Relaxation of the single-slip condition in strain-gradient plasticity

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We consider the variational formulation of both geometrically linear and geometrically nonlinear elasto-plasticity subject to a class of hard single-slip conditions. Such side conditions typically render the associated boundary-value problems non-convex. We show that, for a large class of non-smooth plastic distortions, a given single-slip condition (specification of Burgers vectors) can be relaxed by introducing a microstructure through a two-stage process of mollification and lamination. The relaxed model can be thought of as an aid to simulating macroscopic plastic behaviour without the need to resolve arbitrarily fine spatial scales.

1. Introduction

Complex subgrain dislocation patterns in plastically deformed metallic crystals have been observed by many authors [1–5], and these dislocation microstructures are known to affect greatly several aspects of plastic behaviour, such as work hardening, and the Bauschinger and Hall–Petch effects. Currently, attempts at modelling such behaviour are often phenomenological, which leads to significant problems for specimens on the micrometre scale [6,7]. Nevertheless, predictive models for pattern formation have been proposed, for example in the seminal work by Ortiz & Repetto [8]. Such models are based on a non-quasi-convex energy minimization at each time increment of the evolution, the non-convexity resulting from the assumption of latent hardening in single-crystal plasticity, which means that if dislocations from different slip systems meet, then they form energetically favourable (but sessile) atomistic reaction products, such as the so-called Lomer–Cottrell locks. This increases the dissipation potential for any plastic deformation that does not occur in single slip.
Here, we proceed as in [8], and in addition make the simplifying assumption of infinite latent hardening (i.e. that the material should deform in single slip at each point), thus resulting in a strain-gradient-plasticity model. This is akin to the approach adopted in, say, [9,10], among many other works, or, to take another example, [11], where it was shown that, for a simple shear experiment, the laminate-type microstructures predicted by non-convex single-slip models can indeed be observed in plastically deformed single crystals. An analysis of evolutionary models for such plastic laminates can be found in [12,13]. Despite our emphasis on the single-slip condition (SSC) here, however, it should be noted that the main result, namely that such non-convexities should always be partially relaxed, still holds for finite latent hardening between slip systems in a single-slip plane, at least as long as this is implemented as a direction-dependent dissipation (see remark 4.4).

In incremental models of non-convex crystal plasticity, a strain-gradient penalty term like \( \int |\text{curl}\beta| \), where \( \beta \) is the plastic strain, is commonly added [14] and is actually necessary in some cases if one wants to make the hardening physically realistic [15]. The introduction of a curl-type strain-gradient term in crystal plasticity goes back to Kondo [16,17] and, independently, to Nye [18]. In rectangular shear samples of small size, the strain-gradient energy and single-slip condition together play a dramatic role, as we showed analytically in [19]. Specifically, for a strain-gradient model of a single crystal with B2 symmetry, there are three qualitatively different energy-scaling regimes, which are determined by the aspect ratio, \( L \): when \( L \) is small the energy scales quadratically (i.e. elastically) with the shear, when \( L \) is large one sees easy shear-band formation, and for intermediate values of \( L \) there is a regime in which the energy scales linearly with the applied shear. One consequence of this is that a micrometre-sized sample consisting of only a few grains will easily shear off if a slip system under stress connects free surfaces [6].

In this article, we show that such a strain-gradient penalty term is not sufficient to regularize the problem in the sense of making the associated elasto-plastic energy lower semicontinuous. In some sense, this problem was already discovered in numerical simulations, for example by Hildebrand & Miehe [20]. These authors circumvent the problem by introducing an ultra-fine full-gradient-penalty regularization—this, however, introduces an even finer length scale which is very difficult to treat numerically in an industrial context. Further discussion about models that include a strain-gradient-type penalization of geometrically necessary dislocations (GNDs), and that also potentially admit a non-convexity in the dissipation (implemented through a hardening matrix in the flow rule) can, for example, be found in [21], while a recent solution-method in the framework of finite-element modelling is described in [22]. We believe that some improvement to the numerical stability of these models will be possible if the non-convexity is relaxed according to our prescription, which we now describe.

Continuing in the spirit of Anguige & Dondl [19], and motivated by the remarks above, in this paper we will thus consider the problem of relaxing the geometrically linear energy, subject to a family of Dirichlet conditions at \( \Gamma_D \subset \partial\Omega \) on the vector-valued displacement \( u: \Omega \to \mathbb{R}^3 \), where \( M = \{ m_j; j = 1, \ldots, M \} \) is a family of slip normals, and (SSC) represents a class of single-slip conditions (to be defined shortly) on the matrix-valued plastic deformations, \( \beta = s \otimes m: \Omega \to \mathbb{R}^{3 \times 3} \) with \( m \in M \) almost everywhere. Here, the subscript ‘sym’ denotes taking the symmetric part of the matrix in parenthesis. For simplicity, we have taken the elastic moduli to be unity. Our result, however, can easily be generalized to arbitrary small-strain elasticity (the nonlinear case will be discussed in §5). Note that the first term in the integrand of (1.1) is the linearized elastic energy of the specimen, and the second term penalizes GNDs, which are quantified by the row-wise curl of the Nye tensor multiplied by the line-tension parameter \( \sigma \geq 0 \), taken individually for each slip plane \( m_j \). To be more explicit, assuming that \( \Omega_j \) is the set where \( \beta \)
can be written as $\beta = s \otimes m_i$, with $m_i$ pointing in the $x^1$-direction, and $s \in m_i$, we define
\[
\|\text{curl}_{m_i, \beta}\|_{\Omega} = \int_{\Omega} |\text{curl}_{m_i, \beta}| \, dx^1,
\]
where the planar curl density, $|\text{curl}_{m_i, \beta}|$, is given by
\[
|\text{curl}_{m_i, \beta}|(\Omega \cap \{x^1 = t\}) = \sup_{\phi \in (C^1(\Omega \cap \{x^1 = t\}))^{2\times 2}} \int_{\Omega \cap \{x^1 = t\}} x_\Omega \cdot s_i(y, z)(\text{div} \phi)_i \, dy \, dz,
\]
and where we have used (and often will use) $y$ and $z$ instead of $x^2$ and $x^3$ for brevity. With this in hand, we define
\[
\|\text{curl}_{\mathcal{M}, \beta}\|_{\Omega} = \sum_{j=1}^{M} \|\text{curl}_{m_i, \beta}\|_{\Omega}.
\]
Thus, for $\beta \in L^1$ or $L^2$, say, $\|\text{curl}_{\mathcal{M}, \beta}\|$ is finite iff $\beta$ is a function of bounded variation (BV) on almost every slip plane and the planar total-variation norms are integrable in the normal direction. For convenience, we will sometimes use the total-variation notation $|\text{curl} \beta| = |D_{y,z} \beta|$.

Note that our rather cumbersome definition of the curl was necessary in order to avoid the introduction of spurious dislocation–cancellation effects. On the other hand, we are making the assumption that colliding dislocations from different slip planes do not cancel—for a discussion of this assumption in a simplified scalar model, see ([14] ch. 4). In the two-dimensional model used in [19], such cancellations naturally do not occur.

