( F_i \) acts at midpoints of edges,
- \( F_i \perp \) edges,
- ‘Constant normal loading’ \( F_e = \bar{p}\delta \Sigma \bar{N} \) and
- \( F_e \) acts at some ‘canonical’ point \( r_e \) to be determined.

Here, the ‘pressure’ \( \bar{p} \) is constant, \( \delta \Sigma \) denotes the area of a quadrilateral \([r, r_{(1)}, r_{(12)}, r_{(2)}]\) and \( \bar{N} \) constitutes a unit vector orthogonal to the quadrilateral. A discrete curvature net may then be regarded as a discrete membrane in equilibrium if the equilibrium conditions

\[
\begin{align*}
F_{1(1)} - F_1 + F_{2(2)} - F_2 + F_e &= 0 \\
(r_{(12)} + r_{(1)}) \times F_{1(1)} - (r_{(2)} + r) \times F_1 + (r_{(12)} + r_{(2)}) \times F_{2(2)} - (r_{(1)} + r) \times F_2 + 2 r_e \times F_e &= 0
\end{align*}
\]

may be satisfied. In order to analyse these equilibrium equations, we first introduce a canonical parametrization of discrete curvature nets and generalized barycentric coordinates associated with (cyclic) quadrilaterals.

(a) Discrete curvature nets

A canonical parametrization of discrete curvature nets [6] is obtained by decomposing the edge vectors into their magnitude and direction. Thus, we define ‘metric’ coefficients \( H, K \) and unit ‘tangent’ vectors \( X, Y \) according to

\[
r_{(1)} - r = H X \quad \text{and} \quad r_{(2)} - r = K Y, \quad X^2 = Y^2 = 1,
\]

as indicated in figure 5. Since opposite angles made by the edges of (embedded) quadrilaterals which are inscribed in circles add up to \( \pi \), the circularity condition defining discrete curvature
Figure 5. Decomposition of the tangent vectors of a quadrilateral inscribed in a circle.

nests may be formulated as
\[ X(2) \cdot Y + Y(1) \cdot X = 0. \] (4.3)
The orientation of the tangent vectors is chosen in such a manner that
\[ X(2) \times Y + Y(1) \times X = 0. \] (4.4)
The compatibility condition \( r_{(12)} = r_{(21)} \) applied to the pair (4.2) reveals that there exist functions \( p \) and \( q \) such that, on the one hand,
\[ X(2) = \frac{X + qY}{\Gamma} \quad \text{and} \quad Y(1) = \frac{Y + pX}{\Gamma}, \quad \Gamma = \sqrt{1-pq} \] (4.5)
and, on the other hand,
\[ H(2) = \frac{H + pK}{\Gamma} \quad \text{and} \quad K(1) = \frac{K + qH}{\Gamma}. \] (4.6)
It is noted that circularity condition (4.3) may therefore be formulated as
\[ 2X \cdot Y + p + q = 0. \] (4.7)
A canonical ‘area vector’ \( A \) is defined by
\[ A = \frac{1}{2}(r_{(12)} - r) \times (r_{(2)} - r_{(1)}) \]
\[ = \frac{1}{2}(H_{(2)}K_{(1)} + HK)X \times Y \]
\[ = \frac{1}{2}(H_{(2)}K + K_{(1)}H)X(2) \times Y. \] (4.8)
A quadrilateral is embedded, that is, opposite edges are non-intersecting, if and only if
\[ H_{(2)}HK_{(1)}K > 0. \] (4.9)
In the following, we choose the signs of the unit normal \( \bar{N} \) and the area \( \delta \Sigma \) of a quadrilateral in such a manner that consistency with the convention \( \delta \Sigma \bar{N} = A \) is ensured.

(b) Generalized barycentric coordinates

The classical barycentric coordinates of a point \( V \) with respect to a coplanar non-degenerate triangle \( [V_1, V_2, V_3] \), as displayed in figure 6, are given by Coxeter [12]
\[ \frac{(VV_kV_l)}{(V_1V_2V_3)}, \quad (k,l) \in \{(1,2), (2,3), (3,1)\}, \] (4.10)
that is, \( V \) admits the unique decomposition
\[ V = \frac{(VV_2V_3)V_1 + (VV_3V_1)V_2 + (VV_1V_2)V_3}{(V_1V_2V_3)}, \] (4.11)
Figure 6. The barycentric coordinates of $V$ and $V^*$ coincide. If $V$ is the circumcentre of the triangle $[V_1, V_2, V_3]$ then $V^*$ is the circumcentre of the similar triangle $[V_{23}, V_{31}, V_{12}]$ and, hence, the nine-point centre of $[V_1, V_2, V_3]$. (Online version in colour.)

where $(V V' V')$ denotes the signed area of the triangle $[V, V', V']$ and, as usual, points are identified with their position vectors whenever appropriate. If $V^*$ denotes the image of $V$ under the homothety defined by

$$(V_1, V_2, V_3) \mapsto (V_{23}, V_{31}, V_{12}),$$

(4.12)

where $V_{kl} = V_{lk}$ constitutes the midpoint of the edge $[V_k, V_l]$, then the barycentric coordinates of $V^*$ with respect to the triangle $[V_{23}, V_{31}, V_{12}]$ are evidently the same so that

$$V^* = \frac{(VV_2 V_3)V_{23} + (VV_3 V_1)V_{31} + (VV_1 V_2)V_{12}}{(V_1 V_2 V_3)}.$$  

