In this paper, we study filtering of multiscale dynamical systems with model error arising from limitations in resolving the smaller scale processes. In particular, the analysis assumes the availability of continuous-time noisy observations of all components of the slow variables. Mathematically, this paper presents new results on higher order asymptotic expansion of the first two moments of a conditional measure. In particular, we are interested in the application of filtering multiscale problems in which the conditional distribution is defined over the slow variables, given noisy observation of the slow variables alone. From the mathematical analysis, we learn that for a continuous time linear model with Gaussian noise, there exists a unique choice of parameters in a linear reduced model for the slow variables which gives the optimal filtering when only the slow variables are observed. Moreover, these parameters simultaneously give the optimal equilibrium statistical estimates of the underlying system, and as a consequence they can be estimated offline from the equilibrium statistics of the true signal. By examining a nonlinear test model, we show that the linear theory extends in this non-Gaussian, nonlinear configuration as long as we know the optimal stochastic parametrization and the correct observation model. However, when the stochastic parametrization model is inappropriate, parameters chosen for good filter performance may give poor equilibrium statistical estimates and vice versa; this finding is based on analytical and numerical results on our nonlinear test model and the two-layer Lorenz-96 model. Finally, even when the correct stochastic ansatz is given, it is imperative to estimate the parameters simultaneously and to account for the nonlinear feedback of the stochastic parameters into the reduced filter estimates. In numerical experiments...
on the two-layer Lorenz-96 model, we find that the parameters estimated online, as part of a filtering procedure, simultaneously produce accurate filtering and equilibrium statistical prediction. In contrast, an offline estimation technique based on a linear regression, which fits the parameters to a training dataset without using the filter, yields filter estimates which are worse than the observations or even divergent when the slow variables are not fully observed. This finding does not imply that all offline methods are inherently inferior to the online method for nonlinear estimation problems, it only suggests that an ideal estimation technique should estimate all parameters simultaneously whether it is online or offline.

1. Introduction

Model error is a fundamental barrier to state estimation (or filtering). This problem is attributed to incomplete understanding of the underlying physics and our lack of computational resources to resolve physical processes at various time and length scales. While many numerical approaches have been developed to cope with state estimation in the presence of model errors, most of these methods were designed to estimate only one of the model error statistics, either the mean or covariance, while imposing various assumptions on the other statistics which are not estimated. For example, classical approaches proposed in [1,2] estimate mean model error (which is also known as the forecast bias), assuming that the model error covariance (or the random part) is proportional to the prior error covariance from the imperfect model. Popular approaches are to inflate the prior error covariance estimate with an empirically chosen inflation factor [3–7] or with an adaptive inflation factor [8–17]. All of these covariance inflation methods assume unbiased forecast error (meaning that there is no mean model error). Recently, reduced stochastic filtering approaches to mitigate model errors in multiscale complex turbulent systems were introduced in [18–22]; see also [23,24] for a complete treatment of filtering complex turbulent systems. While many of these computationally cheap methods produce relatively accurate mean estimates, the offline-based methods such as the mean stochastic model (MSM) [24,25] tend to underestimate the error covariance statistics that characterizes the uncertainty of the mean estimate in the nonlinear setting. Similar conclusions were also reported in a comparison study of various approximate filtering methods [26]. There are only handful of numerical results, which suggest that an appropriate stochastic parametrization can improve the filtered covariance estimates at short time [27]. Many studies also show that when the stochastic parameters in the filter are obtained by online fitting as part of the data assimilation scheme [15–17,20–22,28], both the filter mean and the covariance estimates become more accurate. These results suggest that one should treat model error as a stochastic process, rather than estimating model error statistics (the bias term and the random component) separately, as is done in many of the empirical approaches mentioned earlier.

Independent from the data assimilation context, there is a vast literature in modelling unresolved scale processes with stochastic parametrizations [15,17,29–34]. In principle, these approaches were designed to address the predictability of the equilibrium statistics, with climate modelling as a natural application. We should point out that not only are the forms of the stochastic parametrizations of these methods different, their stochastic parameters are determined by various offline/online data fitting methods. In particular, the approach in [15,17] determines the stochastic parameters by fitting the data online with a data assimilation scheme. In [17], it was shown that it is necessary to use a stochastic parametrization model with at least a one-lag memory to obtain reasonably accurate equilibrium statistical prediction of a highly skewed, non-Gaussian distributed dynamical system. When a memory-less stochastic parametrization is used, the equilibrium statistical prediction for the skewness is constrained to zero even when the true equilibrium distribution is highly skewed. However, the trajectory of the filtered state estimates for the observed variables are comparable and they are relatively accurate, regardless of whether the stochastic parametrization with no-lag or one-lag memory is used. This result suggests that a
good reduced stochastic model for filtering may not necessarily be a good model for predicting
equilibrium statistics. Here, we will show that the converse is also true when the form of the
stochastic parametrization is not chosen appropriately.

In this paper, we examine the role of the form of the stochastic parametrization and the method
of parameter estimation. This issue is closely tied to the above hypothesis which suggests treating
model error as a stochastic process in a filtering problem rather than estimating the bias and
random components separately, as is typically done in practice. In particular, we want to address
the following questions:

1. Is it possible to have a stochastic parametrization that will produce, simultaneously,
optimal filtering and equilibrium statistical prediction in the presence of model error?
   If so, when can we expect this hypothesis to prevail?
2. Why is it difficult to find such a stochastic parametrization in practical applications?
   In particular, what could happen when the appropriate stochastic parametrization ansatz is
   not available to us?
3. If we have an appropriate stochastic parametrization ansatz, how should we fit the
   parameters? We will compare the filtering and equilibrium statistical predictive skills of
   an online parameter estimation scheme with those of a standard linear regression-based
   offline parameter estimation method. By online, we mean parameters are estimated as
   part of the filtering procedure and by offline, we mean independent of the filter.

To answer the first question, we develop a linear theory for optimal filtering of multiscale
dynamical systems with model error arising from limitations in resolving the smaller scale
processes. By optimality, we mean the expected state estimate and the error covariance matrix
are as accurate as the true posterior estimates obtained with the perfect model. Ideally, we would
like to have accurate estimates of all higher order moments, but due to practical considerations we
only discuss the accuracy of the first two moments which are already difficult to obtain beyond the
linear and Gaussian setting. Note that this optimality condition is only a minimum requirement
for accurate uncertainty quantification. In order to make a rigorous investigation of state
estimation in the presence of model error, we consider the following prototype continuous-time
filtering problem,

\[
\begin{align*}
  dx &= f_1(x, y; \theta) \, dt + \sigma_x(x, y; \theta) \, dW_x, \\
  dy &= \frac{1}{\epsilon} f_2(x, y; \theta) \, dt + \frac{\sigma_y(x, y; \theta)}{\sqrt{\epsilon}} \, dW_y \\
  dz &= x \, dt + \sqrt{R} \, dV, \quad R > 0.
\end{align*}
\]

(1.1)

Intuitively, the variable \( x \) represents the slow component of the state which we wish to estimate
and predict, while the variable \( y \) which represents the fast component (characterized by small \( \epsilon \))
is either unknown or impractical to estimate. In (1.1), \( W_x, W_y \) and \( V \) are i.i.d. Wiener processes
and \( \theta \) denotes the true model parameters, which may be partially unknown in real applications.
The mathematical analysis in this paper assumes:

(I) Full observations of only the resolved variables \( x \), contaminated by unbiased noise with
a positive definite covariance matrix, \( R \). For general observation models that involve
both the \( x \) and \( y \) variables, such as those considered in [18–20], we recommend that the
reader consult the information criteria for optimality of the filtered solutions [35]. While
their strategy is more general, our analysis (at least in this simpler context) provides
convergence estimates for both the mean and the covariance statistics.

(II) The models for the fast unresolved scales in (1.1) are known in order to find the reduced
model analytically. In the linear case, we will also discuss how to obtain the reduced
model when the fast dynamics in (1.1) are unknown. To make the analysis tractable, our
results assume the filtered solutions based on the full model are stable.
While there are many results concerning the convergence of (1.1) as $\epsilon \to 0$ to an averaged reduced filter for $x$ (such as [36], which also developed a nonlinear theory), we are interested in the case where $\epsilon$ may be $O(10^{-1})$ or even $O(1)$ and we want to understand the structure of the averaged operators $F(X; \Theta)$ and $\sigma_X(X; \Theta)$ corresponding to the reduced filtering problem,

$$
\begin{align*}
&dX = F(X; \Theta) \, dt + \sigma_X(X; \Theta) \, dW_X \\
&dz = X \, dt + \sqrt{R} \, dV, \quad R > 0.
\end{align*}
$$

Ultimately, we would like to find $\Theta$ such that the mean and covariance estimates of the reduced filtering problem in (1.2) are close to the mean and covariance estimates of the true filtering problem with the perfect model in (1.1). In this reduced filtering problem, the observations $z$ in (1.2) are noisy observations of the solutions of the true model in (1.1). We assume that there are no errors in the observation model of the reduced filtering problem, which will allow direct comparison of the filtered estimates from (1.1) and (1.2). The parameters $\Theta$ will depend on the scale gap $\epsilon$ and the unknown true dynamics, including the true parameters $\theta$.