The third term in (1.1) represents a rate-independent dissipation, such that $\tau > 0$ measures the (isotropic) critical resolved shear stress. One could also consider, instead, a more general hardening term of the form $\int |\beta|^\alpha$ with $0 < \alpha \leq 2$, and our results can be generalized, in a straightforward manner, to include this. For the special case $\alpha < 1$, the dissipation relaxes to zero, leading to slip-band formation and the treatment of the singularities that arise here (see remark 4.4). For a recent discussion of such sublinear dissipation in the context of strain-gradient plasticity, we refer to [15]. While beyond the scope of this article, we would also expect a relaxation of the elasto-plastic energy to improve the behaviour of models containing rate-dependent flow rules for the plastic strain.

Relaxation of the energy functional (1.1) was only briefly alluded to in [19], in the course of proving an energy upper bound for a shear experiment on a B2 crystal. Here, we wish to derive the relaxed model rigorously for a class of slip systems which includes the case of B2 symmetry. On general theoretical grounds, one expects the relaxed model to reproduce the correct macroscopic plastic behaviour of the single crystal, and hence to facilitate efficient numerical simulation of this behaviour without the need to resolve arbitrarily fine spatial scales (the latter problem having arisen in [20]). Of course, the scope of our model is restricted to single- or oligo-crystal specimens for which the non-convexity condition and strain-gradient terms are appropriate. Then, a full-gradient penalization of the plastic strain (with a single/similar length scale in all directions) is not physically justified, and our (non-) lower semicontinuity and relaxation results apply.

The slip conditions considered in this article are as follows. We will assume that there are $M$ slip planes available, and, as already mentioned, the set of possible slip-plane normals, $m_i$, $i = 1, 2, \ldots, M$, will be denoted by $\mathcal{M}$. In each $m_i$, there will be two arbitrary normalized Burgers vectors, $b_j, j = 1, 2$, and the single-slip (resp. relaxed-slip) conditions on $\beta = s \otimes m$ read

(SSC) For a.e. $x = (x^1, x^2, x^3) \in \Omega$, there holds $m = m_i(x) \in \mathcal{M}$ and $s(x) \in \bigcup_{j \in \{1, 2\}} \text{Sp}(b_{j, i}(x))$.

(RSC) For a.e. $x = (x^1, x^2, x^3) \in \Omega$, there holds $m = m_i(x) \in \mathcal{M}$ and $s(x) \in m_i(x)^\perp$.

such that (SSC) includes the B2 lattices that were considered in [19]. Note that, in particular, (SSC) requires $s(x)$ to be in the union of the spans of two Burgers vectors (a non-convex set), whereas (RSC) requires $s(x)$ to be in the span of the union of two Burgers vectors (a convex set). The full
The relaxation result is proved rigorously in two stages. First, starting with a sufficiently smooth, relaxed pair \((u, \beta)\), we derive a relaxed energy from (1.1) by laminating between the two Burgers vectors, \(b_j, i\), on each slip patch (the second index will be dropped when working on a fixed slip patch, \(m_i = \text{const.}\)). Next, we note that the relaxed energy also makes sense for a larger class of non-smooth relaxed slips, and the second step of our procedure is to show that such slips can be mollified without increasing the relaxed energy. This is highly non-trivial, since applying a standard Friedrichs mollifier to a non-smooth \(\beta\) can violate the relaxed side condition near slip-patch boundaries.

The paper is organized as follows. In §2, we show how to laminate a smooth, relaxed plastic strain, and we derive the relaxed energy. Section 3 is concerned with mollifying non-smooth slips while preserving (RSC), and in §5, we state a summary of our results in the geometrically linear case. Section 5 consists of a brief treatment of the corresponding results in the geometrically nonlinear case, where the elasto-plastic decomposition of the deformation is multiplicative, rather than additive. Finally, in §6, we state some conclusions and discuss the main remaining open problem.

2. Lamination, relaxed energy

Suppose we have a displacement \(u \in H^1\) on \(\Omega\) satisfying some Dirichlet condition, and a \(C^\infty\), relaxed plastic distortion \(\beta = s \otimes m, s \in m^\perp, m \in M\), which has compact support in a Lipschitz domain \(\Omega_1 \subset\subset \Omega\) and denote by \(c_j, j = 1, 2\), the components of \(s\) with respect to the decomposition along normalized Burgers vectors \(b_j\), so that

\[
s = \sum_{j=1}^{2} c_j b_j,
\]

and \(c_j \in C^\infty\).

Suppose also that we choose Cartesian coordinates \((x^1, x^2, x^3) = (x, y, z)\) in \(\Omega_1\) so that the orthonormal basis vectors, \(e_i\), are arranged with \(e_1 = m\).

(a) Laminating \(\beta\)

Now laminate \(\beta\) by filling \(\Omega_1\) with a stack of bi-layers, each parallel to \(m^\perp\) and having thickness \(1/2^n, n \in \mathbb{N}\), and then defining on each successive bi-layer an alternating (in the \(x^3\)-direction), unrelaxed plastic distortion, \(\beta_n\), by

\[
\beta_n = \begin{cases} 
2c_1 b_1 \otimes m : \text{top slice} \\
2c_2 b_2 \otimes m : \text{bottom slice},
\end{cases}
\]

where the \(c_j\) are evaluated on the centre-plane of bi-layer in (2.2), all slices have the same thickness, and, to be concrete, the \((n + 1)\)th laminate is obtained from the \(n\)th by bisecting each of the
bi-layers along a slip plane. For the moment, we have neglected to describe precisely what happens in the neighbourhood of $\partial \Omega_1$—see, however, below.

Since $\beta$ is assumed smooth, it is easy to see that $\beta_n \to \beta \in L^p(\Omega_1)$ for any $p \in [1, \infty)$ as $n \to \infty$. For example, note that simple functions are dense in $\Omega$ and the corresponding integrand the laminated curl density $\mathbf{W}$ we call the integral on the r.h.s. of (2.3) the lam.$\beta$.

Moreover, as $n \to \infty$, whereby the r.h.s. also makes sense for $|\beta|_{1-\text{lam}}$, the final inequality of (2.6).

Defining $L^1 \lambda = L^1(\Omega_1)$, where $\Omega_1 = \Omega_1 \cap \{x^1 = t\}$, we also immediately see that the strain-gradient energy satisfies

$$\int_{\Omega_1} \|D_{y,z} \beta_n\|_{L^p} \, dt \to \sum_{j=1}^2 \int_{\Omega_1} \|D_{y,z} \beta_j\|_{L^p} \, dt,$$

as $n \to \infty$, whereby the r.h.s. also makes sense for

$$c_j : I_{\Omega_1} \ni t \mapsto \mathrm{BV}(\Omega_1) \quad \text{with} \quad \int_{I_{\Omega_1}} |D_{y,z} c_j| \, dt < \infty, \quad \text{such that} \quad I_{\Omega_1} = \{t \in \mathbb{R} : \Omega_t \neq \emptyset\}. \quad (2.4)$$

We call the integral on the r.h.s. of (2.3) the laminated curl of $\beta$, denoted by $|\mathbf{curl}_m \beta|_{\text{lam}}$, and the corresponding integrand the laminated curl density, $|\mathbf{curl}_m \beta|_{\text{lam}}$: thus, $|\mathbf{curl}_m \beta|_{\text{lam}} = \int |\mathbf{curl}_m \beta|_{\text{lam}} \, dx$.

