Decay of \((p, q)\)-Fourier coefficients

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We show that essentially the speed of decay of the Fourier sine coefficients of a function in a Lebesgue space is comparable to that of the corresponding coefficients with respect to the basis formed by the generalized sine functions \(\sin_{p,q}\).

1. Introduction and preliminaries

The decay properties of classical Fourier coefficients are both well known and important. Consider, for example, a function \(f\) that is Lebesgue-integrable over the interval \(I = [0, 1]\). By the Riemann–Lebesgue lemma, its \(n\)th Fourier sine coefficient \(a_n\) tends to zero as \(n \to \infty\). If \(f\) has some smoothness, then more precise information is available about the rate of decay [1, ch. 3.2]. In this paper, we investigate the relationship between the decay properties of the Fourier sine coefficients of a function and those of the corresponding coefficients when the classical sine functions are replaced by the \(\sin_{p,q}\) functions, where \(p, q \in (1, \infty)\). Interest in these \(\sin_{p,q}\) functions stems not only from their basis properties detailed below, but also from their appearance in connection with the \((p, q)\)-Laplacian [2] and from the fact that their derivatives provide extremal functions for the determination of the norm of the classical Hardy operator acting from \(L^p\) to \(L^q\) (see [3]).

Some notation help us to describe in more detail what is achieved. For each \(p, q \in (1, \infty)\), define \(F_{p,q}: I \to \mathbb{R}\) by

\[
F_{p,q}(x) = \int_0^x (1 - t^q)^{-1/p} \, dt, \quad x \in I.
\]
Because $F_{p,q}$ is strictly increasing, it has an inverse, denoted by $\sin_{p,q}$, that is defined on $[0, \pi_{p,q}/2]$, where

$$\pi_{p,q} = 2 \int_0^1 (1 - t^q)^{-1/p} \, dt = \frac{2 \Gamma(1/p') \Gamma(1/q)}{q \Gamma(1/p + 1/q)} , \quad p' = \frac{p}{p-1}.$$  

This inverse is first extended to $[0, \pi_{p,q}]$ by symmetry about $\pi_{p,q}/2$, then extended to $[-\pi_{p,q}, \pi_{p,q}]$ by oddness, and finally to the whole of $\mathbb{R}$ by $2\pi_{p,q}$-periodicity; the resulting function on $\mathbb{R}$ is again denoted by $\sin_{p,q}$. When $p = q$, we write $\sin_p$ instead of $\sin_{p,p}$ and $\pi_p$ in place of $\pi_{p,q}$; plainly, $\sin_2$ is simply the classical sine function. Additional information about $\sin_{p,q}$ is given in [3].

The classical sine functions $\sin(n\pi \cdot)$ ($n \in \mathbb{N}$) form a basis of every Lebesgue space $L^r(I)$ ($1 < r < \infty$), so do the functions $\sin_p(n\pi \cdot)$ ($n \in \mathbb{N}$) provided that $p \in (p_0, \infty)$, where $p_0$ is defined by the equation $\pi_{p_0} = 2 \pi^2/(\pi^2 - 2)$; $p_0$ is approximately 1.198 (see also [5]). Moreover, it is now known (see [6]) that this basis property is also possessed by the functions $\sin_{p,q}(n\pi_{p,q})$ ($n \in \mathbb{N}$) provided that

$$\frac{p'}{2} < \frac{4}{(\pi^2 - 2)} .$$

In particular, this shows that for all $p \in (1, \infty)$, the functions $\sin_{p,p'}(n\pi_{p,p'})$ ($n \in \mathbb{N}$) form a basis of every $L^r(I)$; note that in contrast to the result for the $\sin_{p,q}$, here $p$ may be arbitrarily close to 1. These basis properties mean that given any $r \in (1, \infty)$ and any $f \in L^r(I)$, then for appropriate $p$ and $q$, there are unique sequences of real numbers $(a_n)$ and $(b_{n,p,q})$ such that (with convergence in $L^r(I)$)