In §2, a linear theory is developed in a linear and Gaussian setting under the assumptions (I) and (II) mentioned earlier. This linear theory will address question (1) mentioned earlier. The results in this section introduce a notion of consistency as a necessary (but not sufficient) condition for filtering with model error. By consistency condition, we mean the error covariance estimate agrees with the actual error covariance; this motivates us to introduce a weak measure to check whether the filter covariance estimate is under- or over-estimating the actual error covariance when optimal filtering is not available. In §3, we study a simple, yet challenging nonlinear problem, where the optimal filter is not available as in practical applications. The ultimate goal is to address the second part of question (1) and question (2). In §4, we will compare numerical results of filtering the two-layer Lorenz-96 models with a one-layer Lorenz-96 model combined with various stochastic parametrization methods. The numerical results in this section confirm the theoretical findings in §§2 and 3, even for larger discrete observation time intervals and sparsely observed slow variables. Furthermore, these results will suggest a promising method to address question (3). We conclude the paper with a short summary and discussion in §5. We also accompany this article with electronic supplementary material that provides the detailed proofs of the analytical results and a detailed description of the online parameter estimation method.

2. Linear theory

The goal in this section is to develop a linear theory for filtering multiscale dynamical systems with model errors. In the presence of model error, even for a linear system, we must carefully differentiate between the actual error covariance of the filtered mean estimate and the error covariance estimate produced by the filtering scheme. The actual error covariance is simply the expected mean squared error of the state estimate produced by the filter, on the other hand, the linear Kalman–Bucy filter [37] produces an estimate of error covariance which solves a Riccati equation. In the perfect model scenario, the Kalman–Bucy solutions are optimal and these two error covariances are identical. When the error covariances agree, we say the filter estimate is consistent. However, when the model used by the filter is not the true model, finding a consistent filter estimate is non-trivial because the covariance solutions of the Riccati equation will typically differ from the actual error of the state estimate.

In the discussion below, we will first show that there are infinitely many choices of parameters, $\Theta$, for the reduced model in (1.2), such that the filter covariance estimate matches the optimal covariance estimate of the true filter in (1.1). However, most of these parameters will not give accurate estimates of the mean and therefore the covariance estimate will be inconsistent with the actual error covariance. In the context of predictability, information theoretic criteria were advocated to ensure consistent covariance estimates [38]. While in the context of filtering, information theoretic criteria were also suggested for optimizing the filtering skill [35]. In the
mathematical analysis below, we will enforce a different criteria which is based on orthogonal projection on Hilbert subspaces (see Theorem 6.1.2 in [39]) to find the unique set of reduced filter parameters that ensures not only consistent but also optimal filtering in the sense of least squares. While this is a useful mathematical tool to understand the structure of the stochastic correction in the linear setting, in general, we do not advocate this criteria as a practical tool for parameter estimation. Moreover, we will show that the same optimal parameters can also be found by matching the system equilibrium covariance statistics and the steady state filter posterior covariance estimates or by matching two equilibrium statistics of the underlying system alone.

Consider a linear model where $f_1 = a_{11} x + a_{12} y$ and $f_2 = a_{21} x + a_{22} y$ with a linear observation which involves only the slow variable, $x$. For this particular case, the full filtering problem in (1.1) becomes

$$
\begin{align*}
\dot{x} &= (a_{11} x + a_{12} y) \, dt + \sigma_x \, dW_x, \\
\dot{y} &= \frac{1}{\epsilon} (a_{21} x + a_{22} y) \, dt + \frac{\sigma_y}{\sqrt{\epsilon}} \, dW_y
\end{align*}
$$

and

$$dz = x \, dt + \sqrt{R} \, dV = H(x, y) \, dt + \sqrt{R} \, dV,$n$$

where we define observation operator $H = (1, 0)$ for convenience. We assume that the matrix $A = (a_{ij})$ is negative definite and $\sigma_x, \sigma_y > 0$ are constants of $O(1)$. We also assume that $\tilde{a} = a_{11} - a_{12} a_{21} a_{22} < 0$, which guarantees the existence of the averaged dynamics in (1.2) for $\epsilon \to 0$; in this case, $F(X) = \tilde{a} X$ and $\sigma_X = \sigma_x$ (see e.g. [40] for detailed derivation).

(a) Expansion of the optimal filter

For the continuous time linear filtering problem in (2.1), the optimal filter estimates (in the sense of minimum variance estimator) are the first- and second-order statistics of a Gaussian posterior distribution that can be completely characterized by the Kalman–Bucy solutions [37]. For this linear and Gaussian filtering problem, the covariance solutions of the filter will converge to a steady-state covariance matrix $\hat{S} = (\hat{S}_{ij})_{i,j=1,2}$, which solves the following algebraic Riccati equation:

$$A_\epsilon \hat{S} + \hat{S} A_\epsilon^\top - \hat{S} H^\top R^{-1} H \hat{S} + Q_\epsilon = 0,$n$$

where

$$A_\epsilon = \begin{pmatrix} a_{11} & a_{12} \\
\frac{a_{21}}{\epsilon} & a_{22} \end{pmatrix} \quad \text{and} \quad Q_\epsilon = \begin{pmatrix} \sigma_x^2 & 0 \\
0 & \sigma_y^2/\epsilon \end{pmatrix}.$$n

We can rewrite the first diagonal component of the algebraic Riccati equation (2.2) for $\hat{s}_{11} := E((x - \hat{x})^2)$ as follows (see the electronic supplementary material, appendix A):

$$- \frac{\hat{s}_{11}^2}{R} + 2\tilde{a}(1 - \epsilon \hat{a}) \hat{s}_{11} + \sigma_x^2 (1 - 2\epsilon \hat{a}) + \epsilon \sigma_y^2 \frac{a_{12}^2}{a_{22}^2} = O(\epsilon^2),$$

where $\tilde{a} = a_{11} - a_{12} a_{21} / a_{22}$ and $\hat{a} = a_{12} a_{21} / a_{22}^2$.

Our goal is to find a one-dimensional model for the slow variable, $x$, which still gives the optimal state estimate. Motivated by the results in [40], and the fact that (2.3) has the form of a one-dimensional Riccati equation, we consider the following one-dimensional linear filtering problem:

$$
\begin{align*}
\dot{X} &= aX \, dt + \sigma_X \, dW_X, \\
\dot{z} &= X \, dt + \sqrt{R} \, dV
\end{align*}
$$

The corresponding steady-state covariance solution for the reduced filter in (2.4) satisfies the following algebraic Riccati equation:

$$- \frac{\hat{s}_{11}^2}{R} + 2\tilde{a} \hat{s} + \sigma_X^2 = 0.$$
Subtracting equation (2.3) from (2.5), we have the following result (see the detailed proof in the electronic supplementary material, appendix A):

**Theorem 2.1.** Let $\hat{s}_{11}$ be the first diagonal component of the algebraic Riccati equation in (2.2) and let $\hat{s}$ be the solution of (2.5). Then $\lim_{\epsilon \to 0}((\hat{s} - \hat{s})/\epsilon) = 0$ if and only if

$$\sigma_2^2 = -2(a - \tilde{a}(1 - \epsilon \tilde{a}))\hat{s}_{11} + \sigma_2^2(1 - 2e\tilde{a}) + e\sigma_y^2 \frac{d^2}{dt^2} + O(\epsilon^2). \quad (2.6)$$

Theorem 2.1 says that there is a manifold of parameters $\Theta = \{a, \sigma_2\}$ for which the steady-state filter covariance estimate $\hat{s}$ produced by the reduced model agrees with the steady-state covariance estimate of the optimal filter $\hat{s}_{11}$, obtained with perfect model. So, for any parameters on the manifold (2.6), the reduced filter mean estimate solves

$$d\hat{x} = a\hat{x} dt + \frac{\hat{s}}{R}(dz - \hat{x} dt), \quad (2.7)$$

while the true filter mean estimate for $x$-variable solves

$$d\hat{x} = HA_x(\hat{x}, \hat{y})^T dt + \frac{\hat{s}_{11}}{R}(dz - \hat{x} dt). \quad (2.8)$$

While the true filter estimate in (2.8) is consistent, meaning that $\hat{s}_{11} = E[(x - \hat{x})^2]$, as shown in the derivation of the Kalman–Bucy equations [37], the reduced filter estimate $\hat{x}$ from (2.7) is not always consistent in the presence of model error. Note that the actual steady-state error covariance, $E_1 = \lim_{\epsilon \to 0} E[(\tilde{e})^2]$, where $e(t) = x(t) - \hat{x}(t)$, is not necessarily equal to the steady-state filter covariance estimate $\tilde{s} = \hat{s}_{11} + O(\epsilon^2)$. In fact, most choices of parameters on the manifold in (2.6) lead to poor filter performance, despite the optimality of $\tilde{s}$ (in the sense of minimum variance), due to the inconsistency of the reduced filter.

Our goal is to specify the parameters such that the filtered solutions are consistent, $E_{11} = \tilde{s} + O(\epsilon^2)$. Unfortunately, this consistency condition is too weak and only specifies the choice of parameters up to order-$\epsilon$. From the general linear theory of Hilbert spaces, the optimal filter mean estimate in the sense of least squares is given by the orthogonal projection onto the subspace spanned by its innovations (see Theorem 6.1.2 and the discussion in Section 6.2 in [39]). This condition implies that the actual error, $e = x - \hat{x}$ is orthogonal to the estimate $\hat{x}$ under the joint probability distribution for $(W_X, V)$, that is $E(e\hat{x}) = 0$.

By requiring the reduced filter estimates to satisfy $E(e\hat{x}) = 0$, we find a unique choice of parameters $\Theta = \{a, \sigma_X\}$ on the manifold in (2.6) which produces optimal filter solutions (see the electronic supplementary material, appendix B, for the detailed proof of theorem 2.2). To obtain these parameters, we apply the following procedure: we write the Lyapunov equation for an augmented state variable, $(x, y, \hat{x})^T$ and find the steady-state solution for $E(e\hat{x})$ up to order-$\epsilon^2$. Then we enforce the condition, $E(e\hat{x}) = 0$, which yields a unique choice of parameters on the manifold in (2.6). Furthermore, we can also use the steady solutions of the same Lyapunov equation to verify that these parameters guarantee consistent filtered solutions, $E_{11} = \tilde{s} + O(\epsilon^2)$. In fact, the same parameters can be obtained by requiring the variance of the reduced model in (2.4) to match the equilibrium variance of the underlying system in (2.1) for variable $x$, in additional to the manifold in (2.6). These results are summarized in the following theorem.