Meanwhile, for the dissipation term we get

$$\int_{\Omega_1} |\beta_n| \, dx \to \sum_{i=1}^2 \int_{\Omega_1} |c_i| \, dx,$$

as $n \to \infty$, and we call the right-hand integral in (2.5) the laminated hardening of $\beta$, denoted by $|\beta|_{1-\text{lam}}$.

We have the following inequalities for the laminated curl and the laminated hardening.

**Proposition 2.1.** If we have a fixed $m_j \in \mathcal{M}$, $s \in \mathrm{BV}((\Omega_1)_t, m_j^{-1})$, some $t \in \mathbb{R}$, then we have, for $\beta = s \otimes m_j$ and $\beta = \sum_i c_i b_i$,

$$|\mathbf{curl}_m \beta| \leq (|b_2^1| + |b_2^1|) |\mathbf{curl}_m \beta|_{\text{lam}} \leq \sqrt{2} |\mathbf{curl}_m \beta|_{\text{lam}} \quad (2.6)$$

and

$$|\mathbf{curl}_m \beta|_{\text{lam}} \leq \sqrt{2} \left(1 + |b_2^1| \right) |\mathbf{curl}_m \beta|. \quad (2.7)$$

Moreover, the same inequalities hold for $|\mathbf{curl}_m \beta|_{\text{lam}}$ relative to $|\mathbf{curl}_m \beta|$ when $|\mathbf{curl}_m \beta|$ is finite on $\Omega_1 \subset \mathbb{R}^3$, and for $|\beta|_{1-\text{lam}}$ relative to $|\beta|_1$ when $\beta \in L^1(\Omega_1)$.

**Proof.** All integrals in the following are taken with respect to $dy \, dz$, at some fixed value of the $x$-coordinate. We may suppose w.l.o.g. that $b_1 = e_2$, which implies, using the notation above, that $s_2 = c_1 + (b_2^2) c_2$ and $s_3 = (b_2^3) c_2$.

Then, using the triangle inequality for the total variation,

$$|\mathbf{curl}_m \beta| = \sup_{c_1 + (b_2^2) c_2, (b_2^3) c_2} \int_{\Omega_1} (c_1 + (b_2^2) c_2) \, \text{div} \, \phi + (b_2^3) c_2 \, \text{div} \, \psi \quad (2.8)$$

$$\leq \sup_{|\phi| \leq 1} \int_{\Omega_1} (c_1 + (b_2^2) c_2) \, \text{div} \, \phi + |b_2^3| \sup_{|\psi| \leq 1} \int_{\Omega_1} c_2 \, \text{div} \, \psi \quad (2.9)$$

$$\leq \sup_{|\phi| \leq 1} \int_{\Omega_1} c_1 \, \text{div} \, \phi + |b_2^2| \sup_{|\psi| \leq 1} \int_{\Omega_1} c_2 \, \text{div} \, \psi \sup_{|\psi| \leq 1} \int_{\Omega_1} c_2 \, \text{div} \, \psi \quad (2.10)$$

$$\leq (|b_2^1| + |b_2^1|) |\mathbf{curl}_m \beta|_{\text{lam}}. \quad (2.11)$$

Since $(|b_2^1| + |b_2^1|)$ is maximized when $b_1$ and $b_2$ meet at an angle of $\pi / 4$, we also obtain the final inequality of (2.6).
Turning to (2.7), since
\[ c_2 = \frac{s_3}{b_2 e_3} \quad \text{and} \quad c_1 = s_2 - \frac{(b_2 e_2)}{(b_2 e_3)} s_3, \] (2.12)
we have
\[
|\text{curl}_{m_j} \beta|_{\text{lam}} = \sup_{|\phi| \leq 1} \int_{\Omega_1} c_1 \text{div} \phi + \sup_{|\psi| \leq 1} \int_{\Omega_1} c_2 \text{div} \psi
\leq \sup_{|\phi| \leq 1} \int_{\Omega_1} s_2 \text{div} \phi + \left(1 + \frac{|b_2 e_2|}{|b_2 e_3|}\right) \sup_{|\psi| \leq 1} \int_{\Omega_1} s_3 \text{div} \psi
\leq \sqrt{2} \left(1 + \frac{|b_1 b_2|}{|b_1 b_2|}\right) |\text{curl}_{m_j} \beta|,
\] (2.13)
as required.

The remaining assertions of the proposition follow in a similar way. ■

Remark 2.2. Inequality (2.7), with \( b_1 b_2 = 0 \), was implicitly used in [19] to get the energy upper bound for a particular shear experiment on a B2 single-crystal.

Next, the laminated curl also has the desirable property of being convex on a given slip patch.

Proposition 2.3. For fixed \( \Omega_1 \subset \mathbb{R}^3 \), \( m_j \in M_j \subset \mathbb{R} \) and \( s \in \text{BV}(\Omega_1, m_j^\perp) \), the mapping \( s \mapsto |\text{curl}(s \otimes m_j)|_{\text{lam}}(\Omega_1) \) is convex. The same assertion holds for the total laminated curl if \( s: \Omega_1 \mapsto m_j^\perp \) is such that \( |\text{curl}_{m_j} \beta||_{\text{lam}}(\Omega_1) < \infty \), and also for the laminated hardening.

Proof. Choose \( m_j \in \mathbb{R}^3 \) and \( s, t \in \text{BV}(\Omega_s, m_j^\perp) \). Set \( r = \lambda s + (1 - \lambda) t \). Then, by linearity, and with (hopefully) obvious notation,
\[
|\text{curl}(r \otimes m_j)|_{\text{lam}} = \sup_{|\phi| \leq 1} \int_{\Omega_1} (\lambda c_1^s + (1 - \lambda) c_1^t) \text{div} \phi + \sup_{|\psi| \leq 1} \int_{\Omega_1} (\lambda c_2^s + (1 - \lambda) c_2^t) \text{div} \psi
\leq \lambda \left( \sup_{|\phi| \leq 1} \int_{\Omega_1} c_1^t \text{div} \phi + \sup_{|\psi| \leq 1} \int_{\Omega_1} c_2^t \text{div} \psi \right)
+ (1 - \lambda) \left( \sup_{|\phi| \leq 1} \int_{\Omega_1} c_1^t \text{div} \phi + \sup_{|\psi| \leq 1} \int_{\Omega_1} c_2^t \text{div} \psi \right)
= \lambda |\text{curl}(s \otimes m_j)|_{\text{lam}} + (1 - \lambda)|\text{curl}(t \otimes m_j)|_{\text{lam}},
\] (2.17)
where we used the triangle inequality for the total variation to get (2.18).