(4.13)

In particular, the circumcentre $V_c$ and the nine-point centre $V_n = V_c^*$ of the triangle $[V_1, V_2, V_3]$ share the barycentric coordinates relative to the triangles $[V_1, V_2, V_3]$ and $[V_{23}, V_{31}, V_{12}]$, respectively. The classical nine-point centre [12] is defined as the point of intersection of the altitudes of a triangle $[V_1, V_2, V_3]$ and coincides with the circumcentre of the triangle $[V_{23}, V_{31}, V_{12}]$. The centroid and the circumcentre define the celebrated Euler line [12] on which both the nine-point centre and the orthocentre lie, whereby the relative position of these four ‘centres’ is independent of the shape of the triangle.

A non-degenerate quadrilateral $[V_1, V_2, V_3, V_4]$ may be decomposed into two pairs of triangles if one takes into account the two diagonals. Thus, if $V$ is an arbitrary coplanar point then there exist four associated points $V_{klm}^*$, which are generated by the homotheties

$$(V_k, V_l, V_m) \mapsto (V_{lm}, V_{mk}, V_{kl})$$

(4.14)

as shown in figure 7. These points are given explicitly by

$$V_{123}^* = \frac{(VV_1 V_2)V_{12} + (VV_2 V_3)V_{23} + (VV_3 V_1)V_{31}}{(V_1 V_2 V_3)},$$

$$V_{341}^* = \frac{(VV_3 V_4)V_{34} + (VV_4 V_1)V_{41} + (VV_1 V_3)V_{13}}{(V_3 V_4 V_1)},$$

$$V_{234}^* = \frac{(VV_2 V_3)V_{23} + (VV_3 V_4)V_{34} + (VV_4 V_2)V_{42}}{(V_2 V_3 V_4)},$$

$$V_{412}^* = \frac{(VV_4 V_1)V_{41} + (VV_1 V_2)V_{12} + (VV_2 V_4)V_{24}}{(V_4 V_1 V_2)}.$$  

(4.15)
Figure 7. Illustration of the map \( V \mapsto V^* \) and the forces \( F_{kl} \). If \( V \) is the circumcentre of the quadrilateral \([V_1, V_2, V_3, V_4]\) then \( V^* \) is the nine-point centre of the triangle \([V^*_1, V^*_2, V^*_3]\) and \( V^* \) constitutes the generalized nine-point centre of the quadrilateral. (Online version in colour.)

The point of intersection \( V^* \) of the (extended) line segments \([V^*_{123}, V^*_{341}]\) and \([V^*_{234}, V^*_{412}]\), as displayed in figure 7, is readily shown to admit the representations

\[
V^* = \frac{(V_1V_2V_3)V^*_{123} + (V_3V_4V_1)V^*_{341}}{(V_1V_2V_3V_4)} = \frac{(V_2V_3V_4)V^*_{234} + (V_4V_1V_2)V^*_{412}}{(V_2V_3V_4V_1)},
\]

(4.16)

where

\[
(V_1V_2V_3V_4) = (V_1V_2V_3) + (V_3V_4V_1) = (V_2V_3V_4V_1) = (V_2V_3V_4) + (V_4V_1V_2)
\]

(4.17)

is the ‘area’ of the quadrilateral \([V_1, V_2, V_3, V_4]\). Furthermore, it may be verified that particular generalized barycentric coordinates of \( V^* \) with respect to the quadrilateral \([V_{12}, V_{23}, V_{34}, V_{41}]\) may be read off the expression

\[
V^* = \frac{(VV_1V_2)V_{12} + (VV_2V_3)V_{23} + (VV_3V_4)V_{34} + (VV_4V_1)V_{41}}{(V_1V_2V_3V_4)},
\]

(4.18)

which constitutes a canonical generalization of the unique decomposition (4.13). It is important to note that the map \( V \mapsto V^* \) is invertible in that, for any given point \( V^* \), (4.18) may be regarded as a linear system of equations for \( V \).

If the quadrilateral \([V_1, V_2, V_3, V_4]\) possesses a circumcentre \( V_c \) then the choice \( V = V_c \) corresponds to four nine-point centres \( V^*_{klm} \) and the point \( V^*_c \) turns out to be the **generalized nine-point centre** for quadrilaterals proposed in [11]. Hence, a cyclic quadrilateral admits another canonical ‘centre’ \( V_n = V^*_c \) which is in one-to-one correspondence with the circumcentre \( V_c \). Before we turn to the physical relevance of the generalized nine-point centre, we observe in passing that, for any quadrilateral, generalizations of the circumcentre and orthocentre likewise exist and, remarkably, these two points together with the centroid and the generalized nine-point centre of the quadrilateral may be shown to lie on a generalized Euler line with fixed relative positions as in the classical case [11].