**Theorem 2.2.** There exists a unique choice of parameters given by $a = \tilde{a}(1 - \epsilon \tilde{a})$ and $\sigma_2^2$ according to theorem 2.1, such that the steady-state-reduced filter (2.4) is both consistent and optimal up to order-$\epsilon^2$. This means that $\tilde{s}$, the steady-state covariance estimate of the reduced filter, is consistent with the steady-state actual error covariance $E_{11} = \lim_{\epsilon \to 0} E[(x(t) - \hat{x}(t))^2]$ so that $\tilde{s} = E_{11} + O(\epsilon^2)$, and also $\tilde{s}$ agrees with the steady-state covariance $\hat{s}_{11}$ from the optimal filter $\hat{s} = \hat{s}_{11} + O(\epsilon^2)$. The unique optimal parameters can also be determined by requiring the covariance of the reduced model to match that of the slow variable from the full model up to order-$\epsilon^2$.

We remark that a result of [41] shows that for $a = \tilde{a} + O(\epsilon)$ and $\sigma_2^2 = \sigma_2^2 + O(\epsilon)$, the reduced filter mean and covariance estimates are uniformly optimal for all time in the following sense:
The filter uses either the full model (2.1) or various reduced models given by parameter choices in (2.4). RSF: $\hat{\epsilon}$ numerical simulation described below. We now verify the accuracy of the filter covariance estimate suggested by theorem 2.2 in the analysis error covariance

$$\hat{\epsilon} \sigma_x(0) \geq \epsilon \sigma_x(0) \geq 0,$$

there are time-independent constants $C$ such that, $E(|\hat{x}(t) - \tilde{x}(t)|^2) \leq C \epsilon$. In fact, we conjecture that the pathwise convergence should be

$$E(|\hat{x}(t) - \tilde{x}(t)|^2) \leq C \epsilon^4$$

for the unique parameters from theorem 2.2 and we confirm this conjecture numerically in the electronic supplementary material, appendix B. However, the proof of this would require solving the Lyapunov equation of a five-dimensional joint evolution of the full model, full filter and reduced filter. As this Lyapunov equation is an algebraic system of 15 equations of 15 variables, it is not illuminating to verify our conjecture analytically.

Comparing this result to the reduced stochastic filter with an additive noise correction (RSFA) computed in [40], theorem 2.2 imposes additional order-$\epsilon$ corrections in the form of linear damping, $-\epsilon \hat{a} \hat{x}$, and additive stochastic forcing, $-2 \epsilon \sigma_x^2 \hat{a} dW_x$. This additive noise correction term was also found in the formal asymptotic derivation of [40] (they denoted the covariance estimate associated with this additive noise correction by $Q_2$), but the absence of the order-$\epsilon$ linear damping correction term in their calculation makes it impossible to match the posterior statistics of the full model to the same level of accuracy. They dropped this additional additive noise term and, subsequently, underestimated the true error covariance (as shown in figure 1).

We now verify the accuracy of the filter covariance estimate suggested by theorem 2.2 in the numerical simulation described below.

In figure 1, we show numerical results comparing the true filter using the perfect model with approximate filter solutions based on three different one-dimensional reduced models of the form (2.4). Here, the model parameters are $a_{11} = a_{21} = 0$; $a_{22} = -1$, $a_{12} = 1$, $\sigma_x^2 = \sigma_y^2 = 2$. The numerical experiments are for discrete time observations at $\Delta t = 1$ with observation noise covariance $\hat{\epsilon} = 0.5$ and the dynamics are solved analytically between observations. The three reduced models include: (i) the simple averaging model (RSF) where $a = \hat{a}$ and $\sigma_x^2 = \sigma_y^2$; (ii) the order-$\epsilon$ reduced model (RSFA) introduced in [40] with $a = \hat{a}$ and $\sigma_x^2 = \sigma_y^2 + \epsilon \sigma_y^2 \hat{a}_{12}^2 / \hat{a}_{22}^2$, and (iii) the order-$\epsilon^2$ optimal reduced filter described in theorem 2.2. Note that only the order-$\epsilon^2$ optimal reduced filter produces mean and covariance estimates that match the true filter solutions. Furthermore, the resulting covariance estimate is consistent, that is, the mean square error, $\hat{E}_{11} := \langle (x - \tilde{x})^2 \rangle$, where $\langle \cdot \rangle$ denotes temporal average (which equals $E_{11}$ for ergodic posterior distribution) matches the asymptotic covariance estimates $\tilde{s}$ (compare the starred data points in figure 1a,b).

**Figure 1.** Mean squared error of the filtered mean (a) and covariance estimates (b) for an observation series produced by (2.1). The filter uses either the full model (2.1) or various reduced models given by parameter choices in (2.4). RSF: $a = \hat{a}$ and $\sigma_x = \sigma_y$; RSFA: $a = \hat{a}$ and $\sigma_x^2 = \sigma_y^2 + \epsilon \sigma_y^2 \hat{a}_{12}^2 / \hat{a}_{22}^2$; optimal one-dimensional filter: $a = \hat{a}(1 - \epsilon \hat{a})$ and $\sigma_x^2 = \sigma_y^2(1 - 2\epsilon \hat{a}) + \epsilon \sigma_y^2 \hat{a}_{12}^2 / \hat{a}_{22}^2$. The observation noise covariance is $\hat{\epsilon} = 0.5$ and observations are at time interval $\Delta t = 1$. Results are averaged over 100 000 assimilation cycles.
In this linear and Gaussian example, we found the optimal stochastic reduced model either by applying an asymptotic expansion to the Kalman–Bucy solutions alone or by applying asymptotic expansion to both the model equilibrium covariance and the filter posterior covariance solutions. In fact, we will show in the next section that the same reduced model can be obtained by applying an asymptotic expansion to the equilibrium statistical solutions of the model alone. We note that the higher order expansion of the filter solution does not require a pathwise expansion of the prior model.

(b) An optimal stochastic parameter estimation method for filtering linear problems

In practical applications, one may have no access to the true dynamics in (2.1) and in this case it is necessary to estimate the parameters in the reduced model in (2.4) to obtain the optimal filtered solutions. Ideally, we would like to find the optimal parameters using some limited information about the marginal statistics of the slow variable, \( x \). For the linear SDE in (2.4) (which is also known as the Ornstein–Uhlenbeck process [42]), the two parameters, namely the linear damping coefficient, \( a \), and the noise amplitude \( \sigma_X \), can be characterized by two equilibrium statistics, variance and correlation time. Theorem 2.2 guarantees that in the linear and Gaussian setting, one can obtain an optimal filtering by specifying the model parameters from these two equilibrium statistics. This parameter estimation strategy was introduced as the MSM in [25] (see also [24] for different stochastic parametrization strategies for the linear SDE in (2.4)). Formally, we have

**Corollary 2.3.** Given the equilibrium variance and the correlation time statistics of the true signal, \( x(t) \), that evolve based on the linear SDE in (2.1), the reduced MSM filter is an optimal filter, in the sense that the posterior mean and covariance estimates differ by an order of \( \epsilon^2 \) from the true filter estimates obtained with the perfect model.

**Proof:** The slow variable, \( x \), from the full model in (2.1) has correlation time \( T_x = -1/(\bar{a}(1 - \epsilon \bar{a})) + O(\epsilon^2) \) which was found in Lemma 2 of the electronic supplementary material, appendix B. Furthermore, the equilibrium variance of the slow variable, \( x \), from (2.1) is

\[
E[x^2] = \frac{\sigma^2_x (1 - 2\epsilon \bar{a}) + \epsilon (a_{12}^2/a_{22}^2) \sigma_y^2}{-2\bar{a}(1 - \epsilon \bar{a})} + O(\epsilon^2),
\]

as shown in Lemma 1 of the electronic supplementary material, appendix B.

The MSM for (2.4) specifies its parameters with the analytical formula for the variance statistics, \( E[x^2] = E[X^2] = -\sigma^2_X / (2\epsilon) \), and correlation times, \( T_x = T_X = a^{-1} [24,25] \), and from these equations, we obtain \( a = \bar{a}(1 - \epsilon \bar{a}), \quad a_X^2 = \sigma_X^2 (1 - 2\epsilon \bar{a}) + \epsilon (a_{12}^2/a_{22}^2) \sigma_y^2, \) which are the parameters in theorem 2.2 that give the optimal filter solutions up to order-\( \epsilon^2 \).

This result suggests that in the linear and Gaussian setting, it is possible to find the parameters for optimal filtering without using the filter, by using the MSM [24,25]. Furthermore, these parameters also give the optimal equilibrium statistics, up to order-\( \epsilon^2 \). In the nonlinear setting, however, filtering with the MSM can produce an accurate mean estimate but typically underestimates the covariance statistics [26]. In §3, we will explain how this issue arises.