The two last assertions of the proposition are, once again, trivial. ■

(b) The displacement
We now construct a zig-zag perturbation to the displacement \( u \) which accommodates the laminated plastic distortion, for a given \( \beta = s \otimes m \) and decomposition \( s = \sum_{i=1}^2 c_i b_i \), with negligible addition of elastic energy. This perturbation, denoted by \( \hat{u}_n \), is constructed as follows. On a given bi-layer of the \( \beta_n \)-laminate, we set \( \hat{u}_n = 0 \) on the bottom boundary, then on the \( b_j \)-slice we set
\[
\frac{\partial \hat{u}_n}{\partial x^j} = s - 2c_j b_j,
\] (2.20)
where the r.h.s. of (2.20) is evaluated on the centre-plane of the bi-layer. By construction, this implies that \( \hat{u}_n = 0 \) on top of the bi-layer, and so this procedure can be carried out consistently on
the whole slip patch $\Omega_1$. Also, by the smoothness of $\beta$, both $\hat{u}_n$ and $\nabla_{\hat{y},\hat{z}} \hat{u}_n$ are $O(1/2^n)$, uniformly on $\Omega_1$. Thus, by defining $u_n = u + \hat{u}_n$ and $\hat{\beta}_n = \beta_n - \beta$, we get

$$
\nabla \hat{u}_n = \hat{\beta}_n + O\left(\frac{1}{2^n}\right),
$$

and hence that the linearized elastic energy of $(u_n, \beta_n)$ converges to that of $(u, \beta)$ as $n \to \infty$, whereby $u_n \in H^1$ and $\beta_n \in L^\infty$ for all $n$. An illustration of the resulting laminate construction is given in figure 1a–c.

The above applies to $(u, \beta)$ away from $\partial \Omega_1$. Since we are assuming $\text{supp} \beta$ to be Lipschitz and compactly included in $\Omega_1$, we can, near $\partial \Omega_1$, make a small perturbation to $\hat{u}_n$ to ensure that the construction can be carried out on the whole of $\Omega$ while $u_n$ still satisfies the original Dirichlet condition. In particular, and in a completely standard way, for each $n$ and each laminate bi-layer, we clip off the bi-layer in the region $\Omega_1 \setminus \text{supp} \beta$ in order to make a thin cylinder which is compactly included in $\Omega_1$ and $o(1)$-distant from $\partial \Omega_1$, which can be done if $n$ is large enough. Then, for each value of $x^1$ on $\Omega_1$ we linearly interpolate $\hat{u}_n$ down to zero over a distance of magnitude $o(1)$, such that $\hat{u}_n$ has a long tent shape on each bi-layer, and such that $\nabla \hat{u}_n = O(1)$.
outside \text{supp } \beta. Clearly, the perturbation to the elastic energy due to this boundary correction is \( o(1) \), and so the elastic energy of the laminate still converges to that of \((u, \beta)\) as \( n \to \infty \). This is illustrated in figure 1d.

(c) The relaxed energy

From the above considerations, we see that the correct expression for the relaxed energy, which makes sense for non-smooth \( \beta \), and, for example, \( u \in H^1 \), is just

\[
E_{\text{rel}}(u, \beta) = \begin{cases} 
\int_\Omega |(\nabla u - \beta)_{\text{sym}}|^2 \, dx + \sigma \|\text{curl}_M \beta\|_{\text{lam}} + \tau \|\beta\|_{1-\text{lam}} : & \text{(RSC) holds,} \\
+\infty : & \text{otherwise,}
\end{cases}
\]

(2.22)

where \( \|\text{curl}_M \beta\|_{\text{lam}} \) and \( \|\beta\|_{1-\text{lam}} \) denote the laminated curl and laminated hardening, such that, by analogy with \( \|\text{curl}_M \beta\|_1 \), \( \|\text{curl}_M \beta\|_{\text{lam}} \) is obtained by summing the laminated curls over all possible slip planes.

Given this, we can summarize the results of our lamination procedure as follows.

Proposition 2.4. For any pair \((u, \beta)\), \( u \in H^1 \), \( \beta \) smooth, on a Lipschitz domain \( \Omega \), such that \( \beta = s \otimes m_j \) \((m_j \in \mathcal{M})\) on \( \Omega_j \) and \( \text{supp} \mathcal{O}_j \subset \subset (\Omega_j \cup (\partial \Omega \cap \partial \Omega_j)) \) is Lipschitz, there exists a sequence \((u_n, \beta_n)\) such that \( u_n \in H^1 \), \( u_n = u \) on \( \partial \Omega \), \( \beta_n \in L^\infty \) satisfies the non-relaxed side condition \((\text{SSC})\), and

\[
E(u_n, \beta_n) \to E_{\text{rel}}(u, \beta).
\]

(2.23)

Moreover, the relaxed energy (2.22) is convex on each \( \Omega_j \), as a function of \( s \).

We now wish to show that a larger class of (possibly non-smooth) pairs \((u, \beta)\) for which the relaxed energy (2.22) is finite can be approximated by smooth functions which respect (RSC) and the boundary conditions, without increasing the curl.

3. Smoothing

In order to perform the lamination of §2, we must first smooth (the Burgers components of) a given, relaxed \( \beta \) without increasing the relaxed energy (2.22), which is a little involved, due to the relaxed side condition on \( \beta \), as noted in the Introduction.

Our basic procedure on a given slip-patch \( \Omega_j \) is to mollify the \( c_i \) once, thereby perhaps violating the relaxed slip condition, but reducing the curl, then to cut off the resulting functions on a strictly interior approximation to \( \Omega_j \), such that the curl barely increases, and finally to apply a standard mollifier, which re-establishes the relaxed slip condition if the convolution kernel is chosen fine enough, and which once more decreases the curl.

First of all, it is no loss of generality to assume boundedness of the \( c_i \), which will be useful in the sequel. More explicitly:

Proposition 3.1. Suppose we have a domain \( \Omega_1 \subset \Omega \) and scalar functions \( f \) and \( g \) on \( \Omega_1 \) such that \( gf \in L^p(\Omega_1) \) for some \( p \in [1, \infty) \), \( f = 0 \) on \( \Omega \setminus \Omega_1 \) and \( \|\text{curl}f\|_p(\Omega) < \infty \). Then \( \exists \) a sequence \( f_n \in L^\infty(\Omega) \) such that \( f = 0 \) on \( \Omega \setminus \Omega_1 \), \( gf^n \to gf \in L^p(\Omega) \) as \( n \to \infty \), and \( \|\text{curl}f_n\|_p(\Omega) \leq \|\text{curl}f\|_p(\Omega), \forall n \in \mathbb{N} \).