(c) Quadrilaterals in equilibrium

In order to gain insight into the moment equation (4.1)2, it is instructive to focus on a single quadrilateral \([V_1, V_2, V_3, V_4]\) on which forces \( F_{kl} \) perpendicular to the edges \([V_k, V_l]\) and an external force \( F_e \) perpendicular to the quadrilateral act (cf. figure 7). If \( F_e \) acts at the point \( V_e \) and \( F_{kl}^{||}, F_{kl}^{\perp} \) denote the components of \( F_{kl} \) which are parallel and orthogonal to the quadrilateral,
respectively, then the equilibrium equations may be decomposed into
\[
\begin{align*}
F_{12}^\parallel + F_{23}^\parallel + F_{34}^\parallel + F_{41}^\parallel &= 0, \\
F_{12}^\perp + F_{23}^\perp + F_{34}^\perp + F_{41}^\perp + F_e &= 0
\end{align*}
\tag{4.19}
\]
and
\[
V_{12} \times F_{12}^\perp + V_{23} \times F_{23}^\perp + V_{34} \times F_{34}^\perp + V_{41} \times F_{41}^\perp + V_e \times F_e = 0.
\tag{4.19}
\]

It is noted that the vanishing total force guarantees that the moment equilibrium condition is independent of the origin of the coordinate system, and hence the moment balance equation \((4.19)_3\) does not involve the tangential components of \(F_{kl}\) since we may choose the circumcentre of the quadrilateral as the origin. Furthermore, it is useful to observe that the midpoints \(V_{kl}\) of the respective edges satisfy the identity
\[
V_{12} + V_{34} = V_{23} + V_{41}.
\tag{4.20}
\]

As pointed out in the preceding, for any given point \(V_e\), there exists a unique point \(V\) such that \(V^* = V_e\). The relationship between \(V\) and \(V_e\) is encapsulated in \((4.18)\) so that
\[
V_e = Q_{12} V_{12} + Q_{23} V_{23} + Q_{34} V_{34} + Q_{41} V_{41},
\tag{4.21}
\]
where the generalized barycentric coordinates \(Q_{kl}\) are given by
\[
Q_{kl} = \frac{(V V_e V_l)}{(V_1 V_2 V_3 V_4)}.
\tag{4.22}
\]

Insertion into the moment balance equation \((4.19)_3\) then leads to
\[
\begin{align*}
V_{12} \times (F_{12}^\perp + Q_{12} F_e) + V_{23} \times (F_{23}^\perp + Q_{23} F_e) + V_{34} \times (F_{34}^\perp + Q_{34} F_e) + V_{41} \times (F_{41}^\perp + Q_{41} F_e) &= 0 \\
\end{align*}
\tag{4.23}
\]
and elimination of \(V_{41}\) by means of the identity \((4.20)\) yields
\[
\begin{align*}
(V_{23} - V_{12}) \times [F_{23}^\perp + F_{34}^\perp + (Q_{23} + Q_{34}) F_e] + (V_{23} - V_{34}) \times [F_{12}^\perp + F_{23}^\perp + (Q_{12} + Q_{23}) F_e] &= 0,
\end{align*}
\tag{4.24}
\]
where the force balance equation \((4.19)_2\) and normalization property
\[
Q_{12} + Q_{23} + Q_{34} + Q_{41} = 1
\tag{4.25}
\]
have been exploited. As the points \(V_{12}, V_{23}, V_{34}\) are not collinear, it follows that
\[
F_{12}^\perp + F_{23}^\perp + (Q_{12} + Q_{23}) F_e = 0
\tag{4.26}
\]
and
\[
F_{23}^\perp + F_{34}^\perp + (Q_{23} + Q_{34}) F_e = 0
\tag{4.26}
\]
By construction, this pair of equations is equivalent to the moment balance equation \((4.19)_3\) and the force balance equation \((4.19)_2\) confirms that, by symmetry, an equivalent pair of equations is given by
\[
F_{34}^\perp + F_{41}^\perp + (Q_{34} + Q_{41}) F_e = 0
\tag{4.27}
\]
and
\[
F_{41}^\perp + F_{12}^\perp + (Q_{41} + Q_{12}) F_e = 0.
\tag{4.27}
\]

In fact, any of the above four equations may be associated with a pair of forces \(F_{kl}^\perp, F_{lm}^\perp\) acting on a corresponding pair of edges which share the vertex \(V_l\) and any two equations which do not correspond to opposite pairs of edges are independent and imply the other two equations modulo the force balance equation \((4.19)_2\). Moreover, we conclude that a quadrilateral is in equilibrium if the total tangential internal force \((4.19)_1\) vanishes and the perpendicular components \(F_{kl}^\perp\) of the internal forces are compatible with the \textit{area-weighted decomposition} \((4.26)\) and \((4.27)\) of the external force \(F_e\) (figure 8a). Any one of the latter four equations may be regarded as the discrete
and (may be chosen at will. We begin with the equilibrium equation (4.29) and state that elementary force averaged over any two adjacent edges is tangential, that is,

\[ F_e \]  