### 3. Extension of the linear theory to a nonlinear test model

In this section, we consider a simple nonlinear continuous-time filtering problem,

\[
\begin{align*}
\text{d}u &= \left[-(y + \lambda_u)u + b\right] \text{d}t + \sigma_u \, \text{d}W_u, \\
\text{d}b &= -\frac{\lambda_b}{\epsilon} b \, \text{d}t + \frac{\sigma_b}{\sqrt{\epsilon}} \, \text{d}W_b \\
\text{d}y &= -\frac{\lambda_y}{\epsilon} y \, \text{d}t + \frac{\sigma_y}{\sqrt{\epsilon}} \, \text{d}W_y
\end{align*}
\]

(3.1)
and
\[ dz = h(u) \, dt + \sqrt{R} \, dV = u \, dt + \sqrt{R} \, dV. \] (3.2)

The discrete observation-time analogue of this nonlinear filtering problem was introduced as SPEKF, which stands for ‘stochastic parameterized extended Kalman filter’ in [21,22], in which filter estimates for SPEKF are obtained by applying a Kalman update to the exactly solvable prior statistical solutions of the full model in (3.1). The nonlinear system in (3.1) has several attractive features as a test model. First, it has exactly solvable statistical solutions which are non-Gaussian. This fact has allowed evaluation of non-Gaussian prior statistics conditional to the Gaussian posterior statistical solutions of a Kalman filter, which verified certain uncertainty quantification methods [43,44]. Second, the results in [45] suggest that the system in (3.1) can reproduce signals in various turbulent regimes such as intermittent instabilities in a turbulent energy transfer range, a dissipative range and for laminar dynamics. Third, the system in (3.1) was also used as a test bed for investigating the consistency of the statistical solutions for various imperfect models in the context of long-term predictability [38]. Our goal here is to verify the existence of an ‘accurate’ reduced filter for this simple test model and to determine whether the corresponding reduced filter model produces accurate long-term statistical prediction. Then we will close this section with a simple example which shows what can go wrong when an insufficient reduced stochastic model is used.

In contrast to the linear filtering problem in §2, the optimal solution to a nonlinear filtering problem is not available in practice, as it requires solving an infinite-dimensional stochastic system. In particular, the true posterior distribution, \( p(u, t) = P(u, t | z(\tau), 0 \leq \tau \leq t) \) for \( u = (u, b, \gamma) \), solves the Kushner equation [46]
\[ dp = \mathcal{L}^* p \, dt + p(h - \mathbb{E}[h])^\top R^{-1} dw_{\bar{u}}, \] (3.3)
where \( \mathcal{L}^* \) is the Fokker–Planck operator for the state variables \( u \). The term \( dw_{\bar{u}} = dz - \mathbb{E}[h] \, dt \) is called the innovation process, and it represents the difference between the actual observation \( z \) and the expected observation \( \mathbb{E}[h] \) with respect to \( p \). As in the linear example above, we will assume that \( h(u) = u \) so that only the slow variable is observed, thus allowing fair comparison with a reduced model for the slow variable. The Kushner equation is a stochastic partial differential equation (SPDE) which is easily solved when both the dynamics and observation process are linear, in which case one recovers the Kalman–Bucy equations. As most practical methods that are being used for assimilating high-dimensional nonlinear problems are linear (or Kalman)-based methods, we restrict our study to the Gaussian approximation of the first two moments, \( \hat{u} = \int u p \, du \) and \( \hat{S} = \int (u - \hat{u})^2 p \, du \), for the slow variable, \( u \), of the conditional density which solves (3.3).

In particular, substituting the Kushner expression for \( dp \) in \( d\hat{u} = \int u \, dp \, du \) and applying integration by parts with the assumption that \( p \) has fast decay at infinity, we find that
\[ d\hat{u} = (-\lambda \hat{u} - \bar{u} \gamma + \bar{b}) \, dt + \hat{S} R^{-1} \, dw_{\bar{u}}, \]
where \( \bar{u} \gamma = \int u \gamma \, dp \, du \) and \( \bar{b} = \int b p \, du \). By differentiating these terms and applying the expansion \( p = p_0 + \epsilon p_1 \), we can explicitly approximate these terms up to order-\( \epsilon^2 \). The full details of this expansion are found in the electronic supplementary material, appendix C, where we find that evolution of \( \hat{u} \) and \( \hat{S} \) reduces to
\[ d\hat{u} = -\left( \lambda \hat{u} - \frac{\epsilon \sigma^2_\gamma}{2 \lambda \gamma (\lambda \gamma + \lambda u)} \right) \hat{u} \, dt + \hat{S} R^{-1} \, dw_{\bar{u}} + O(\epsilon^2) \]
and
\[ d\hat{S} = \left[ -2 \left( \lambda \hat{u} - \frac{\epsilon \sigma^2_\gamma}{\lambda \gamma (\lambda \gamma + \lambda u)} \right) \hat{S} + \frac{\epsilon \sigma^2_\gamma}{\lambda \gamma (\lambda \gamma + \lambda u)} \hat{u}^2 + \frac{\epsilon \sigma^2_\gamma}{\lambda \gamma (\lambda \gamma + \lambda u)} \hat{S} \right] \, dt \]
\[ + \left[ \int (u - \hat{u})^3 \, dp \, du \right] R^{-1} \, dw_{\bar{u}} + O(\epsilon^2). \] (3.4)
These equations approximate the exact solutions for the evolution of the first two statistics of the posterior distribution, \( p \), up to order \( \epsilon^2 \); however, they are not closed because the skewness \( \int (u - \hat{u})^3 p \, du \) appears in the evolution of the covariance \( \hat{S} \). We close these equations by assuming that the posterior distribution is Gaussian, or effectively, \( \int (u - \hat{u})^3 p \, du = 0 \). While the equilibrium statistics of the dynamics in (3.1) have zero skewness, this is not necessarily the case for the posterior distribution given a noisy observation sequence [44]. Note that this closure is different from the Gaussian closure filter introduced in [45], which applies a Gaussian closure on the prior dynamics of the slow variable \( u \). These equations approximate the exact solutions for the evolution of the first two statistics of the posterior distribution governed by the Kushner equation for (3.6), are given by

\[
\begin{aligned}
d\hat{u} &= - \left( \lambda_u - \frac{\epsilon \sigma_u^2}{2 \lambda_y (\lambda_u + \lambda_y)} \right) \hat{u} \, dt + \hat{S} R^{-1} \, dw_{\hat{u}} + \mathcal{O}(\epsilon^2) \\
d\hat{S} &= -2 \left( \lambda_u - \frac{\epsilon \sigma_u^2}{\lambda_y (\lambda_u + \lambda_y)} \right) \hat{S} + \frac{\epsilon \sigma_u^2}{\lambda_y (\lambda_u + \lambda_y)} \hat{u}^2 \\
&\quad + \sigma_u^2 + \frac{\epsilon \sigma_u^2}{\lambda_b (\lambda_b + \lambda_u \epsilon)} - \hat{S} R^{-1} \hat{S} + \mathcal{O}(\epsilon^2).
\end{aligned}
\] (3.5)

We refer to these statistical estimates as the continuous-time SPEKF solutions for the variable \( u \). To obtain the full continuous-time SPEKF solution, one can compute the mean and covariance matrix of the full state \( u \), with similar computations via the Itô calculus. In this sense, the original SPEKF that was introduced in [21,22] is a discrete-time analogue of the continuous-time SPEKF because it implicitly truncates the higher order moments of the posterior statistics through a discrete-time Kalman update.

Motivated by the results in [40,43,44], we now propose the following reduced filter model to approximate the filtering problem in (3.1):

\[
\begin{aligned}
dU &= -\alpha U \, dt + \beta U \circ dW_\nu + \sigma_1 \, dW_u + \sigma_2 \, dW_b, \\
&= - \left( \alpha - \frac{\beta^2}{2} \right) U \, dt + \beta U \circ dW_\nu + \sigma_1 \, dW_u + \sigma_2 \, dW_b, \quad (3.6)
\end{aligned}
\]

and

\[
dz = h(U) \, dt + \sqrt{R} \, dV = U \, dt + \sqrt{R} \, dV.
\]

The evolution of the first two moments of (3.6), \( \bar{u} = \int U \pi \, dU \) and \( \bar{S} = \int (U - \bar{u})^2 \pi \, dU \), where \( \pi \) is the posterior distribution governed by the Kushner equation for (3.6), are given by

\[
\begin{aligned}
d\bar{u} &= - \left( \alpha - \frac{\beta^2}{2} \right) \bar{u} \, dt + \hat{S} R^{-1} \, dw_{\bar{u}}, \\
d\hat{S} &= -2(\alpha - \beta^2)\bar{S} + \beta^2 \bar{u}^2 + \sigma_1^2 + \sigma_2^2 - \hat{S} R^{-1} \hat{S},
\end{aligned}
\] (3.7)

where \( dw_{\bar{u}} = dz - \bar{u} \, dt \) denotes the innovation process and Gaussian closure is imposed by setting the skewness to zero (see the electronic supplementary material, appendix C, for detailed derivation). We can specify the parameters in (3.6) by matching coefficients in the equations governing the evolution of the mean and covariance in the filters (3.5) and (3.7) which yields

\[
\begin{aligned}
\alpha &= \lambda_u, \quad \sigma_1^2 = \sigma_u^2, \\
\sigma_2^2 &= \frac{\epsilon \sigma_u^2}{\lambda_b (\lambda_b + \epsilon \lambda_u)} \quad \text{and} \quad \beta^2 = \frac{\epsilon \sigma_u^2}{\lambda_y (\lambda_u \epsilon + \lambda_y)}.
\end{aligned}
\] (3.8)
We refer to the solutions of (3.7) with parameters in (3.8) as the continuous-time reduced SPEKF solutions of the filtering problem (3.6). With this choice of coefficients, we have the following result (see the electronic supplementary material, appendix C, for detailed proof).

**Theorem 3.1.** Let \( \lambda_u > 0 \), and let \( z \) be noisy observations of the state variable \( u \) which solves the full model in (3.1). Given identical initial statistics, \( \tilde{u}(0) = \hat{u}(0) \) and \( \tilde{S}(0) = \hat{S}(0) > 0 \), the mean and covariance estimates of a stable continuous-time reduced SPEKF in (3.6) with parameters (3.8) agree with mean and covariance of a stable continuous-time SPEKF for variable \( u \) in the following sense. There exist time-independent constants, \( C_1, C_2 \), such that,

\[
|\tilde{S}(t) - \hat{S}(t)| \leq C_1 \epsilon
\]

and

\[
\mathbb{E}[(\hat{u}(t) - \tilde{u}(t))^2] \leq C_2 \epsilon^2.
\]

Furthermore, the reduced filtered solutions are also consistent, up to order-\( \epsilon \).