Proof. First define

\[
\tilde{f}^n = \begin{cases} 
f : f < n \\
n : f \geq n
\end{cases}
\]

(3.1)

Then \((\tilde{g}f^n)^p \not\to (gf)^p\) pointwise, and therefore monotone convergence implies \( \|\tilde{f}^n\|_p \to \|gf\|_p \) as \( n \to \infty \).
Let $\Omega_c = \Omega \cap \{x_1 = c\}$. By the co-area formula applied to $F_{t,c}^n = \{x' \in \Omega_c : \tilde{f}^n(x') > t\}$, we get
\[
|D_{y,z}\tilde{f}^n| (\Omega_c) = \int_{-\infty}^{\infty} \|\partial F_{t,c}^n\| (\Omega_c) \, dt
\]
and
\[
= \int_{-\infty}^{\infty} \|\partial F_{t,c}\| (\Omega_c) \, dt
\]

\[
\leq |D_{y,z}f| (\Omega_c). \tag{3.4}
\]
Applying the same argument to
\[
f^n = \begin{cases} \tilde{f}^n : f^n > -n \\ -n : f^n \leq -n, \end{cases}
\]
we arrive at
\[
|D_{y,z}f^n| (\Omega_c) \leq |D_{y,z}f| (\Omega_c), \tag{3.6}
\]
and, by integrating in the $x_1$-direction, $\|\text{curl}f^n\| (\Omega) \leq \|\text{curl}f\| (\Omega)$. Clearly, we also have $g f^n \to g f$, $|g f^n|^p \to |g f|^p$, pointwise, and therefore by monotone convergence, $\|g f^n\|_{L^p} \to \|g f\|_{L^p}$. Together with, say, Proposition 1.3.3 of [23], we also get $g f_n \to g f \in L^p$ as $n \to \infty$, as required.

We now show that a bounded $\beta$ supported on a slip-patch $\Omega_1 \subset \Omega$ can be mollified without enlarging the support, and such that the laminated curl essentially decreases, provided $\Omega_1$ satisfies an additional boundary-regularity condition. This condition states that $\Omega_1$ should have finite perimeter in $\Omega$, and that the perimeter should be equal to the area of the topological boundary: $\mathcal{H}^2(\partial \Omega_1) = F(\Omega_1)$, which is equivalent to the statement $\mathcal{H}^2(\partial \Omega_1 \setminus F\Omega_1) = 0$, where $F\Omega_1$ is the reduced boundary [23], and which very roughly means that $\Omega_1$ has no cuts. The same condition was employed by Schmidt [24] in order to obtain strictly interior approximations of BV domains and functions. Our approach to obtaining the desired cut-off is somewhat different to (and less technical than) that of Schmidt, and only works in the scalar case, which is enough for us since the components of $\beta$ ($c_1$ in (2.1)) can be mollified separately when considering the relaxed energy. The advantage of our method is that we require less regularity on $\beta$ than does Schmidt, namely just $\beta \in L^p(\Omega)$, rather than $\beta \in BV \cap L^\infty(\Omega)$. The case where $\partial \Omega_1$ intersects the sample boundary, $\partial \Omega$, will be considered separately towards the end of this section.

**Proposition 3.2.** Let $\Omega_1 \subset \subset \Omega \subset \mathbb{R}^3$ be a bounded domain with finite perimeter satisfying $\mathcal{H}^2(\partial \Omega_1 \setminus F\Omega_1) = 0$, and let $f \in L^p(\Omega)$ with $\text{supp} f \subset \Omega_1$ be such that $\int |D_y z f| \, dx^1 < \infty$. Then there exists a family $f_\varepsilon \in C_0^\infty (\Omega_1)$, $\varepsilon > 0$, such that
\[
f_\varepsilon \to f \in L^p (\Omega_1) \quad \text{as} \quad \varepsilon \to 0 \tag{3.7}
\]
and
\[
\int_{\Omega_1} |D_y z f_\varepsilon| \, dx^1 \leq \int_{\Omega_1} |D_y z f| \, dx^1 + \varepsilon. \tag{3.8}
\]

**Proof.** By proposition 3.1, we may assume w.l.o.g. that $f$ is bounded.

The first stage of the proof is to construct a strictly interior approximation to $\Omega_1$. To do this, we employ the Structure Theorem for sets of finite perimeter [25]. Thus, since $F\Omega_1$ is $\mathcal{H}^2$-almost all of $\partial \Omega_1$, we have that $\forall \varepsilon > 0$ there exist $M(\varepsilon) \in \mathbb{N}$ and compact, disjoint sets $K^\varepsilon_k \subset \partial \Omega_1$, $k = 1, 2, \ldots M(\varepsilon)$, such that each $K^\varepsilon_k$ is a subset of a $C^1$-surface, $\nu_{\Omega_1} |_{K^\varepsilon_k}$ is normal to $K^\varepsilon_k$, and
\[
\partial \Omega_1 = \left( \bigcup_{k=1}^{M(\varepsilon)} K^\varepsilon_k \right) \cup N, \quad \text{where} \quad \mathcal{H}^2 (N) \leq \varepsilon, \tag{3.9}
\]
for some $\mathcal{H}^2$-measurable set $N \subset \partial \Omega_1$, and where $\nu_{\Omega_1} |_{K^\varepsilon_k}$ is the generalized outward unit normal to $\partial \Omega_1$.

By compactness, the $K^\varepsilon_k$ are pairwise separated by a distance $d(\varepsilon) > 0$, and we may assume that $K^\varepsilon_k = \phi_k (\bar{U}_k)$, where $U^\varepsilon_k \subset U_k$ is an open coordinate patch in $\mathbb{R}^2$ (as is $U_k$), $\phi_k$ is a $C^1$-function and the overbar denotes topological closure.

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Now, with $\varepsilon > 0$ fixed, for each $k = 1, 2, \ldots, M$, cover $U_k^\varepsilon$ with an $\hat{\varepsilon}$-coordinate mesh, where $\hat{\varepsilon} > 0$ is small, and remains to be determined. Let $S_{ij}^k$ be the open mesh squares, with centres denoted by $y_{ij}^k$, and define the slightly enlarged open squares $\tilde{S}_{ij}^k = (1 + \hat{\varepsilon})S_{ij}^k$ by scaling about the $y_{ij}^k$ (here, $i$ and $j$ index the mesh lines in the $y^1$ and $y^2$ coordinate directions). Thus, the $\phi_k(\tilde{S}_{ij}^k \cap U_k^\varepsilon)$ comprise a relatively open cover of $K_k^\varepsilon$.

By compactness of the $U_k^\varepsilon$ and the fact that $\phi_k$ is $C^1$, we have that for every $\delta > 0$ there exists an $\hat{\varepsilon}(\delta, M)$ such that, for $y, \tilde{y} \in U_k^\varepsilon$,

$$|y - \tilde{y}| < 2\hat{\varepsilon}(\delta, M) \Rightarrow |\nabla \phi_k(y) - \nabla \phi_k(\tilde{y})| < \delta. \quad (3.10)$$

For fixed $\varepsilon$ and $\delta$, we will henceforth choose the $\hat{\varepsilon}$-mesh on the $U_k^\varepsilon$ so fine that (3.10) holds.