$$f = \sum_{k=1}^{\infty} a_k e_k = \sum_{k=1}^{\infty} b_{k,p,q} f_{k,p,q} ,$$

where $e_k = \sin(k\pi \cdot)$ and $f_{k,p,q} = \sin_{p,q}(k\pi_{p,q})$. The $b_{k,p,q}$ may be thought of as $p, q$-Fourier coefficients of $f$: note that when $p \neq 2$, the lack of orthogonality of the $f_{k,p,q}$ means that the $b_{k,p,q}$ do not have the attractive integral representation possessed by the $e_k$.

Proofs of these basis assertions rely on a technique originally adopted in [4] and which we now outline. Given any $f: I \to \mathbb{R}$, extend it to $\tilde{f}: [0, \infty) \to \mathbb{R}$ by setting $\tilde{f} = f$ on $[0, 1]$, and $\tilde{f}(t) = -\tilde{f}(2k - t)$ for $t \in (k,k + 1)$, $k \in \mathbb{N}$. For each $m \in \mathbb{N}$, define $M_m: L^r(I) \to L^r(I), r \in (1, \infty)$, by $M_m g(t) = \tilde{g}(mt)$. Then, $M_me_n = e_{mn}$, $m, n \in \mathbb{N}$ and $M_m$ is a linear isometry, that is, $\|M_m g\|_{L^r(I)} = \|g\|_{L^r(I)}$. Because each $F_{n,p,q}$ is differentiable, it has a Fourier sine expansion

$$f_{n,p,q}(t) = \sum_{k=1}^{\infty} \widehat{f_{n,p,q}}(k) \sin(k\pi t),$$

where

$$\widehat{f_{n,p,q}}(k) = 2 \int_0^1 f_{n,p,q}(t) \sin(k\pi t) \, dt.$$  

It is shown in [6] that

$$\widehat{f_{n,p,q}}(k) = \begin{cases} f_{1,p,q}(m) & \text{if } k = mn \text{ and } m \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

Put

$$\tau_m = \tau_m(p, q) = f_{1,p,q}(m) ,$$

and note that $\tau_m = 0$ if $m$ is even. Then, the map $T$ defined by

$$Tg = \sum_{m=1}^{\infty} \tau_m(p, q) M_m g$$

is a bounded linear map of $L^r(I)$ to itself with the property that

$$Te_n = f_{n,p,q}, \quad n \in \mathbb{N}.$$
Further, $T: L'(l) \to L'(l)$ is a homeomorphism for all $r \in (1, \infty)$ if

$$
\sum_{k=1}^{\infty} |\tau_{2k+1}(p, q)| < |\tau_1(p, q)|. 
$$

(1.5)

This condition is satisfied if

$$
\frac{p'}{q} < \frac{4}{(\pi^2 - 8)} \quad \text{or} \quad p = q \in (p_0, \infty).
$$

(1.6)

If $T: L'(l) \to L'(l)$ is a homeomorphism, then $(f_{n,p,q})_{n \in \mathbb{N}}$ is a basis of $L'(l)$. For proofs of these claims, see [6].

Our basic result is expressed in terms of sequence spaces. We recall that given any $\lambda \in (1, \infty)$, the space $l_{\lambda,\infty}$ is the Lorentz space of all sequences $c = (c_k)_{k \in \mathbb{N}}$ such that

$$
\|c\|_{\lambda,\infty} = \sup_{k \in \mathbb{N}} k^{1/\lambda} c_k^{*} < \infty,
$$

where $(c_k^*)$ denotes the non-increasing rearrangement of the sequence $c$ (for more details, see, e.g. [7]). As usual, $l_1$ will stand for the space of all sequences $c$ such that $\|c\|_1 = \sum_{k=1}^{\infty} |c_k| < \infty$.