Theorem 3.1 shows that the continuous-time reduced SPEKF solutions in (3.7) are consistent up to order-\( \epsilon \) and match the first two moments of the continuous-time SPEKF solutions for the slow variable \( u \) up to order-\( \epsilon \). Moreover, theorem 3.1 implies that in the context of Gaussian closure on the posterior distribution, accounting for truncation of fast time scales in a nonlinear model with only additive noise requires a multiplicative noise correction term in the reduced model.

We note that the term \( \epsilon \lambda_u \) appearing in the denominator of the parameters \( \sigma_2 \) and \( \beta \) in (3.8) is technically an order-\( \epsilon^2 \) adjustment; however, this term arises naturally in the derivation of the continuous-time SPEKF solutions for (3.1) in the electronic supplementary material, appendix C, and is important, as we will discuss below. We should point out that these extra order-\( \epsilon^2 \) correction terms were not found in the white noise limit approximation [40,43,44].

(a) **Numerical experiments: assessing the mean and covariance filter estimates**

In the numerical experiments below, we show results for two regimes (as defined in [45]) for (3.1). Regime I corresponds to the turbulent energy transfer range, in which \( \gamma \) decays faster than \( u \). The parameters for this regime are: \( \lambda_u = 1.2 - 1.78i, \lambda_b = 0.5 - i, \lambda_y = 20, \sigma_u = 0.5, \sigma_b = 0.5, \sigma_y = 20 \) and \( \epsilon = 1 \). Regime II, as defined in [45], is an extremely difficult regime corresponding to the dissipative range, where the dynamics of \( u(t) \) exhibits intermittent burst of transient instabilities, followed by quiescent phases. The parameters are: \( \lambda_u = 0.55 - 1.78i, \lambda_b = 0.4 - i, \lambda_y = 0.5, \sigma_u = 0.1, \sigma_b = 0.4, \sigma_y = 0.5 \) and \( \epsilon = 1 \). In this regime, the decaying time scales for \( u \) and \( y \) are comparable. Note that the accuracy of the closure in (3.6) is up to order-\( \epsilon \) when the parameters in the full model in (3.1) are all order-one. As the parameters in Regime I are defined without \( \epsilon \) in [45] and not all of them are order-one, by taking the ratio of the damping coefficients \( \Re(\lambda_u) = 1.2 \) and \( \lambda_y = 20 \), the implicit time-scale separation is approximately \( \epsilon \approx \lambda_u / \lambda_y = 0.05 \). In Regime II, the implicit time-scale separation is approximately \( \epsilon \approx \lambda_u / \lambda_y = 1.1 \).

In these numerical experiments, we apply all the filters with discrete time observations at time \( \Delta t = 0.5 \) and noise covariance \( R = 50\% \text{Var}(u) \). Here, we numerically compare the full SPEKF solutions with

- The reduced stochastic filter (RSF) which assumes \( \alpha = \lambda_u, \sigma_1^2 = \sigma_2^2, \sigma_2 = 0 \) and \( \beta = 0 \).
- The reduced stochastic filter with additive correction (RSFA) which assumes \( \alpha = \lambda_u, \sigma_1^2 = \sigma_2^2, \sigma_2^2 = \epsilon \sigma_b^2 / \lambda_b^2 \) and \( \beta = 0 \).
Figure 2. Posterior error covariance estimates, corresponding to the mean solutions in table 1, as functions of time. In this numerical simulations, we consider observation time interval $\Delta t = 0.5$ and $R = 0.5 \text{Var}(u)$.

Table 1. Average RMSE (and empirical consistency measure) for various filtered mean estimates in Regimes I and II over 20 000 assimilation cycles for observation time $\Delta t = 0.5$ and $R = 0.5 \text{Var}(u)$.

<table>
<thead>
<tr>
<th>scheme</th>
<th>Regime I</th>
<th>Regime II</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPEKF</td>
<td>0.47 (1.27)</td>
<td>2.29 (11.50)</td>
</tr>
<tr>
<td>RSF</td>
<td>0.84 (9.42)</td>
<td>10.53 (1.22 × 10^4)</td>
</tr>
<tr>
<td>RSFA</td>
<td>0.54 (1.52)</td>
<td>9.54 (106.76)</td>
</tr>
<tr>
<td>RSFC</td>
<td>0.47 (0.90)</td>
<td>3.00 (0.60)</td>
</tr>
<tr>
<td>RSPEKF</td>
<td>0.47 (1.10)</td>
<td>2.02 (3.37)</td>
</tr>
<tr>
<td>$\sqrt{R}$</td>
<td>0.5866</td>
<td>5.2592</td>
</tr>
</tbody>
</table>

The reduced SPEKF solutions with white-noise limit parameters [40,43,44], $\alpha = \lambda_u, \sigma^2_1 = \sigma^2, \sigma^2_2 = \epsilon \sigma^2 b^2, \gamma$ and $\beta^2 = \epsilon \sigma^2 / \lambda^2_\gamma$. We will denote this by RSFC, following the notation in [40].

The RSPEKF solutions with parameters in (3.8).

In table 1, we show the average RMSEs, averaged over 20 000 assimilation cycles. In Regime I, the accuracy of the filtered mean estimates of SPEKF, RSFC and RSPEKF are roughly similar. On the other hand, RSF and RSFA are less accurate; in particular, the average RMSE of RSF is larger than the observation noise error, $\sqrt{R}$. In Regime II, RSPEKF has the smallest error, even smaller than SPEKF, followed by RSFC. The linear filters without multiplicative noise, RSF and RSFA, are not accurate at all, their errors are roughly twice the observation noise error. We do not show the pathwise filtered solutions compared to the true signals because they look very similar to those of figs 7 and 8 of [40]. Instead, we examine the filter covariance estimates (figure 2). Note that in both regimes, the covariance estimates of both RSFC and RSPEKF are larger than that of SPEKF. The differences between RSFC and SPEKF in Regime II are even more pronounced. The differences between RSPEKF and SPEKF are of order $\epsilon$ in both regimes, where $\epsilon = 0.05$ for Regime I and $\epsilon = 1.1$ in Regime II. The covariance estimates of the other two linear filters, RSF and RSFA, converge to constant solutions, as expected; RSF underestimates the covariance, while RSFA covariance estimates are closer to RSPEKF.

From these covariance estimates, we cannot conclude which of them over- or underestimate the actual error covariance because we have no access to the optimal filter solutions; even SPEKF solutions are sub-optimal because they are the Gaussian approximation of the first two-moments.
of (3.3). Motivated by the result in theorem 2.2, where the optimal filter guarantees a consistency condition in the sense that the filter covariance estimate matches the actual filter error, we propose the following metric as an empirical measure to determine whether the filter covariance estimates are consistent.

**Definition 3.2 (Consistency of covariance).** Let $\tilde{x}(t) \in \mathbb{R}^n$ and $\tilde{S}(t) \in \mathbb{R}^{n \times n}$ be a realization of the solution to a filtering problem for which the true signal of the realization is $x(t) \in \mathbb{R}^n$. The consistency of the realization is defined as

$$C(x, \tilde{x}, \tilde{S}) = \frac{1}{n} \langle (x(t) - \tilde{x}(t))^\top \tilde{S}(t)^{-1} (x(t) - \tilde{x}(t)) \rangle,$$

where $\langle \cdot \rangle$ denotes temporal average. We say that a filter is consistent if $C = 1$ almost surely (independent of the realization). The filter covariance underestimates the actual error covariance when $C > 1$ ($C < 1$).

This metric is simply the signal part of the relative entropy measure of two Gaussian distributions [35]. With this definition, it is obvious that an optimal filter is always consistent. However, it is not the only consistent filter and not every consistent filter is accurate (see the electronic supplementary material, appendix D, for trivial examples). In parallel to the consistency condition in (2.2), this consistency measure is only a necessary (or weak) condition for the covariance to be meaningful. It should be used together with the mean squared error measure. However, this measure has the following useful property: a consistent filter which produces posterior mean estimates close to the true posterior mean estimates also has a covariance close to the true posterior covariance (see the electronic supplementary material, appendix D, for detail).

We should point out that although this measure is much weaker than the pattern correlation measure advocated in [35], we shall see that many suboptimal filters are not even consistent in the sense of definition 3.2.

In table 1, we record the numerically computed empirical consistency for the corresponding filtering experiments. The consistency results show that in Regime I, almost all filtering methods except RSFC are underestimating the actual error covariance. In this regime, both RSFC and RSPEKF produce the most consistent covariance estimates. In Regime II, both linear filters (RSF and RSFA) significantly underestimate the actual error covariance. RSFA improves both mean and covariance estimate (it reduces the consistency measure from $C \approx 10^4$ to $10^2$) by an additive covariance inflation factor $\sigma_b^2 = \epsilon \sigma^2 / \lambda^2$. In this regime, SPEKF, which produces reasonably accurate filtered solutions, also underestimates the actual error covariance ($C \approx 11.5$). Even RSPEKF underestimates the covariance ($C \approx 3.37$). We suspect that the underestimation of these covariances in SPEKF and RSPEKF are due to the Gaussian closure approximation. Additionally, the underestimation of the actual error covariance in the full SPEKF solutions can be attributed to a combination of the following issues: (i) the full SPEKF has sparse observations of only $u$, and (ii) the prior statistical solutions of the full SPEKF involve quadrature approximation of various integral terms.