Now, denoting by $\text{Tan}(S, x)$ the tangent plane of a manifold $S$ at the point $x$, for each $i, j, k$, we take the orthogonal projection of $\phi_k(\tilde{S}_{ij}^k)$ onto $\text{Tan}(K_k^\varepsilon, x_{ij}^k)$, where $x_{ij}^k = \phi_k(y_{ij}^k)$, and call the result $p_{ij}^k$, then construct the cylinder

$$C_{ij}^k = p_{ij}^k \times \hat{t}_{ij}^k, \quad \text{where } \hat{t}_{ij}^k = \{x_{ij}^k + t\hat{\varepsilon} \nu_{ij}^k; t \in [0, 1]) \}, \quad (3.11)$$

which contains $\phi_k(\tilde{S}_{ij}^k)$ if $\hat{\varepsilon}(\delta, M)$ is chosen as above, and is such that the outer face, defined by

$$O_{ij}^k = p_{ij}^k \times \{x_{ij}^k + \hat{\varepsilon} \nu_{ij}^k \} \quad (3.12)$$

does not intersect $\Omega_1$, which can always be arranged if $\delta$ is small enough, by compactness. Also, if $\delta$ is small enough, then any given $C_{ij}^k$ intersects only those other $C_{pq}^k$ such that $S_{pq}^k$ is one of the (at most eight) immediate neighbours of $S_{ij}^k$.

Next, by the definition of Hausdorff measure, we may cover the $H^2$-small set $N$ with countably many balls, $B_{\rho_i}(q_i)$ with $\rho_i \leq 1$, such that $\sum_{i=1}^{\infty} \rho_i^2 \leq 2\varepsilon$. Moreover, by compactness and (3.9), we can cover $\partial\Omega_1$ with the $C_{ij}^k$ and a finite sub-collection of the $B_{\rho_i}(q_i)$, say $B_{\rho_{m_i}}(q_{m_i})$ for $i = 1, 2, \ldots, M_1$, so that, defining

$$\Omega_\varepsilon = \Omega_1 \setminus \left( \bigcup_{i=1}^{M_1} B_{\rho_i}(q_i) \right) \cup \left( \bigcup_{ik} C_{ij}^k \right), \quad (3.13)$$

we see that

$$\Omega_\varepsilon \subset \subset \Omega_1 \quad \text{and} \quad |\Omega_1 \setminus \Omega_\varepsilon| = O(\varepsilon + \hat{\varepsilon}\delta). \quad (3.14)$$

For each $C_{ij}^k$, we now subtract the shadow of its (thin) lateral portion, along with the shadow of the intersection with its (at most eight) neighbouring $C_{pq}^k$ and the $B_{\rho_i}(q_i)$, the shadow being taken with respect to the projection of $\nu_{ij}^k$ onto the $(x^2, x^3)$-plane. Denote by $\hat{C}_{ij}^k$ the resulting disjoint sets, and observe that the sections $C_{ij}^k \cap \{x^1 = \text{const.}\}$ are necessarily rectangular, by construction.

It follows from the $C^1$-property of the $\phi_k$ and (3.10) that the amount of projected area lost from $\partial\Omega_\varepsilon$ by removing shadows in this way is bounded by $C_{M(\varepsilon)}(\hat{\varepsilon}(\delta, M(\varepsilon)) + \delta) + 2\varepsilon$ for some $C_{M(\varepsilon)}$, and this can be made less than $3\varepsilon$ for a given $\varepsilon$, if $\delta$ and then $\hat{\varepsilon}$ are made sufficiently small.

Next, suppose that we smooth $f$ with a Friedricks mollifier so fine that the resulting function $\tilde{f}$ has support contained in $\Omega_1 \cup (\cup_{ik} \hat{C}_{ij}^k) \cup (\cup_i B_{\rho_{m_i}}(q_{m_i}))$. By a standard result on the mollification of measures, namely Thm.2.2(b) of [23], this reduces the curl, and $\tilde{f}$ can be taken arbitrarily close to $f$ in $L^p$, by standard convolution theory. Also, by what we just said above, along with Young’s inequality, if we cut off $\tilde{f}$ on $\partial\Omega_\varepsilon$, then the amount of curl on $\partial\Omega_\varepsilon$ which goes unaccounted for by neglecting shadows is bounded by $3\varepsilon \|f\|_{L^\infty}$ and it remains to estimate the curl generated on the flat faces $C_{ij}^k \cap \partial\Omega_\varepsilon$. 


Thus, on each $\hat{C}_ij$ we consider the horizontal slices $x^1 = t$, which, as we already noted, are rectangles. Since $\tilde{f}$ is a smooth scalar function and $\tilde{f} = 0$ on the outer edge of $\hat{C}_ij \cap \{x^1 = t\}$, we have
\[
\sum_{ijk} \int dt \int |D_{y,z}f|((x^1 = t) \cap \hat{C}_ijk) \geq \sum_{ijk} \int_{\partial \Omega_t \cap \hat{C}_ijk \cap \{x^1 = t\}} |\tilde{f}| d\ell, \tag{3.15}
\]
where $d\ell$ is the Euclidean line element.

The curl generated by cutting off $\tilde{f}$ on $\partial \Omega_\varepsilon$ is just the r.h.s. of (3.15) plus a contribution coming from the parts of $\partial \Omega_\delta$ lost upon subtracting shadows, which we know is bounded by $3\varepsilon \|f\|_\infty$ if $\varepsilon$ and $\delta$ are chosen small enough. Thus, for appropriate $\varepsilon$ and $\delta$, we may perform the cut-off and then (Friedrichs) mollify once more to obtain a smooth $f_\varepsilon$ with smooth support which satisfies (3.7) and (3.8), modulo a relabelling of $\varepsilon$.

Proposition 3.2 can also be used to smooth $\beta$ on slip patches which intersect the sample boundary, $\partial \Omega$. Here, we must take care not to introduce spurious curl when mollifying the slip near $\partial \Omega$; a key tool in this connection is standard extension theory for BV-functions [23].