Theorem 2.3 shows that if $f \in L'(l) \ (1 < r < \infty)$, $\lambda \in (1, \infty)$ and (1.5) holds, then the Fourier sine coefficients $a_k$ and their $(p,q)$-counterparts $b_{kp,q}$ (both appearing in (1.2)) have the same rate of decay: $(a_k)_{k \in \mathbb{N}} \in l_{\lambda,\infty}$ if and only if $(b_{kp,q})_{k \in \mathbb{N}} \in l_\lambda$. Moreover, $(a_k)_{k \in \mathbb{N}} \in l_1$ and only if $(b_{kp,q})_{k \in \mathbb{N}} \in l_1$. In the first of these results, the $k$th rearranged coefficient decays like $k^{-1/\lambda}$, where $\lambda > 1$. It is natural to ask what happens when the decay of the coefficients is of the form $k^{-\alpha}$, where $\alpha$ may be greater than 1. Theorem 2.4 provides some information when the sequences $(|a_k|)_{k \in \mathbb{N}}$ and $(|b_{kp,q}|)_{k \in \mathbb{N}}$ are non-increasing: it shows that if $\alpha \in (0,2)$, then $|a_k| \leq k^{-\alpha}$ if and only if $|b_{kp,q}| \leq k^{-\alpha}$. The requirement that both sequences be non-increasing is very strong: without it, we show, in remark 2.5, that results can be obtained that involve a loss of sharpness of the exponents.

We hope that the similarities uncovered here between properties of the classical basis $(e_k)$ and that generated by the $\sin_{p,q}$ functions will stimulate further work on and applications of the generalized sine functions.

## 2. The main results

Throughout this section, the norm on $L'(l)$ is denoted by $\| \cdot \|_r$, or even by $\| \cdot \|$ if the context is clear. We begin by showing that the inverse of the map $T$ defined above has properties similar to those of $T$.

**Lemma 2.1.** Let $p, q, r \in (1, \infty)$, let $T$ be as in (1.4) and suppose that (1.5) holds. Then, there are constants $\beta_{2k+1} = \beta_{2k+1}(p, q), k \in \mathbb{N}$, with

$$
\sum_{k=1}^{\infty} |\beta_{2k+1}| < \infty,
$$

(2.1)

such that $T^{-1}: L'(l) \to L'(l)$ has the representation

$$
T^{-1} = \frac{1}{\tau_1} \text{Id} + \sum_{k=1}^{\infty} \beta_{2k+1} M_{2k+1},
$$

where $\text{Id}$ is the identity map of $L'(l)$ to itself.

**Proof:** From (1.4) and (1.5), we see that $T: L'(l) \to L'(l)$ is a homeomorphism given by

$$
T = \sum_{k=0}^{\infty} \tau_{2k+1} M_{2k+1}.
$$
Define
\[ T_1 = \frac{1}{\tau_1} T = \sum_{k=0}^{\infty} \frac{\tau_{2k+1}}{\tau_1} M_{2k+1} \]
and set (observing that \( M_1 = \text{Id} \))
\[ A = \text{Id} - T_1 = \sum_{k=1}^{\infty} \frac{-\tau_{2k+1}}{\tau_1} M_{2k+1} = \sum_{k=1}^{\infty} \frac{\tau_{2k+1}}{\tau_1} M_{2k+1}, \quad \bar{\tau}_{2k+1} = \frac{-\tau_{2k+1}}{\tau_1}. \]

Using (1.5),
\[ \|A\| \leq \sum_{k=1}^{\infty} |\bar{\tau}_{2k+1}| = \delta < 1, \]
and so, by C. Neumann’s theorem [8, theorem II.1.2],
\[ (T_1)^{-1} = \text{Id} + \sum_{k=1}^{\infty} A^k, \]
where the infinite sum is considered as \( \lim_{n \to \infty} \sum_{k=1}^{n} A^k f \) for any \( f \in L^r(I) \), with convergence in the norm of \( L^r(I) \).