**b) Numerical experiments: assessing the predictive skill of the covariance estimates**

In this section, we compare the evolution of the covariance of the stochastic variable $u$ in the nonlinear model in (3.1) with the evolution of the covariance of $U$ in the approximate model in (3.6) for the following parameter sets:

- the RSFC or white-noise limit parameters [40, 43, 44] and
- the RSPEKF parameters obtained in (3.8).

In this numerical experiment, we solve the evolution of the true covariance of $u$ and the two reduced models analytically as in [24] and in the appendix of [40], respectively. We assume an
Figure 3. Covariance solutions of $u$ for the true model in (3.1), the reduced model in (3.6) with parameters specified as in RSFC and RSPEKF.

Each model is then used to evolve these initial conditions forward in time and the resulting covariances are shown in figure 3. Note that in both regimes, the covariance estimates from the parameters of RSFC are not accurate at all. In Regime I, the absolute error of the final covariance estimates shown in figure 3 is about $|\text{Var}(u) - \text{Var}(U)| = 0.7171 \approx 14\epsilon$ for $\epsilon = 0.05$. In Regime II, the covariance estimate of the RSFC is unstable since the stability condition,

$$\Xi_2 = -2\lambda_u + \epsilon \frac{2\sigma^2_v}{\lambda^2_u} < 0,$$

(3.10)

is not satisfied. The order-$\epsilon^2$ correction terms in (3.8) yield significant improvement in both regimes. In Regime I, the absolute error of the covariance estimate from RSPEKF, $|\text{Var}(u) - \text{Var}(U)| = 0.3 = 6\epsilon$, is much smaller than that of RSFC. In Regime II, the RSPEKF correction terms ensure the stability of the prior covariance estimate, that is, it provides the following stability condition:

$$\tilde{\Xi}_2 = -2\lambda_u + \epsilon \frac{2\sigma^2_v}{\lambda^2_u (\lambda_v + \epsilon\lambda_u)} < 0.$$

(3.11)

Moreover, the corresponding absolute error in this regime is $|\text{Var}(u) - \text{Var}(U)| = 0.5979 = 0.5\epsilon$, where $\epsilon = 1.1$.

The RSPEKF example shows that there exists a parameter set, given by (3.8), that produces reasonably accurate and consistent filtered estimates as well as relatively accurate covariance solutions, up to order-$\epsilon$. Interestingly, the stochastic noise terms in the moment equations of (3.5) and (3.7) do not affect the determination of the parameters in (3.8). In fact, we can obtain the same stochastic parameters by applying the asymptotic expansion to the first two moments of the marginal distribution of $u$ which solves the deterministic part of SPDE in (3.3) (which is simply the Fokker–Planck equation). We also note that by a straightforward (but tedious and lengthy) calculation, one can verify that, for this particular example, the parameters (3.8) also match the third moments of SPEKF and RSPEKF, when one does not apply the Gaussian closure in (3.4). The coefficients in the higher order moment equations still satisfy the same constraints which yield (3.8). Thus, as long as no model error is committed in the observation operator $h$ in the reduced filter of (3.6), the stochastic term in (3.3) will not produce extra constraints.
Figure 4. Contour plot of the average RMSE and the weak consistency measure for filtering noisy observations in (3.1) with the reduced filter model in (3.12) for various parameters $Re(\alpha)$ and $\sigma_1^2 + \sigma_2^2$ and fixed frequency $Im(\alpha)$ determined by the equilibrium statistics through MSM method (corollary 3.1) in Regime I. The ‘+’ sign denotes the parameters determined by the equilibrium statistics through MSM method.

In real applications, however, it is typically difficult to find an appropriate ansatz which can give both accurate filter solutions and accurate long-term statistical prediction with the same parameters. In this case, it is possible to have more constraints than the number parameters in the reduced model. To illustrate this point, suppose we choose the following stochastic ansatz for the reduced filter model:

$$dU = -\alpha U dt + \sigma_1 dW_u + \sigma_2 dW_b, \quad (3.12)$$

ignoring the multiplicative noise component in (3.6). By comparing the Kalman–Bucy solutions of (3.12) to the Gaussian closure moments in (3.5), it is clear that the accuracy of the filtered mean and covariance estimates are not within order-$\epsilon^2$. With this stochastic ansatz, we can fit the equilibrium variance and correlation time (with the MSM method as mentioned in corollary 3.1) to obtain $\alpha = 0.7683 - 0.9971i$ and $\sigma_1^2 + \sigma_2^2 = 2.1147$ (shown with the ‘+’ sign in figure 4 for Regime I). In figure 4, we compare the average RMSEs and consistency measure of filtering noisy observations of (3.1), with the stochastic ansatz in (3.12) as the filter model, for a wide range of parameters. We set the frequency parameter, $Im(\alpha) = -0.9971i$ to be exactly the value determined by MSM. In these numerical simulations, the filtered performance is quantified over 20,000 assimilation cycles for observation noise $R = 0.5 \text{Var}(u)$ and time $\Delta t = 0.5$. Note that the MSM parameters do not produce the best filtered solutions; they yield an average RMSE close to 0.85 and consistency measure close to 1.4. Moreover, in this parameter range, the average RMSE is much larger than 0.54 which was produced by RSFA (table 1), which is also using the ansatz in (3.12) with parameters in (3.8), except for $\beta = 0$. Conversely, the parameters associated with RSFA in table 1 produce inaccurate equilibrium statistics; the correlation time and the equilibrium variance are significantly underestimated by 52% and 83%, respectively. This example illustrates the importance of having the appropriate ansatz. Moreover, when the ansatz is not appropriate, parameters which are chosen to give good equilibrium statistics may give poor filter performance and vice versa.

4. Stochastic parametrizations for the two-layer Lorenz-96 model

In the previous sections, we were able to obtain an optimal stochastic parametrization ansatz which compensates for the unresolved scales, because the full dynamics are known and the two test problems are quite simple. Our results showed that it is critical to use the correct stochastic ansatz in order to simultaneously obtain accurate filtering and accurate equilibrium statistical estimates. In practical applications, it is rather difficult to derive the correct parametric form for
the reduced model, especially when the dynamics of the unresolved variables are not completely known. Motivated by our results in §§2 and 3, as well as the normal forms for reduced climate models deduced in [29,33], we propose a stochastic parametrization ansatz which includes a linear damping term and a combined, additive and multiplicative, stochastic forcing to account for the unresolved scales in filtering nonlinear multiscale problems.

In this section, we present numerical results from filtering the two-layer Lorenz-96 model [47], which has been widely used for testing stochastic parametrization methods [28,30–32]. The two-layer Lorenz-96 model is an $N(j + 1)$-dimensional ODE given by

$$\frac{dx_i}{dt} = x_{i-1}(x_{i+1} - x_{i-2}) - x_i + F + h_x \sum_{j=(i-1)j+1}^{ij} y_j,$$

and

$$\frac{dy_j}{dt} = 1 = \frac{1}{\epsilon} (ay_{j+1}(y_{j-1} - y_{j+2}) - y_j + y_{j} \sigma_{\text{ceil}(i/j)}),$$

where $x = (x_i)$ and $y = (y_j)$ are vectors in $\mathbb{R}^N$ and $\mathbb{R}^{NJ}$, respectively, and the subscript $i$ is taken modulo $N$ and $j$ is taken modulo $NJ$. To generate the observations, we integrate this model using the Runge–Kutta method (RK4) with a time-step $\delta t$ and take noisy observations $z_k$ at discrete times $t_k$ with spacing $\Delta t = t_{k+1} - t_k$ given by

$$z_k = h(x(t_k)) + \eta_k = x(t_k) + \eta_k,$$

where $z_k \in \mathbb{R}^M$ and $R$ is a symmetric positive definite $M \times M$ matrix. The main goal of this section is to show that when the ‘correct’ ansatz is known, a natural way to estimate the parameters is online (as part of the filtering procedure). To achieve this, we separate the discussion into three subsections. In the first subsection, we will provide a short review of the online parametrization method that was introduced in [16] (we accompany this section with a more detail discussion in the electronic supplementary material, appendix E). Second, we compare the filter performance of various choices of ansatz with the online parameter estimation scheme. The main point here is to empirically find the ‘correct’ ansatz, because the complexity of the full model makes it difficult to analytically derive such an ansatz. In the third part of this section, we compare the filter and the equilibrium statistical estimates of the online parametrization method with an offline stochastic parameter estimation method proposed in [31,34]. To make a fair comparison, we will use the same stochastic parametric form (ansatz).

(a) Review of the online parameter estimation method

We consider the following reduced stochastic model to approximate the filtering problem in (4.1) and (4.2):

$$\frac{dx_i}{dt} = x_{i-1}(x_{i+1} - x_{i-2}) - ax_i + F + \left( -\alpha x_i + \sum_{j=1}^{N} \sigma_{ij} \tilde{W}_j + \sum_{j=1}^{N} \beta_{ij} x_j \circ \tilde{V}_j \right).$$

As we pointed out earlier, such a stochastic parametrization was motivated by our results in §§2 and 3, as well as by earlier work that suggested a normal form for reduced climate modelling [29,33]. The filter model in (4.3) is simply the one-layer Lorenz-96 model augmented with an additional linear damping term $\alpha$, an additive noise term which amplitude is given by the matrix $\sigma = (\sigma_{ij})$ and a multiplicative noise term with coefficient $\beta = (\beta_{ij})$. The notations $\tilde{W}, \tilde{V}$ denote standard i.i.d. white noise. In the remainder of this paper, we will set $\beta = 0$ as the multiplicative noise seems to play little role based on the study in [34] and we suspect that an online estimation method for the multiplicative noise amplitude may involve a more expensive Markov chain Monte Carlo algorithm, which is beyond the scope of this paper.