which is defined in the same manner as that of corresponding edges, we introduce the parametrization \( \vec{r} \mathbb{R}^3 \) which enters the decomposition (4.26) and (4.27). In the case of a cyclic quadrilateral, the circumcentre \( Q_{ij} \) is to be determined, and hence it is natural to select the generalized nine-point centre \( V_n = V_n^e \) as the point at which the external force \( F_e \) acts. Accordingly, if \( r_e : F \mathbb{R}^3 \) encodes the set of circumcentres of the discrete curvature net then we make the choice \( r_e = r_c^e = r_n \). Since the equilibrium equations (4.19) for a single quadrilateral are equivalent to (4.19) and any three of the four equations (4.26), (4.27), the equilibrium equations (4.1) are completely encoded in

\[
\begin{align*}
(F_{1(1)} - F_1 + F_{2(2)} - F_2) \times N &= 0, \\
- (F_1 + F_2) \cdot A + F_e \cdot (A[r_{(2)}, r] + A[r, r_{(1)}]) &= 0, \\
(F_{2(2)} - F_1) \cdot A + F_e \cdot (A[r_{(12)}, r_{(2)}] + A[r, r_{(1)}]) &= 0, \\
(F_{1(1)} - F_2) \cdot A + F_e \cdot (A[r, r_{(1)}] + A[r_{(1)}, r_{(12)}]) &= 0,
\end{align*}
\]

(4.29)

and

where \( A[\vec{r}, \vec{r}] \) designates the area vector associated with the triangle \( [\vec{r}, \vec{r}, r_c] \), the orientation of which is defined in the same manner as that of \( A \).

In order to adhere to the requirement that the internal forces be orthogonal to the corresponding edges, we introduce the parametrization

\[
F_1 = Y \times X \quad \text{and} \quad F_2 = U \times X,
\]

(4.30)

which turns out to be convenient as the components of \( U \) and \( V \) parallel to \( X \) and \( Y \), respectively, may be chosen at will. We begin with the equilibrium equation (4.29) and state that elementary geometry leads to the expression

\[
A[r_{(2)}, r] + A[r, r_{(1)}] = \left[ \frac{1}{2} HK - \frac{1}{4} (H^2 + K^2) \right] X \cdot Y \frac{X \times Y}{(X \times Y)^2}
\]

(4.31)

so that \( F_e = \vec{p} A \) with \( A \sim X \times Y \) yields

\[
U \cdot Y + V \cdot X + \frac{1}{2} \vec{p} HK - [(U \cdot X + \frac{1}{4} \vec{p} H^2) + (V \cdot Y + \frac{1}{4} \vec{p} K^2)] X \cdot Y = 0.
\]

(4.32)
Hence, if we choose the gauge
\[ \mathbf{U} \cdot \mathbf{X} = - \frac{1}{4} \bar{p} H^2 \quad \text{and} \quad \mathbf{V} \cdot \mathbf{Y} = - \frac{1}{4} \bar{p} K^2 \] (4.33)
then this equilibrium condition reduces to
\[ \mathbf{U} \cdot \mathbf{Y} + \mathbf{V} \cdot \mathbf{X} = - \frac{1}{2} \bar{p} H K. \] (4.34)

Similarly, the equilibrium equations (4.29)\textsuperscript{3,4} may be decomposed into
\[ \mathbf{U}(2) \cdot \mathbf{X}(2) = - \frac{1}{4} \bar{p} H^2_{(2)} \quad \text{and} \quad \mathbf{V}(1) \cdot \mathbf{Y}(1) = - \frac{1}{4} \bar{p} K^2_{(1)} \] (4.35)
and
\[ \mathbf{X}(2) \cdot [(1 - 2YY^T)\mathbf{V} - \mathbf{U}(2) \cdot \mathbf{Y} = - \frac{1}{2} \bar{p} H_{(2)} K] \] (4.36)
and
\[ \mathbf{Y}(1) \cdot [(1 - 2XX^T)\mathbf{U} - \mathbf{V}(1) \cdot \mathbf{X} = - \frac{1}{2} \bar{p} H_{(1)} K] \]
since the conditions (4.35) are identical to (4.33), regarded as gauge conditions valid for the entire discrete membrane rather than a single quadrilateral.

The remaining equilibrium equation (4.29)\textsuperscript{1} may be formulated as
\[ (\mathbf{U}(2) \cdot \mathbf{N})\mathbf{X}(2) - (\mathbf{U} \cdot \mathbf{N})\mathbf{X} - (\mathbf{V}(1) \cdot \mathbf{N})\mathbf{Y}(1) + (\mathbf{V} \cdot \mathbf{N})\mathbf{Y} = 0 \] (4.37)
so that decomposition into the \(X\) and \(Y\) components gives rise to the pair
\[ (\mathbf{U}(2) - \bar{p}\mathbf{V}(1) - \Gamma\mathbf{U}) \cdot (\mathbf{X} \times \mathbf{Y}) = 0 \] (4.38)
and
\[ (\mathbf{V}(1) - \bar{q}\mathbf{U}(2) - \Gamma\mathbf{V}) \cdot (\mathbf{X} \times \mathbf{Y}) = 0 \]
where \(\mathbf{N} \sim \mathbf{X} \times \mathbf{Y}\) has been taken into account. If one now regards the six equations (4.35), (4.36) and (4.38) as a linear system for the vectors \(\mathbf{U}(2)\) and \(\mathbf{V}(1)\) then it is easily verified that its solution is unique and admits the compact representation stated in the following theorem.