As \( M_n M_n = M_{2n} \) \( (m, n \in \mathbb{N}) \), it follows that
\[ A^2 = \sum_{k_1=1}^{\infty} \bar{\tau}_{2k_1+1} M_{2k_1+1} \left( \sum_{k_2=1}^{\infty} \bar{\tau}_{2k_2+1} M_{2k_2+1} \right) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \bar{\tau}_{2k_1+1} \bar{\tau}_{2k_2+1} M_{2(k_1+1)(2k_2+1)}, \]
and more generally,
\[ A^n = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left( \prod_{i=1}^{n} \bar{\tau}_{k_i+1} \right) M_{\prod_{i=1}^{n} (2k_i+1)}, \quad n \in \mathbb{N}. \]

For every odd \( k \geq 3 \), define
\[ D(k) = \left\{ (k_1, \ldots, k_n); \ k = \prod_{i=1}^{n} k_i, \ k_i \geq 3 \text{ and odd for all } i = 1, \ldots, n \right\}. \]

Then,
\[ (T_1)^{-1} = \text{Id} + \sum_{k=1}^{\infty} A^k = \text{Id} + \sum_{k=1}^{\infty} \left( \sum_{(k_1, \ldots, k_n) \in D(2k+1)} \left( \prod_{i=1}^{n} \bar{\tau}_{k_i} \right) \right) M_{2k+1} \]
because, owing to (2.2),
\[ \sum_{k=1}^{\infty} \left( \sum_{(k_1, \ldots, k_n) \in D(2k+1)} \left( \prod_{i=1}^{n} |\bar{\tau}_{k_i}| \right) \right) \leq \frac{\delta}{1-\delta} < \infty. \]

Putting
\[ \alpha_{2k+1} = \sum_{(k_1, \ldots, k_n) \in D(2k+1)} \left( \prod_{i=1}^{n} \bar{\tau}_{k_i} \right), \quad k \in \mathbb{N}, \]
we have
\[ (T_1)^{-1} = \text{Id} + \sum_{k=1}^{\infty} \alpha_{2k+1} M_{2k+1} \]
with
\[ \sum_{k=1}^{\infty} |\alpha_{2k+1}| < \infty. \]

Now, rescaling using \( \tau_1 \) gives the desired representation of \( T^{-1} \) with \( \beta_{2k+1} = (1/\tau_1) \alpha_{2k+1}, \)
\( k \in \mathbb{N}. \)
The above result enables techniques used below and based on properties of $T$ to be applied with equal facility to $T^{-1}$, the role of (1.5) with respect to $T$ being played by the inequality (2.1) as regards $T^{-1}$.

Next, we establish a limited rearrangement property of the basis elements.

**Lemma 2.2.** Let $p, q, r \in (1, \infty)$, let $T$ be as in (1.4) and suppose that (1.5) holds. Let $f \in L^r(I)$ be given in terms of the bases $(e_k)_{k \in \mathbb{N}}$ and $(f_{k,p,q})_{k \in \mathbb{N}}$ by

$$f = \sum_{k=1}^{\infty} a_k e_k = \sum_{k=1}^{\infty} b_k f_{k,p,q}. \quad (2.3)$$

Then,

$$a_k = \sum_{m,n \in \mathbb{N}} b_n \tau_m. \quad (2.4)$$

**Proof.** This result would be obvious if we knew that the bases were unconditional, but as this is not the case some care is needed. Given $f \in L^r(I)$, $T^{-1}f \in L^r(I)$ and may be represented in terms of the basis $(e_n)$ by

$$T^{-1}f = \sum_{k=1}^{\infty} b_k e_k. \quad (2.5)$$

For each $\ell \in \mathbb{N}$, let $P_\ell$ be the canonical projection associated with the basis $(e_k)$, so that if $\sum_{k=1}^{\infty} c_k e_k \in L^r(I)$, then

$$P_\ell \left( \sum_{k=1}^{\infty} c_k e_k \right) = \sum_{k=1}^{\ell} c_k e_k.$$