The core of the online parameter estimation scheme considered in this paper is the ensemble Kalman filter (EnKF) algorithm with an adaptive noise estimation scheme proposed in [16]. For the application in this paper, we will combine this scheme with the classical online state
Observability issues can also be mitigated with an alternative algorithm in the EnKF framework, as described in detail in the electronic supplementary material, appendix E. We should point out that this approach applies to the h_x for example, where the covariance in the model error should only depend on the distance between the slow variables, so we introduce a cyclic parametrization of Q for half of the slow variables. When J/2 are observed, and because of the spatial homogeneity of the problem, we must parametrize the Q matrix. In §4c, we consider a sparse observation where only half of the slow variables are observed, and because of the spatial homogeneity of the problem, we introduce a cyclic parametrization of Q. The idea of the cyclic parametrization is that the covariance in the model error should only depend on the distance between the slow variables, so for example Q_{12} = Q_{23} = \cdots = Q_{NJ}. The cyclic parametrization reduces the number of parameters in Q from \(N^2\) to \(\text{ceil}(N/2)\) and can be used whenever \(\text{ceil}(N/2) \leq M^2\); the full details are described in detail in the electronic supplementary material, appendix E. We should point out that this observability issue can also be mitigated with an alternative algorithm in the EnKF framework, which uses more than one-lag covariances of the innovation vectors to estimate Q and R [17].

(b) The role of damping and stochastic forcing in the reduced Lorenz-96 model

In this section, we compare multiple choices of ansatz which are reduced forms of (4.3) across a wide range of time-scale separations, \(2^{-7} \leq \epsilon \leq 1\). The goal of this example is to compare the filtering skill of various stochastic parametrization ansatz when the unresolved scales dynamics in (4.1) are ignored. We generate the truth data from (4.1) and observation data from (4.2) using a short observation time \(\Delta t = \min[0.01, \epsilon/10]\) and an integration time step \(\delta t = \Delta t/10\). The \(\epsilon\) dependence in the time step is necessary due to the stiffness of the problem as \(\epsilon \to 0\). We use the parameters from Regime 2 of [28] where \(N = 9, J = 8, a = 1, F = 10, R = 0.1 \times I_{N \times N}\), \(h_x = -0.1\) and \(h_y = 1\). Note that there are 81 total variables, only nine of which are observed. For diagnostic purposes, we first consider the idealized case where the full model (4.1) and all the parameters are known exactly. We apply an EnKF based on the full ensemble transform [49] with 162 ensemble members (double the total state variables, \(2N(J + 1)\)), each of which are integrated 10 times between observations. We will refer to this scheme as the full model; see figure 5 for the average RMSE and consistency measure of the filtered solutions based on this full model.

Numerically, the reduced filter models are all integrated with \(\delta t = \Delta t\), thus using 10 times fewer integration steps than the full model, because the numerical stiffness disappears when the fast processes are removed. Moreover, as the reduced models have significantly fewer variables, \(N = 9\), we consider an ensemble of \(2N = 18\) members (or 20 members, when \(\alpha\) is estimated), which is much fewer than the 162 ensemble members used for the full model. In this section, all \(N = 9\) slow variables are observed which allows us to estimate the full \(9 \times 9\) matrix \(Q\), although this requires a long time series. Thus, we use a series of 80,000 observations and each filter uses the first 20,000 observations to estimate their parameters so that only the last 60,000 observations are used in the computation of the averages in the RMSE and the consistency (shown in figure 5).

To evaluate the effectiveness of the additional damping and additive noise in the reduced model, we consider four separate cases. First, we set \(\alpha = 0\) and \(\sigma = 0\), which we call the reduced deterministic filter (RDF) since the slow variables are unchanged and the fast variables have simply been truncated. As shown in figure 5, the RDF has very poor performance for all but extremely small values of \(\epsilon\). In fact for \(\epsilon \geq 0.125\), the truncated model’s filtered estimate is actually worse than the observation. Next, we consider the RDF with an additional damping correction (RDFD)
Figure 5. Filter performance measured in terms of RMSE (a) and consistency measure (b) for various time-scale separations $\epsilon$ with an integration time step of $\delta t = \Delta t/10$ and observations at time intervals $\Delta t = \min(0.01, \epsilon/10)$ with observation noise $R = 0.1$. Results are averaged over 60,000 assimilation cycles in Regime 2 from [28], where $N = 9$, $J = 8$, $\alpha = 1$, $F = 10$, $R = 0.1$, $h_x = -0.8$ and $h_y = 1$. All filters use the ensemble transform Kalman filter with dynamics integrated with the Runge–Kutta (RK4) method. The full model filter uses (4.1), the same model used to generate the data. The remaining models use (4.3) where RDF sets $\sigma_{ij} = \alpha = 0$, RDFD sets $\sigma_{ij} = 0$, RSFA sets $\alpha = 0$ and RSFAD fits both parameters simultaneously.

where $\sigma = 0$ and the reduced stochastic filter with an additive noise correction (RSFA) where $\alpha = 0$. As shown in figure 5, the damping improves the filter accuracy for small $\epsilon$, whereas the additive noise stochastic forcing improves the filter accuracy for large $\epsilon$. Finally, we combine both damping and additive stochastic forcing in RSFAD, which shows the improvement that is achievable with this simple stochastic parametrization of model error compared with simply neglecting unresolved scales.

Of course, estimating the accuracy of the posterior mean filter solutions is only part of filter performance, the filter also quantifies the uncertainty in the mean state estimate $\tilde{x}(t)$ via the estimated covariance matrix $\tilde{S}(t)$. We would like to know whether the filter is doing a good job of determining $\tilde{S}(t)$; however, judging the accuracy of $\tilde{S}(t)$ is difficult as we do not have access to the optimal filter (even our full model simply uses a Gaussian update). Thus, we compute the empirical measure of filter consistency introduced in §3. As shown in §3 and the electronic supplementary material, appendix D, the consistency quantifies the degree to which the actual error covariance of the suboptimal estimate $\tilde{x}(t)$ agrees with the filtered covariance estimate, $\tilde{S}(t)$. Moreover, if $\tilde{x}(t)$ is a good estimate of the true posterior mean, consistency close to one implies that $\tilde{S}(t)$ is close to the true posterior covariance. In figure 5, we show that consistency is significantly improved by the additive noise term characterized by the parameters $\sigma_{ij}$. When these stochastic parameters are included, the reduced model is consistent with $C \approx 1$, compared with the order $10^4$ underestimation of the actual error covariance without this stochastic parametrization.

(c) Comparison of online and offline stochastic parametrization methods

In this section, we compare the online parameter estimation scheme in §4a with the linear regression-based offline scheme from [34]. We will consider their parameter regime, which can be written as (4.1) with parameters $N = 8$, $J = 32$, $\epsilon = 0.25$, $F = 20$, $a = 10$, $h_x = -0.4$ and $h_y = 0.1$. Here, we only consider the regime of [34] with the smaller time-scale separation ($c = 1/\epsilon = 4$ in their parameters). In this section, we will vary the observation time $\Delta t$ and we use an integration time step of $\delta t = 0.001$ for the full model (to avoid numerical instability) and all the reduced models will use an integration step of $\delta t = 0.005$. We consider a more challenging regime for the stochastic parametrization with sparse observations of every other grid point of the slow variables so that
\( M = 4 \). Each observed variable is perturbed by a mean zero Gaussian noise with variance 0.1 so that the covariance matrix of the observation noise is \( R = 0.1 \times I_{M \times M} \). The filter performance is assessed with the average RMSE and the consistency measure, computed over all \( N = 8 \) slow variables and 7000 assimilation cycles.

We also consider one of the stochastic parametrization ansatz that was studied in [34], which we refer to as Cubic + AR(1),

\[
\frac{dx_i}{dt} = x_{i-3} - x_{i-2} - x_i + F - \left(b_0 + b_1 x_i + b_2 x_i^2 + b_3 x_i^3 + e_i(t)\right)
t\]

and

\[
e_i(t) = \phi e_i(t - \delta t) + \hat{\sigma} (1 - \phi^2) z_i(t),
\]

where \( b_j, \phi, \hat{\sigma} \) are scalars to be determined from the data and \( z_i \) denotes standard i.i.d. white noise. This model fits the bias component of the model error to a cubic polynomial and the random component of the model error to an AR(1) model. Following [34], we integrate this model with the stochastic integration method described in [50] with integration step \( \delta t = 0.005 \) and hold the stochastic term, \( e_i(t) \), constant over each integration step.

The parameters for (4.4) are found offline in [34] from a noiseless time series of the slow variables \( \{x_i\}_{i=1}^{8} \). In particular, given a training time series of \( x_i(t) \), the offline estimation first constructs

\[
U(x_i, t) = x_{i-1}(t)(x_{i+1}(t) - x_{i-2}(t)) - x_i(t) + F - \left(\frac{x_i(t + \delta t) - x_i(t)}{\delta t}\right),
\]

which represents the model error from using the truncated one-layer Lorenz-96 model. The errors \( U \) are then fitted to the cubic polynomial, \( U(x_i, t) \approx b_0 + b_1 x_i(t) + b_2 x_i(t)^2 + b_3 x_i(t)^3 \) using a standard least-squares method. Finally, the residual, \( e_i(t) = U(x_i, t) - (b_0 + b_1 x_i(t) + b_2 x_i(t)^2 + b_3 x_i(t)^3) \), is fitted with an AR(1) stochastic model to find the parameters \( \phi \) and \( \hat{\sigma} \). As the model is spatially homogeneous, we fit a single set of parameters for all \( i \). The parameters of [34] are \( b_0 = -0.198, b_1 = 0.575, b_2 = -0.0055, b_3 = -0.000223, \phi = 0.993 \) and \( \hat{\sigma} = 2.12 \), which we verified using the above procedure, and we use these parameters for the Cubic + AR(1) reduced model. We found that the filtered solutions with this model diverge catastrophically (the average RMSE goes to numerical infinity) for the sparse observation. For the fully observed slow dynamics case, we found that the average RMSE of this stochastic parametrization is slightly below the observation error but they were no where close to that of the full model or the other reduced models considered.