**Theorem 4.1.** In terms of the parametrization of §4a, a discrete curvature net \(\Sigma\) may be regarded as a discrete shell membrane in equilibrium with \(r_\mathbf{e} = r_n\) if the shear-free internal forces \(\mathbf{F}_1\) and \(\mathbf{F}_2\) may be parametrized according to
\[ \mathbf{F}_1 = \mathbf{Y} \times \mathbf{V}, \quad \mathbf{V} \cdot \mathbf{Y} = - \frac{1}{4} \bar{p} K^2 \] (4.39)
and
\[ \mathbf{F}_2 = \mathbf{U} \times \mathbf{X}, \quad \mathbf{U} \cdot \mathbf{X} = - \frac{1}{4} \bar{p} H^2, \]
where the vector fields \(\mathbf{U}\) and \(\mathbf{V}\) obey the linear system
\[ \mathbf{U}(2) = \frac{\mathbf{U} + \bar{p}\mathbf{V} - 2[(\mathbf{U} + \bar{p}\mathbf{V}) \cdot \mathbf{Y}]\mathbf{Y}}{\Gamma} \] (4.40)
and
\[ \mathbf{V}(1) = \frac{\mathbf{V} + \bar{q}\mathbf{U} - 2[(\mathbf{V} + \bar{q}\mathbf{U}) \cdot \mathbf{X}]\mathbf{X}}{\Gamma} \]

**Remark 4.2.** Even though the above system for \(\mathbf{U}\) and \(\mathbf{V}\) is linear and the normalizations (4.39)\textsuperscript{2,4} are preserved by the evolution equations (4.40) modulo the constraint (4.41), the latter couples the equilibrium equations to the discrete ‘Gauss–Weingarten’ equations (4.3)–(4.6) which, in turn, renders the complete set of governing equations nonlinear.

### 5. Integrable structure

In §3, it has been demonstrated that the relevant class of discrete shell membranes is integrable in the sense that the underlying system of discrete equations is amenable to the techniques of soliton theory. Moreover, in the case of vanishing external force \(\mathbf{F}_e\), the discrete shell membranes investigated in the previous section have been shown to be likewise integrable [13]. This has been achieved by using a method which has come to be known as ‘consistency on multi-dimensional lattices’ [7]. Here, we show that integrability is still present in the case of a non-vanishing external force.
force by identifying the discrete shell membranes as particular $O$ surfaces. The latter class of integrable surfaces has been introduced in [5,6] and captures a great variety of classical and novel integrable surfaces both continuous and discrete. As a by-product, a link between the analysis presented in [13] for $\bar{p} = 0$ and $O$ surface theory is established.

(a) An $O$ surface connection

We begin with the observation that any discrete curvature net $\Sigma$ admits a two-parameter family of privileged Combescure transforms $\Sigma_\circ$ which may be regarded as discrete spherical representations of $\Sigma$ [6,34] (figure 9). Indeed, we first choose an arbitrary point on the unit sphere with position vector $N$ which we regard as a ‘normal’ associated with a fixed vertex $r$ of $\Sigma$. A discrete curvature net $N: \mathbb{Z}^2 \to S^2$ on the unit sphere is then uniquely determined by successively drawing lines which are parallel to the corresponding edges of $\Sigma$ and defining $N$ as the position vectors of the points of intersection with $S^2$. It is important to note that this procedure is well defined as the quadrilaterals of $\Sigma$ are inscribed in circles. Algebraically, $N$ may be constructed by solving the compatible linear system

$$N(1) = (1 - 2XX^T)N \quad \text{and} \quad N(2) = (1 - 2YY^T)N.$$

(5.1)

Since $N$ constitutes a Combescure transform of $r$, the quantities

$$H_\circ = -2N \cdot X \quad \text{and} \quad K_\circ = -2N \cdot Y,$$

(5.2)

which may be read off (5.1) formulated as

$$N(1) - N = H_\circ X \quad \text{and} \quad N(2) - N = K_\circ Y$$

(5.3)

may be verified to obey compatibility conditions (4.6), that is,

$$H_\circ(2) = \frac{H_\circ + pK_\circ}{\Gamma} \quad \text{and} \quad K_\circ(1) = \frac{K_\circ + qH_\circ}{\Gamma}.$$  (5.4)

Now, the key idea is to consider three linearly independent normals $N_1, N_2$ and $N_3$, that is, three linearly independent solutions of (5.1) and proceed with the expansions

$$U = \sum_{i=1}^{3} H_i N_i \quad \text{and} \quad V = \sum_{i=1}^{3} K_i N_i,$$