From the definition of $T$, we see that

$$T(T^{-1}f) = \sum_{m=1}^{\infty} \tau_m M_m (T^{-1}f) = \sum_{m=1}^{\infty} \tau_m \left( \sum_{n=1}^{\infty} b_n e_{mn} \right).$$

Thus,

$$P_\ell (T(T^{-1}f)) = P_\ell \left( \sum_{m=1}^{\infty} \tau_m \left( \sum_{n=1}^{\infty} b_n e_{mn} \right) \right).$$

Because

$$\sum_{m=1}^{\infty} \tau_m \left( \sum_{n=1}^{\infty} b_n e_{mn} \right) = \sum_{m=1}^{\ell} \tau_m \left( \sum_{n=1}^{\infty} b_n e_{mn} \right) + \sum_{m=\ell+1}^{\infty} \tau_m \left( \sum_{n=1}^{\infty} b_n e_{mn} \right),$$

and

$$P_\ell \left( \tau_m \left( \sum_{n=1}^{\infty} b_n e_{mn} \right) \right) = 0 \quad \text{when} \ m > \ell,$$
it follows that
\[ P_\ell \left( \sum_{m=\ell+1}^{\infty} \tau_m \left( \sum_{n=1}^{\infty} b_n e_{mn} \right) \right) = 0. \]

Hence,
\[ P_\ell(T(T^{-1}f)) = P_\ell \left( \sum_{m=1}^{\ell} \tau_m \left( \sum_{n=1}^{\infty} b_n e_{mn} \right) \right) = \sum_{m=1}^{\ell} \tau_m \left( \sum_{n \in \mathbb{N}, mn \leq \ell}^{\infty} b_n e_{mn} \right) = \sum_{k=1}^{\ell} \left( \sum_{mn \leq \ell} b_n \tau_m \right) e_k. \]

Use of the isometric property of \( M_m \) together with (1.5) and (2.5) shows that
\[ \left\| \sum_{k=1}^{\ell} \left( \sum_{mn=k} b_n \tau_m \right) e_k \right\| = \left\| \sum_{m=1}^{\ell} \tau_m \left( \sum_{mn \leq \ell}^{\infty} b_n e_{mn} \right) \right\| = \left\| \sum_{m=1}^{\ell} \tau_m M_m \left( \sum_{mn \leq \ell} b_n e_n \right) \right\| \leq \sum_{m=1}^{\ell} |\tau_m| \left\| M_m \left( \sum_{mn \leq \ell} b_n e_n \right) \right\| \leq B \sum_{m=1}^{\ell} |\tau_m| \leq 2B|\tau_1|. \]

Because the \((e_k)\) form a basis of the reflexive space \( L'(l) \), it follows from a theorem of James [9, theorem 1.2.13] that \((e_k)\) is boundedly complete, so that \( \sum_{k=1}^{\infty} (\sum_{mn=k} b_n \tau_m) e_k \) exists and has norm \( R \leq 2B|\tau_1| \). Because we have, for each \( \ell \in \mathbb{N} \),
\[ P_\ell(T(T^{-1}f)) = \sum_{k=1}^{\ell} \left( \sum_{mn=k} b_n \tau_m \right) e_k, \]
we obtain
\[ f = (T(T^{-1}f)) = \sum_{k=1}^{\infty} \left( \sum_{mn=k} b_n \tau_m \right) e_k. \]

Finally, we present the main results.

**Theorem 2.3.** Let \( p, q, r, \lambda \in (1, \infty) \) be such that (1.5) holds and let \( f \in L'(l) \) have the representations given in (2.3):
\[ f = \sum_{k=1}^{\infty} a_k e_k = \sum_{k=1}^{\infty} b_k f_{k,p,q}. \]

Let \( b = (b_k)_{k \in \mathbb{N}} \) and \( a = (a_k)_{k \in \mathbb{N}} \). Then,

(i) \( b \in l_{1,\infty} \) if and only if \( a \in l_{1,\infty, r} \),

(ii) \( b \in l_1 \) if and only if \( a \in l_1 \).