To make a fair comparison between the online and offline parameter estimation schemes, we consider a simplified version of the parametric form in (4.4) with \( b_0 = b_2 = b_3 = \phi = 0 \). This is equivalent to using (4.3) with a diagonal diffusion matrix, \( \sigma_{ij} = \hat{\sigma} \delta(i-j) \). In this sense, both resulting filter models will have only two parameters in their stochastic parametrization, namely \( \alpha = b_1 \) and \( \hat{\sigma} \). For the offline fit, we obtain the parameters with the same linear regression-based offline estimation technique described earlier as in [31,34] by fitting to a large dataset of \( x_i(t) \) (2 \times 10^5 time steps); the resulting parameters are \( \alpha = b_1 = 0.481 \) and \( \hat{\sigma} = 2.19 \). In order to produce online estimates of the reduced model parameters, we ran the adaptive EnKF described in the electronic supplementary material, appendix E, using the RSFAD reduced model of (4.3) on 10^6 noisy observation of \( M = 4 \) of the \( N = 8 \) slow variables. The RSFAD scheme estimates the parameters \( \alpha \) and \( \sigma_{ij} \) on-the-fly, as described in the electronic supplementary material, appendix E, using the cyclic parametrization of \( \sigma_{ij} \). We define the online fit reduced model by taking \( \alpha \) and \( \hat{\sigma} = \frac{1}{8} \sum_{i=1}^{8} \sigma_{ii} \) from the RSFAD scheme. The parameters from online fit and offline fit are shown in figure 6. We should emphasize that the parameter estimation scheme of [34] takes place offline, using a noiseless dataset of \( x_i(t) \) and fits the deterministic and stochastic parts of the model error separately. In contrast, the online data assimilation scheme of [16] with RSFAD uses a much shorter and spatially sparse time series of noisy observations without knowing the noise error covariance, \( R \), and simultaneously updates the mean and covariance parameters of the reduced stochastic model (4.3).
In Figure 6, we compare the performance on the filtering problem and we see that the offline fit parameters give worse performance than the observation in terms of RMSE. On the other hand, the online fit parameters produce filtered solutions with a RMSE which is relatively close to that of the full model. Note that the consistency of the offline fit is very good, which agrees with the results in [34] which compares ensemble spread to ensemble error in the prior model. However, as shown in the electronic supplementary material, appendix D, a good consistency result is meaningless when the mean estimate is not accurate; so while the actual error and filter error estimate agree, they are both very large. In contrast, the online fit is underestimating the covariance slightly (compare the scale of the y-axis to that in Figure 5) but the RMSE and consistency are
close to those of the full model. Moreover, in order to make a fair comparison, the online fit only uses the diagonal part of the covariance matrix estimated by the RSFAD and the additional covariance of the off-diagonal terms is probably needed to produce a more consistent filter. In figure 7, we compare the equilibrium marginal density and correlation function of the online fit and offline fit to those of the slow variables of the full model. In this regime, the online fit produces very good agreement with both the equilibrium density and the correlation function over a very long time (note that four model time units corresponds to 800 integration steps for the reduced model). In contrast, the offline fit and even the full Cubic + AR(1) models showed some deviations, notably underestimating the variance and overestimating the lag correlations at the later times.

As the online fit gives good filter performance and also closely matches the equilibrium statistics of the full model, we conclude that the ansatz (4.3) is sufficient for this regime of the two-layer Lorenz-96. We should emphasize that there is no inherent problem with offline parameter estimation. The problem with the linear regression-based estimation scheme of [34] is that the deterministic parameter, $\alpha = b_1$, and diffusion amplitude, $\hat{\sigma}$, in the stochastic parametrization model are estimated separately. So, when a parameter in (4.3) is independently perturbed, the nonlinear feedback of this perturbation is not appropriately accounted in the filtered estimates. A successful offline parameter estimation scheme would need to simultaneously account for all of these nonlinear feedback relationships, rather than with two separate least-squares estimates.

5. Summary and discussion

In this paper, we studied two simple examples to understand how model error from unresolved scales affects the state estimation and uncertainty quantification of multiscale dynamical systems, given noisy observations of all the resolved scale components alone. From the mathematical analysis of these simple test problems, we learned that for a continuous time linear model with Gaussian noise, there exists a unique choice of parameters in a linear reduced model for the slow variables which gives optimal filtering when only the slow variables are observed. Moreover, these parameters simultaneously give the best equilibrium statistical estimates, and as a consequence they can be estimated offline from equilibrium statistics of the true signal. In particular, this shows that in the linear setting the MSM introduced in [24, 25] is the optimal reduced model.

By examining the continuous-time nonlinear SPEKF problem, we showed that the linear theory extends to this non-Gaussian, nonlinear configuration as long as we know the optimal stochastic parametrization ansatz and there is no error in the observation model. We confirmed this finding by noticing that the stochastic terms in the Kushner equations do not produce additional constraints to determine the reduced model parameters as long as the observation model has no additional error. This implies that one would get the same parameters by matching the first two-moments of the corresponding Fokker–Planck equations (ignoring the stochastic terms in the Kushner equations). Although we only show the Gaussian closure approximate filter in this paper, we found no additional constraints when the Gaussian closure approximation is ignored (the coefficients of the higher moments satisfy the same constraints). Numerically, we show that the additional correction terms that we found in our formal asymptotic expansion produces accurate filtering as well as accurate long-term covariance prediction. Moreover, we reinforce this result numerically on a complex nonlinear system in §4 by numerical estimation of a reduced stochastic model for the two-layer Lorenz-96 model. Once again, given the ‘right’ stochastic parametrization (chosen based on our analysis and the earlier work [29, 33]) we find a single set of parameters which simultaneously produces reasonably accurate filter estimates and equilibrium statistical estimates.

When the stochastic parametrization ansatz is insufficient, parameters chosen for good filtering may give poor equilibrium statistics and vice versa. This is shown analytically and numerically in §3 for the SPEKF filtering problem when the stochastic parametrization does
not include a multiplicative stochastic forcing. This is also reinforced numerically in §4b for the two-layer Lorenz-96 model, where we show that neither linear damping (RDFD) nor additive stochastic forcing (RSFA) alone are sufficient to give accurate filtering across multiple time-scale separations, and the ‘right’ parametrization requires both of these terms (RSFAD). Moreover, in §4c we show that the parameters estimated by RSFAD match the equilibrium statistics of the full model and give good filter performance even for sparse observations at long observation times with small time-scale separation.

Finally, even when the correct stochastic ansatz is known, it is imperative to estimate the parameters simultaneously and to account for the nonlinear feedback of the stochastic parameters in the parameter estimation technique. In particular, in §4c we compare an offline parameter estimation technique proposed in [31], and used by Arnold et al. [34], to an online parameter estimation technique introduced in [16]. We find that the online parameters give good filter performance and match the equilibrium statistics, whereas the offline parameters yield filter estimates which are worse than the observations. In our numerical results, the online stochastic parameter estimation scheme produces extremely accurate filter and equilibrium statistical estimates even when the data assimilation method only takes noisy observations of only half of the slow variables, while the offline technique uses a much longer dataset of noiseless observations of all of the slow variables.

The weakness of the offline technique that we tested [31,34] is that the deterministic and stochastic parameters are estimated separately based on a linear regression fitting on a training dataset. Such a scheme does not account for the feedback of these parameters on the reduced model, which is particularly important when the underlying dynamics are nonlinear. We emphasize that offline estimation methods are not inherently worse than online estimation; however, a successful estimation technique must estimate all parameters simultaneously. As pointed out in [35], the design of an adequate offline estimation method for accurate filtering with model error may involve solving a constrained optimization problem with the constraints given by the three information measures to account for the statistics of the filter error, including the mean biases, and the correlations between the truth and the filter estimates. It would be interesting to see whether such method produces parameters that give accurate equilibrium statistical estimates, assuming that the optimal stochastic parametrization is known. On the other hand, there are, for example, other offline methods based on information theoretic criteria which were shown to improve long-term uncertainty quantification [38,43,51–53]; it would also be interesting to check whether these methods can also give accurate filtered solutions when the optimal stochastic ansatz is given. In the linear and Gaussian setting, the MSM in [24,25] is an example of an offline estimation method that produces optimal filtering as well as equilibrium statistical prediction. In this special setting, the offline technique is sufficient because of two special circumstances, first, the parameters can be completely characterized by two equilibrium statistics and, second, the underlying signal is stationary. In a nonlinear setting, the online technique gives a natural way for the parameters to evolve via small perturbations while the feedback from these perturbations on the reduced model are compared to the observations, adaptively. Moreover, it does not require a training dataset and can work directly with the noisy observations as they become available.

The online parameter estimation methods also have some weaknesses, they sometimes are limited by issues of observability of parameters (see Chapter 13 of [24]) and they also required careful design to avoid expensive computational costs [11,16,17]. Extending these techniques to more general stochastic parametrizations and high-dimensional problems is therefore an important task for future research. To make stochastic parametrization more useful in real applications, many open problems remain to be solved, including how to determine the ‘right’ stochastic parametrization if the one we proposed is inadequate, and how to simultaneously estimate the multiplicative noise coefficients in addition to the linear damping and additive noise considered in §4. Another important issue is to understand whether the linear theory holds when the observation model also depends on the fast variables. We plan to address these questions in our future research.
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