(5.5)
wherein the coefficients $H_i$ and $K_i$ are to be determined. Insertion into the pair (4.40) leads to

$$
\sum_{i=1}^{3} \left[ H_{i(2)} N_{i(2)} - \frac{H_i + pK_i}{\Gamma} (1 - 2 Y Y^T) N_i \right] = 0
$$

and

$$
\sum_{i=1}^{3} \left[ K_{i(1)} N_{i(1)} - \frac{K_i + qH_i}{\Gamma} (1 - 2 X X^T) N_i \right] = 0
$$

so that comparison with (5.1) results in

$$
H_{i(2)} = \frac{H_i + pK_i}{\Gamma} \quad \text{and} \quad K_{i(1)} = \frac{K_i + qH_i}{\Gamma},
$$

the structure of which once again coincides with that of the linear system (4.6). Accordingly, as in the continuous case, the internal forces are encoded in Combescure transforms of the shell membrane. However, in the current discrete case, the coefficients $H_i$ and $K_i$ may be regarded as the ‘metric’ coefficients of three Combescure transforms $\Sigma_i$ of $\Sigma$. Finally, the remaining equilibrium equations (4.39)$_{2,4}$ and (4.41) imply that the seven Combescure transforms $\Sigma, \Sigma_{cd}$ and $\Sigma_i$ are linked by compact but nonlinear relations.

**Theorem 5.1.** A discrete curvature net $\Sigma$ constitutes a discrete shell membrane in equilibrium with shear-free internal forces and constant normal loading $\bar{p}$ acting at the generalized nine-point centres of the quadrilaterals (cf. theorem 4.1) if and only if there exist three spherical representations $\Sigma_{cd}$ and three Combescure transforms $\Sigma_i$ of $\Sigma$ such that the associated metric coefficients $H, K, H_{cd}, K_{cd}$ and $H_i, K_i$ obey the integrable ‘orthogonality conditions’

$$
\langle H, H \rangle = 0, \quad \langle K, K \rangle = 0 \quad \text{and} \quad \langle H, K \rangle = 0,
$$

where the inner product $\langle , \rangle$ is taken with respect to the symmetric matrix $\Lambda$ and

$$
H = \begin{pmatrix}
H_1 \\
H_2 \\
H_3 \\
H_{12} \\
H_{23} \\
H_{13}
\end{pmatrix}, \quad K = \begin{pmatrix}
K_1 \\
K_2 \\
K_3 \\
K_{12} \\
K_{23} \\
K_{13}
\end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix}
0 & 0 & 1 \\
0 & -\bar{p} & 0 \\
1 & 0 & 0
\end{pmatrix}.
$$

**Proof.** The orthogonality conditions (5.8)$_{1,2}$ and (5.8)$_{3}$ are merely a reformulation of the normalization conditions (4.39)$_{2,4}$ and the constraint (4.41), respectively. The integrability of these quadratic conditions is a consequence of the following remark.

**Remark 5.2.** The above orthogonality conditions may be reformulated as

$$
\langle H, H \rangle = \alpha(n_1), \quad \langle K, K \rangle = \beta(n_2) \quad \text{and} \quad \langle H_{(2)}, K + (K_{(1)}, H) = 0
$$

with

$$
H_{(2)} = \frac{H + pK}{\Gamma} \quad \text{and} \quad K_{(1)} = \frac{K + qH}{\Gamma}.
$$

and $\alpha = \beta = 0$. For arbitrary non-vanishing functions $\alpha$ and $\beta$, an arbitrary number of Combescure transforms and an arbitrary constant symmetric matrix $\Lambda$, one obtains the general class of integrable discrete O surfaces proposed in [6]. However, if $\alpha = \beta = 0$, the theory developed therein is still valid, and hence the above discrete shell membranes may indeed be identified as particular discrete O surfaces. It is observed that one may regard the conditions (5.10) as the analogue of the conditions

$$
X \cdot X = \alpha(n_1), \quad Y \cdot Y = \beta(n_2) \quad \text{and} \quad X_{(2)} \cdot Y + Y_{(1)} \cdot X = 0
$$

which characterize discrete curvature nets since $\alpha = \beta = 1$ without loss of generality. In fact, the conditions (5.10) may be interpreted as the ‘circularity’ conditions for discrete conjugate nets in a pseudo-Euclidean dual space with metric $\Lambda$. 


(b) The continuum limit: a discrete Gauss equation and special discrete shell membranes

The preceding analysis demonstrates how the non-uniqueness of the spherical representation of a discrete curvature net may be exploited to derive a compact form of the equilibrium equations. In order to relate the corresponding parametrization of the internal forces to the classical continuous case and identify discrete analogues of the particular classes of shell membranes listed in §2, we first establish the natural continuum limit in which the continuous theory may formally be recovered.