**Proof.** (i) First assume that \( b \in l_{1,\infty} \). We define \( c = (c_k)_{k \in \mathbb{N}} \), where, for each \( k \in \mathbb{N} \),
\[ c_k = \sum_{mn \in \mathbb{N}, mn = k} |b_n| |\tau_m|. \]

Without loss of generality, we may suppose that \( \|b\|_{l_{1,\infty}} \leq 1 \), so that \( b_k^* \leq k^{-1/\lambda}, k \in \mathbb{N} \). Assume that \( \|c\|_{l_{1,\infty}} > D \), where \( D > 2(1 + \lambda')|\tau_1| \). Then, there exists \( N \in \mathbb{N} \) such that \( c_k^* \geq DN^{-1/\lambda} \) whenever
$k \leq N$. There is a bijective map $P : \mathbb{N} \to \mathbb{N}$ which enables elements of the sequence $c$ to be related to those of the corresponding sequence $(c_k^P)$: for each $k \in \mathbb{N}$, $c_k = c_{P(k)}$. Thus,

$$DN^{1 - 1/\lambda} \leq \sum_{k=1}^{N} c_k^* = \sum_{k=1}^{N} c_{P(k)} = \sum_{k=1}^{N} \sum_{P(k) = mn} |b_n| \tau_m,$$

$$= \sum_{k=1}^{N} \left( \sum_{P(k) = mn, |b_n| \geq N^{-1/\lambda}} |b_n| \tau_m \right) + \sum_{P(k) = mn, |b_n| < N^{-1/\lambda}} \sum_{|b_n| \geq N^{-1/\lambda}} |b_n| \tau_m$$

$$\leq \sum_{k=1}^{N} \left( \sum_{P(k) = mn, |b_n| \geq N^{-1/\lambda}} |b_n| \tau_m \right) + N^{1 - 1/\lambda} \sum_{m=1}^{\infty} |\tau_m|$$

$$\leq 2(1 + \lambda')|\tau_1| N^{1 - 1/\lambda},$$

where we used the convergence of $\sum_{m=1}^{\infty} |\tau_m|$ and the assumption $b \in l_{1, \infty}$. This contradicts the second assumption about $D$ and shows that if $D < \|c\|_{l_{1, \infty}}$, then $D \geq 2(1 + \lambda')|\tau_1|$. Hence, $c \in l_{1, \infty}$ and, by (2.4), $a \in l_{1, \infty}$. The proof of the reverse implication is similar, using the representation of $T^{-1}$ given by lemma 2.1 and (2.1) in place of the convergence of $\sum_{m=1}^{\infty} |\tau_m|$.

(ii) Suppose that $b \in l_1$ and put $\|b\|_1 = D$. Then, with the help of (1.5),

$$\sum_{k=1}^{\ell} c_k = \sum_{k=1}^{\ell} \sum_{mn=k} |b_n| \tau_m = \sum_{m=1}^{\ell} |\tau_m| \sum_{mn \leq \ell} |b_n| \leq D \sum_{m=1}^{\ell} |\tau_m| \leq 2D|\tau_1|,$$

which implies $c \in l_1$, and so $a \in l_1$. As before, the proof of the reverse implication is similar. 

Note that (see (1.6)) the conclusion holds if $p'/q < 4/(\pi^2 - 8)$ or $p = q \in (p_0, \infty)$. For this result to apply, the decay of the $n$th coefficients must be of the form $n^{-\alpha}$, where $\alpha \in (0, 1)$. Progress can be made when $\alpha$ lies outside this interval, as we now show. (If $(c_k), (d_k)$ are non-negative sequences, then we use the notation $c_k \lesssim d_k$ to mean that $c_k \leq Cd_k$ with some constant $C \in (0, \infty)$ independent of $k$.)