**Proposition 3.3.** Suppose the sample domain $\Omega$ is Lipschitz, and that we have a slip patch $\Omega_1 \subset \Omega$ satisfying the regularity condition $H^2(\partial \Omega_1 \setminus \mathcal{F}(\Omega_1)) = 0$. Let $f \in L^p(\Omega)$, with supp $f \subset \Omega_1$, be such that $\|\text{curl} f\|_\Omega < \infty$. Then there exists a family $f_\varepsilon \in C^\infty(\Omega)$, $\varepsilon > 0$, with $f_\varepsilon = 0$ on $\Omega \setminus \Omega_1$, such that
\[
dist(\text{supp } f_\varepsilon, \partial \Omega \setminus \partial \Omega_1) \geq \varepsilon, \tag{3.16}
\]
\[
f_\varepsilon \rightarrow f \text{ in } L^p(\partial \Omega) \text{ as } \varepsilon \rightarrow 0 \tag{3.17}
\]
and
\[
\int_{\Omega_t} |D_{y,z}f_\varepsilon| \, dx^1 \leq \int_{\Omega_t} |D_{y,z}f| \, dx^1 + \varepsilon. \tag{3.18}
\]

**Proof.** Once more, we may assume that $f$ is bounded. Now focus attention on a slice $\Omega_t$ through $\Omega$. By Proposition 3.21 of [23], for a.e. $t$ we can extend $f$ on $\Omega_t$, continuously in BV, to a function $\tilde{f} \in \text{BV}(\mathbb{R}^2)$, with compact support, such that $|D_{y,z}\tilde{f}|(\partial \Omega_t) = 0$ (i.e. the sample boundary remains unchanged). Also, by inspecting the proof of this proposition, one can see that the BV operator norm of the extension map $f \mapsto \tilde{f}$ is controlled by the Lipschitz constant of $\partial \Omega_t$, which in turn is controlled by that of $\partial \Omega$. In other words,
\[
\exists L(\partial \Omega) > 0 : \|\tilde{f}\|_{\text{BV}(\mathbb{R}^2)} \leq L \|f\|_{\text{BV}(\Omega_t)}, \quad \text{for a.e. } t \in \mathbb{R} : \Omega_t \neq \emptyset. \tag{3.19}
\]

We also use $\tilde{f}$ to denote the whole family of extensions over all $\Omega_t$.

Next, applying a three-dimensional Friedrichs mollifier to $\tilde{f}$, and denoting the result by $\tilde{f}_\varepsilon$, we get
\[
\int_{\Omega} |D_{y,z}\tilde{f}_\varepsilon| \leq \int_{\Omega'} |D_{y,z}\tilde{f}| \tag{3.20}
\]
\[
= \int dt \int_{(\Omega')_t} |D_{y,z}\tilde{f}| \, dy \, dz, \tag{3.21}
\]
where $\Omega'$ is the $\varepsilon$-neighbourhood of $\Omega$, and the first inequality follows from Theorem 2.2(b) of [23].

Furthermore, for each $t$, it follows that
\[
|D_{y,z}\tilde{f}|((\Omega'_t)) \cap |D_{y,z}\tilde{f}|(\bar{\Omega}_t) = |D_{y,z}f|(\Omega_t), \tag{3.22}
\]
as $\varepsilon \rightarrow 0^+$, since $D_{y,z}\tilde{f}$ does not charge $\partial \Omega$.

Thus, we get, by monotone convergence and (3.19),
\[
\limsup_{\varepsilon \rightarrow 0^+} \int_{\Omega} |D_{y,z}(\tilde{f}_\varepsilon)| \, dt \, dy \, dz \leq \int \int |D_{y,z}f|(\Omega_t), \tag{3.23}
\]
as required. $L^p$-convergence of the mollified extensions is once again completely standard.
Once we have our smooth \( \tilde{f}_\varepsilon \) on \( \Omega \), we can find an \( \Omega_1^\varepsilon \subset \Omega_1 \) such that \( 2\varepsilon \leq \text{dist}(\Omega_1^\varepsilon, \partial \Omega_1 \setminus \partial \Omega) \leq 3\varepsilon \), and then cut off \( \tilde{f}_\varepsilon \) on \( \partial \Omega_1^\varepsilon \setminus \partial \Omega \) for an \( \Omega_1^\varepsilon \subset \Omega_1 \), by precisely the method of proposition 3.2, thereby essentially reducing the curl, and barely changing the \( L^p \)-norm.

Finally, we repeat the first part of the proof (extend across the domain boundary then smooth), but this time applied to the interior cut-off of \( \tilde{f}_\varepsilon \), rather than the initial \( f \). This gives the required result if the mollifier is chosen fine enough.

**Remark 3.4.** The smoothed extension constructed above can be used to laminate \( \beta \) in the neighbourhood of the domain boundary, on a slip patch \( \Omega_1 \), in the case when \( \partial \Omega_1 \) intersects \( \partial \Omega \).

### 4. The full relaxation statement

By applying propositions 3.2 or 3.3 separately to each of the Burgers components of the plastic distortion \( \beta \), for a decomposition \( s = \sum_i c_i b_i \), we can cut-off and laminate \( \beta \) independently on neighbouring slip patches, and it follows that our lamination and smoothing results can be combined to arrive at the desired relaxation theorem.

**Theorem 4.1.** Suppose we have a Lipschitz domain \( \Omega \in \mathbb{R}^3 \) and that we have \((u, \beta)\) on \( \Omega \), such that \( u \in H^1 \) satisfies a Dirichlet condition on a Lipschitz subset of \( \partial \Omega \), \( \beta = s \otimes m \) satisfies (RSC) with \( m = m_i \in \mathcal{M} \) on each \( \Omega_i \), and the relaxed energy (2.22) is finite. Assume furthermore that the sets \( \{ \Omega_i \}_{i=1}^M \) where \( \beta = s \otimes m_i \) satisfy the regularity condition \( H^2(\partial \Omega_i \setminus \mathcal{F}_\partial) = 0 \).

Then, for each \( \varepsilon > 0 \), there exists a pair of test functions \((u_\varepsilon, \beta_\varepsilon)\) satisfying the same Dirichlet condition and (SSC), such that \( u_\varepsilon \in H^1 \), \( \beta_\varepsilon \in L^\infty \) and

\[
E(u_\varepsilon, \beta_\varepsilon) \leq E_{\text{rel}}(u, \beta) + \varepsilon. \tag{4.1}
\]

**Remark 4.2.** Of course, what we would really like is to find the quasi-convex envelope of \( E \) for general domains \( \Omega \), and then prove the existence of minimizers. Note that, for the simple case where a slip plane connects free surfaces, one can achieve \( E_{\text{rel}} = 0 \) with a simple, relaxed shear (see, for example, the \( L > 2 \) result from [19])—hence our theorem shows that the required envelope is in fact zero in this case.

**Remark 4.3.** Note that, while the class of \( \beta \) we are considering here is rather large (keeping in mind that changes to \( \beta \) on a null set are also allowed), we could, instead, allow those \( \beta \) that can be approximated by smooth, compactly supported plastic distortions. As shown in §3 (which we think is of some independent interest), this class is no smaller than the one used in the above theorem—it is, however, much harder to characterize. In the context of the previous remark, we believe that minimizers of the relaxed energy (with sufficiently smooth data) would satisfy the requirements of theorem 4.1.

**Remark 4.4.** A \( \int |\beta|^\alpha \) penalty term with \( \alpha < 1 \) relaxes to zero, as one can see quite easily. Thus, for a given single-slip one can laminate between the same kind of slip and zero slip, which strictly reduces the energy. Iterating this procedure to get ever-finer laminates gives zero energy in the limit. It follows that if we replace \( \int |\beta| \) with \( \int |\beta|^\alpha \) then \( E_{\text{rel}} \) should just contain the elastic energy and the laminated curl.