(i) A discrete Gauss equation

In order to perform the continuum limit, it is necessary to introduce lattice parameters \( \epsilon \) and \( \delta \) into the discrete Gauss–Weingarten and Gauss–Mainardi–Codazzi equations (4.2)–(4.7). Thus, if we apply the scaling

\[
(H, p) \rightarrow \epsilon(H, p) \quad \text{and} \quad (K, q) \rightarrow \delta(K, q)
\]

then (4.5) and (4.6) become

\[
X(2) = \frac{X + \delta Y}{\Gamma}, \quad Y(1) = \frac{Y + \epsilon X}{\Gamma}
\]

and

\[
H(2) = \frac{H + \delta K}{\Gamma}, \quad K(1) = \frac{K + \epsilon H}{\Gamma}
\]

so that the limit \( x = \epsilon n_1, y = \delta n_2, \epsilon, \delta \to 0 \) leads to the standard pairs

\[
X_y = qY, \quad Y_x = pX, \quad H_y = pK \quad \text{and} \quad K_x = qH.
\]

as discussed in §2b. Furthermore, if we single out the discrete ‘normal’ \( N := N_3 \), say, and scale \( H := H_{o,3} \) and \( K := K_{o,3} \) in the same manner as \( H \) and \( K \) then, in the continuum limit, (5.3) and (5.4) reduce to

\[
N_x = H_0 X, \quad N_y = K_0 Y, \quad H_{oy} = pK_0 \quad \text{and} \quad K_{ox} = qH_0.
\]

Accordingly, since \((X, Y, N)\) constitutes an orthonormal triad, the complete set of Gauss–Weingarten equations (2.10) together with the Mainardi–Codazzi equations (2.8)\(_{1,2}\) are recovered.

In order to retrieve the important Gauss equation \([16] (2.8)_3\), we first observe that the linear equations (5.1) imply that if we choose the triad \((N_1, N_2, N_3)\) to be orthonormal at a single point then the triad is orthonormal throughout the lattice. However, geometrically, it is evident that the normals \(N_1\) and \(N_2\) do not possess continuum limits unless the normals are ‘flipped’ in such a manner that pairs of normals along edges become ‘aligned’ in the limiting process. Indeed, if we use the change of variables

\[
(N_1, H_{o,1}, K_{o,1}) \rightarrow (-1)^{n_1}(N_1, H_{o,1}, \delta K_{o,1})
\]

then the analogues of (5.3) and (5.4) become

\[
-N_{1(1)} = N_1 + H_{o,1} X, \quad N_{1(2)} = N_1 + \delta K_{o,1} Y
\]

and

\[
H_{o,1(2)} = \frac{H_{o,1} + \delta p K_{o,1}}{\Gamma}, \quad -K_{o,1(1)} = \frac{K_{o,1} + q H_{o,1}}{\Gamma}
\]

which, in the continuum limit, reduce to

\[
H_{o,1} = -2, \quad K_{o,1} = q, \quad N_1 = X \quad \text{and} \quad X_y = qY.
\]

Hence, it turns out that the ‘normal’ \(N_1\) coincides with the tangent vector \(X\) in the natural continuum limit. Similarly, the scaling

\[
(N_2, H_{o,2}, K_{o,2}) \rightarrow (-1)^{n_2}(N_2, \epsilon H_{o,2}, K_{o,2})
\]

leads to

\[
H_{o,2} = p, \quad K_{o,2} = -2, \quad N_2 = Y \quad \text{and} \quad Y_x = pX
\]

so that \(N_2\) becomes the unit tangent \(Y\).
The discrete Gaussian curvature $K$ of a discrete curvature net $\Sigma$ with respect to a spherical representation $\Sigma_0$ may be defined as [6,13]

$$K = \frac{H_{(2)} K_{(2)} + K_{(1)} H_{(1)}}{H_{(2)} + K_{(1)}}. \tag{5.22}$$

This is the natural analogue of the definition in the classical continuous case since $K$ constitutes the ratio of the areas of corresponding quadrilaterals of $\Sigma_0$ and $\Sigma$, that is,

$$A_0 = KA, \tag{5.23}$$

where the area vector $A_0$ associated with $\Sigma_0$ is defined as in (4.8). A discrete version of the Gauss equation (2.8) may then be formulated as follows.

**Theorem 5.3.** The Gaussian curvatures $K_i$ of a discrete curvature net $\Sigma$ associated with an orthonormal triad $(N_1, N_2, N_3)$ of Gauss maps obeys the identity

$$\sum_{i=1}^{3} K_i = 0, \tag{5.24}$$

which may be regarded as a discrete analogue of the classical Gauss equation

$$q_x + p_y + H_0 K_0 = 0. \tag{5.25}$$

**Proof.** As $(N_1, N_2, N_3)$ constitutes an orthonormal triad, the tangent vectors $X$ and $Y$ may be expanded according to

$$X = \sum_{i=1}^{3} (X \cdot N_i) N_i \quad \text{and} \quad Y = \sum_{i=1}^{3} (Y \cdot N_i) N_i. \tag{5.26}$$

The discrete orthogonality condition

$$2X \cdot Y + qX^2 + pY^2 = 0 \tag{5.27}$$

may then be formulated as

$$\sum_{i=1}^{3} (2H_{oi} K_{oi} + qH_{oi}^2 + pK_{oi}^2) = 0 \tag{5.28}$$

by virtue of $2N_i \cdot X = -H_{oi}$ and $2N_i \cdot Y = -K_{oi}$. It is readily verified that the above identity is equivalent to

$$\sum_{i=1}^{3} (H_{oi(2)} K_{oi} + K_{oi(1)} H_{oi}) = 0 \tag{5.29}$$

so that assertion (5.24) is made good. Moreover, application of the scalings (5.17) and (5.20)–(5.29) results in

$$2H_{c3} K_{c3} - H_{c1} K_{c1x} - H_{c2y} K_{c2} = 0 \tag{5.30}$$

in the limit $\epsilon, \delta \to 0$. The latter coincides with the Gauss equation (5.25) by virtue of identifications (5.19) and (5.21).
(ii) Special discrete shell membranes