**Theorem 2.4.** Let $p, q, r \in (1, \infty)$ be such that (1.5) holds and let $f \in L'(I)$ have the representations (2.3). Suppose that the sequences $(|a_k|)_{k \in \mathbb{N}}$ and $(|b_k|)_{k \in \mathbb{N}}$ are non-increasing and let $\alpha \in (0, 2)$. Then, $|a_k| \lesssim k^{-\alpha}$ if and only if $|b_k| \lesssim k^{-\alpha}$.

**Proof.** First suppose that $|b_k| \lesssim k^{-\alpha}$. By lemma 2.2,

$$a_k = \sum_{m, n \in \mathbb{N}, mn = k} b_n \tau_m, \quad k \in \mathbb{N},$$

which we write as

$$a_k = \sum_{n|k} b_n \tau_{k/n}, \quad k \in \mathbb{N},$$

where the summation is over all $n \in \mathbb{N}$ which divide $k$, written $n|k$. As $(|a_k|)_{k \in \mathbb{N}}$ is non-increasing, it is sufficient to investigate the decay when $k = 2^\ell$, $\ell \in \mathbb{N}$, and in this case

$$a_{2^\ell} = \sum_{j=1}^{\ell} b_{2^j} \tau_{2^{\ell-j}}.$$
Use of the fact that $|\tau_m| \lesssim m^{-2}, m \in \mathbb{N}$ (see [6, proof of proposition 4.1]), together with the given decay of the $(b_n)$ now shows that

$$|a_{2^\ell}| \lesssim \sum_{j=1}^{\ell} 2^{-\alpha j/2} 2^{-2(\ell-j)} \lesssim 2^{-\alpha \ell}.$$ 

The reverse implication follows similarly. ■

**Remarks 2.5.** The assumption of monotonicity made in theorem 2.4 is very strong. To some extent, we can do without it, but at the expense of a weaker conclusion. For example, suppose that $|b_k| \lesssim k^{-\alpha}$ for some $\alpha > 0$. Let $d(k)$ be the number of divisors of $k$, including 1 and $k$:

$$d(k) = \sum_{m|k} 1;$$

and put

$$\sigma_\beta(k) = \sum_{m|k} m^\beta,$$

so that $\sigma_0(k) = d(k)$. Then,

$$|a_k| = \left| \sum_{n|k} b_n \tau_{k/n} \right| \lesssim \sum_{n|k} n^{-\alpha} (k/n)^{-2} = k^{-2} \sigma_{2-\alpha}(k).$$

If $\alpha < 2$, it is known (see [10, p. 86]) that given any $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\sigma_{2-\alpha}(k) \leq k^{(2-\alpha)(1+\varepsilon)}$$

for all $n \geq N(\varepsilon)$, and hence

$$|a_k| \lesssim k^{-\alpha + \varepsilon},$$

and so, compared with the hoped-for result, there is an arbitrarily small loss of sharpness of the exponent. If $\alpha > 2$, then because (see [10, p. 85])

$$\sigma_{2-\alpha}(k) = k^{2-\alpha} \sigma_{\alpha-2}(k),$$

we obtain an estimate of the form

$$|a_k| \lesssim k^{-2+\varepsilon}$$

for any $\varepsilon > 0$. This time the loss of sharpness is greater. For more precise estimates, in which the perturbing terms involve logarithms instead of powers of $k$, we refer to [10, p. 86]. When $\alpha = 2$, it turns out that

$$|a_k| \lesssim k^{-2} d(k) \leq 2^{(1+\varepsilon) (\log k)/(\log \log k)} k^{-2}$$

for all $\varepsilon > 0$ (see [11, p. 262]).

### 3. Concluding remarks

Our results partially answer some questions of Boulton [12] concerning approximation properties of $(p, q)$-sine bases and the rate of the decay of their Fourier coefficients. As we can see from theorems 2.3, 2.4 and remarks 2.5, one cannot expect faster decay of the Fourier coefficients $b_k$ as $k \to \infty$ than that of the coefficients $a_k$ in the expansions (2.3).

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