### 5. Nonlinear elasto-plasticity

We now show that the foregoing analysis of the geometrically linear problem can be carried over, with little additional work, to the nonlinear case. Here, we have in mind the usual multiplicative decomposition of the deformation gradient, \( F = F^e(F^p)^{-1} \), so that the elastic deformation is given by

\[
F^e(u, \beta) = (I + \nabla u)(I - \beta), \tag{5.1}
\]

where \( u \) is once again the material displacement, and, on a given slip patch, \( \beta = s \otimes m \) (with \( s \in m^\perp \)) is the plastic slip, such that the family of possible slip systems has the same form as before.
The non-relaxed, nonlinear energy to be considered is now

\[
E^{(nl)}(u, \beta) = \begin{cases} 
\int_{\Omega} W_e(F^{el}) \, dx + \sigma \|\text{curl}M\beta\|_{L^\infty} + \tau \int_{\Omega} |\beta| \, dx : & \text{(SSC) holds,} \\
+\infty : & \text{otherwise,}
\end{cases}
\tag{5.2}
\]

for some elastic-energy function \(W_e\), and the relaxed, nonlinear energy is

\[
E^{(nl)}_{\text{rel}}(u, \beta) = \begin{cases} 
\int_{\Omega} W_e(F^{el}) \, dx + \sigma \|\text{curl}M\beta\|_{L^\infty} + \tau \|\beta\|_{1-L^\infty} : & \text{(RSC) holds,} \\
+\infty : & \text{otherwise,}
\end{cases}
\tag{5.3}
\]

Here, we still use the linearized version of the density measure of the GNDs. For an in-depth discussion on the controversy about which term is correct in the geometrically nonlinear setting we refer to [26,27].

**Theorem 5.1.** Suppose that \(W_e: \mathbb{R}^{3\times 3} \mapsto [0, \infty)\) is continuous and satisfies the \(p\)-growth condition

\[-c_1 + c_2|F|^p \leq W_e(F) \leq C_1 + C_2|F|^p. \tag{5.4}\]

Suppose furthermore, as in theorem 4.1, that we have a Lipschitz domain \(\Omega \subset \mathbb{R}^3\) and that we have \((u, \beta)\) on \(\Omega\), such that \(u \in W^{1,p}\) satisfies a Dirichlet condition on a Lipschitz subset of \(\partial \Omega\), \(\beta = s \otimes m\) satisfies (RSC) with \(m = m_i \in \mathcal{M}\) on each \(\Omega_i\), and the relaxed energy (5.3) is finite. Assume again that the sets \(\{\Omega_i\}_{i=1}^M\) where \(\beta = s \otimes m_i\) satisfy the regularity condition \(H^2(\partial \Omega_i \setminus \mathcal{F} \Omega_i) = 0\).

Then, for each \(\varepsilon > 0\), there exists a pair of test functions \((u_\varepsilon, \beta_\varepsilon)\) satisfying the same Dirichlet condition and (SSC), such that \(u_\varepsilon \in W^{1,p}\), \(\beta_\varepsilon \in L^\infty\) and

\[
E^{(nl)}(u_\varepsilon, \beta_\varepsilon) \leq E^{(nl)}_{\text{rel}}(u, \beta) + \varepsilon. \tag{5.5}\]

**Proof.** First, suppose we have a pair of relaxed test functions \((u, \beta)\) on a slip patch \(\Omega_1\) of the type stipulated above, with finite energy, such that \(\beta = s(x) \otimes m\) is smooth. Now laminate exactly as before to get a new function \(\beta_n = \beta + \hat{\beta}_n\) and perturbations \(\hat{u}_n\) and \(\hat{\beta}_n\), such that (2.21) holds.

Put \(y_n = y \circ \hat{y}_n \Leftrightarrow \nabla y_n = (\nabla y)(\nabla \hat{y}_n)\), such that \(\hat{y}_n\) is defined by \(\hat{y}_n = Ix + \hat{u}_n\). If we set \(u_n = -Ix + y_n\), then, away from an \(o(1)\)-thin neighbourhood of \(\partial \Omega_1\), we get

\[
F^{el}(u_n, \beta_n) = (I + \nabla u_n)(I - \beta_n) = (\nabla y)(\nabla \hat{y}_n)(I - \beta - \hat{\beta}_n) = \nabla y(I + \nabla \hat{u}_n)(I - \beta - \hat{\beta}_n) = \nabla y(I - \beta) - (\nabla y)(\nabla \hat{u}_n)\beta + O\left(\frac{1}{2^n}\right) = F^{el}(u, \beta) + O\left(\frac{1}{2^n}\right),
\tag{5.6}
\]

since \((\nabla \hat{u}_n)\beta = (\hat{\beta}_n + O(1/2^n))\beta\) and \((\hat{\beta}_n)(\beta) = 0\). Also, we may interpolate \(u_n\) to zero near \(\partial \Omega_1\) in the same way as before, so that \(|\nabla u_n|\) is \(O(1)\) on a set of Lebesgue measure \(o(1)\). Hence, by (5.4), dominated convergence and continuity, we get, by taking an a.e. converging subsequence, \(\int_{\Omega} W_e(u_n, \beta_n) \rightarrow \int_{\Omega} W_e(u, \beta)\) as \(k \rightarrow \infty\).

Next, we need to show that, given relaxed \((u, \beta)\) with finite energy and \((I + \nabla u)(I - \beta) \in L^p\), we can smooth \(\beta\) without increasing the energy. By proposition 3.1 and dominated convergence, \(\beta\) may be assumed bounded, and then we know that our previous smoothing method gives a uniformly bounded, smooth family \(\beta_\varepsilon\) which takes care of the curl and hardening terms. It is also easy to see that the method gives \(\beta_\varepsilon \rightarrow \beta \in L^q\) as \(\varepsilon \rightarrow 0\). Thus, possibly passing to a subsequence \(\varepsilon_k\) to get a.e. convergence, dominated convergence gives \(\int_{\Omega} W_e(u, \beta_{\varepsilon_k}) \rightarrow \int_{\Omega} W_e(u, \beta)\) as \(\varepsilon_k \rightarrow 0\) since we have \(I + \nabla u \in L^p\).

This completes the proof. ■
6. Conclusion and open problems

In this article, we have shown that in both geometrically linear and geometrically nonlinear models for strain-gradient crystal plasticity with cross-hardening, a single-slip side condition should always be relaxed to a single-plane condition. In other words, a simple-shear deformation in any slip plane, and in any slip direction, can be approximated by a laminate deformation such that the slip is parallel to a Burgers vector at every point, and such that an explicitly computable addition to the strain-gradient term arises.

The main open question remaining is whether our (at least partially) relaxed model does in fact admit a minimizer. For some simple—but experimentally relevant—situations, it is easy to show that a minimizer for our relaxed model exists, since a test function with vanishing energy can be explicitly constructed [19]. More generally, it is possible to see that fine oscillations between potential wells of our non-convex relaxed slip condition are penalized by the convex curl-term in the relaxed energy. An attempt to turn this observation into a rigorous existence proof is currently underway.

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