Consistency with the quadratic constraints (5.8) shows that the scalings (5.13), (5.17) and (5.20) induce the scalings

\[
(H_1, H_2, H_3) \rightarrow \epsilon((-1)^{h_1} \epsilon H_1, (-1)^{h_2} \delta H_2, H_3)
\]

and

\[
(K_1, K_2, K_3) \rightarrow \delta((-1)^{h_1} \epsilon K_1, (-1)^{h_2} \delta K_2, K_3)
\]

de the coefficients associated with the internal forces. In the limit \(\epsilon, \delta \rightarrow 0\), the linear system (5.7) then reveals that \(H_2 = K_1 = 0\) and \(H_{1y} = K_{2x} = 0\) and the constraints (5.8) adopt the form

\[
\langle H, H \rangle = \alpha, \quad \langle H, K \rangle = 0 \quad \text{and} \quad \langle K, K \rangle = \beta,
\]

wherein the vectors \(H, K\) and the matrix \(\Lambda\) are now defined by

\[
H = \begin{pmatrix} H_3 \\ H \\ H_{c3} \end{pmatrix}, \quad K = \begin{pmatrix} K_3 \\ K \\ K_{c3} \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} 0 & 0 & 1 \\ -\bar{p} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

and \(\alpha = -4H_1, \beta = -4K_2\). Hence, we retrieve the O surface constraint (2.22) which characterizes the shell membranes in the classical case. It is noted that the additional conditions (5.32)_{1,3} are redundant since \(H\) and \(K\) constitute vector-valued solutions of the pair (5.15)_{3,4} so that the orthogonality condition (5.32)_{2} implies that \(\langle H, H \rangle_y = \langle K, K \rangle_x = 0\). Finally, at the level of the internal forces \(F_1\) and \(F_2\), we obtain the limits

\[
\delta^{-1}F_1 \rightarrow K_3Y \times N_3 = K_3X \quad \text{and} \quad \epsilon^{-1}F_2 \rightarrow H_3N_3 \times X = H_3Y,
\]

which are consistent with the standard parametrization of the internal stresses in the classical case [2] if one makes the identification \(\tilde{H} = H_3\) and \(\tilde{K} = K_3\) in (2.19).

As in the continuous case, we may now make the restrictive but admissible assumption that the Combesure transform \(\Sigma_3\) is a ‘linear combination’ of the shell membrane \(\Sigma\) and the spherical representation \(\Sigma_{c3}\), that is,

\[
2(H_3, K_3) = \mu(H, K) - \lambda(H_{c3}, K_{c3}).
\]

In this case, the orthogonality condition (5.8)_{3} in the form (5.10) may be reformulated as

\[
\lambda K_3 + \mu H_3 + \bar{p} = \frac{\langle \hat{H}(2), \hat{K} \rangle + \langle \hat{K}(1), \hat{H} \rangle}{H(2)K + K(1)H},
\]

where

\[
\hat{H} = \begin{pmatrix} H_1 \\ H_2 \\ H_{c1} \end{pmatrix}, \quad \hat{K} = \begin{pmatrix} K_1 \\ K_2 \\ K_{c1} \end{pmatrix} \quad \text{and} \quad \hat{\Lambda} = \begin{pmatrix} 0 & 1 & 0 \\ \bar{p} & 0 & 0 \end{pmatrix}
\]

and the mean curvature \(\mathcal{H}\) of a discrete curvature net \(\Sigma\) with respect to a spherical representation \(\Sigma_{c}\) is defined by Schief [6]

\[
\mathcal{H} = -\frac{H(2)K_{c} + K(1)H_{c} + H_{c1}K + K_{c1}H}{2(H(2)K + K(1)H)}.
\]

The geometric meaning and significance of the discrete mean curvature in terms of parallel surfaces and the associated discrete version of the classical Steiner formula may be found in Schief [34]. Since, in the continuum limit, the right-hand side of (5.36) is readily shown to vanish, the particular discrete shell membranes governed by (5.36) constitute integrable discretizations of the classical linear Weingarten surfaces alluded to in §2. It is emphasized that the standard integrable discretization of linear Weingarten surfaces is defined by an identically vanishing right-hand side of (5.36) which, in the current context, may be shown to be inconsistent. In particular, in the case of shell membranes of constant Gaussian or mean curvature, that is, if \(\mu = 0\) or \(\lambda = 0\), respectively, the ‘physical’ discretization of these membranes is encapsulated in (5.36) and is, in general, not captured by the standard ‘mathematical’ discretization of Bobenko & Pinkall [8,27].
In general, as demonstrated in the preceding, even though the determination of the class of continuous shell membranes may be formulated as a purely geometric problem, the natural geometric discretization of this problem does not admit a consistent physical interpretation. Thus, we conclude that the geometric and physical properties of the shell membranes need to be taken into account simultaneously in order to obtain a physically meaningful discretization of shell membranes in equilibrium.

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**References**