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On uniqueness in the problem of gravity–capillary water waves above submerged bodies

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In this paper, we consider the two-dimensional linear problem of wave–body interaction with surface tension effects being taken into account. We suggest a criterion for unique solvability of the problem based on symmetrization of boundary integral equations. The criterion allows us to develop an algorithm for detecting non-uniqueness (finding trapped modes) for given geometries of bodies; examples of numerical computation of trapped modes are given. We also prove a uniqueness theorem that provides simple bounds for the possible non-uniqueness parameters.

Keywords: water waves; surface tension; submerged body; trapped modes; uniqueness; solvability

1. Introduction

In this paper, we consider the two-dimensional linear problem that describes interaction between an ideal unbounded fluid and bodies located under the free surface of the fluid. In particular, it can be the radiation of waves by forced motion of rigid bodies or diffraction of waves by fixed bodies. The problem appears within the framework of the surface wave theory under the assumptions that the motion of fluid is harmonic in time, irrotational and the oscillations have small amplitudes.

Our main interest here will be the question of uniqueness. It is notable that this question (despite its long history and importance) is not fully answered even in the case when the surface tension is not taken into account; reviews of the existing results can be found in the book by Kuznetsov *et al.* (2002) and in the paper by Linton & McIver (2007). Renewed interest in uniqueness in recent years has been stimulated by finding examples of non-uniqueness (trapped modes) by McIver (1996), and later by many other authors who used the so-called inverse scheme suggested in McIver (1996). (The existence of the examples shows that a general uniqueness theorem is unobtainable.) Furthermore, in recent years,

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a number of new theorems for uniqueness have been proved; among them, we mention the approaches of Motygin & McIver (2003) and of Motygin (2006), which will be used and developed in the present paper.

As far as we are aware, in the case when surface tension effects are taken into account, uniqueness theorems are not known at the moment. First examples of trapped modes for the problem in question have been obtained quite recently by Harter *et al.* (2007, 2008) by using the inverse procedure (McIver 1996, 2000). These examples of non-uniqueness were obtained for surface-piercing bodies (which also needed some attention to the statement itself, because in this case some additional conditions, describing motion of the fluid at the contact points, should be added) and for totally submerged bodies that, however, are very close to the free surface. To the present authors' knowledge, direct methods for finding trapped modes for *given* bodies have not been described previously in the problem under consideration. In the case when surface tension is neglected, a direct method was developed in Porter (2002; although the paper lacks a proof that the used conditions for trapping are sufficient).

In this paper, we shall obtain a number of results concerning unique solvability and non-uniqueness for the problem of gravity–capillary water waves above submerged bodies. We shall derive a new criterion for uniqueness based on the symmetrization of integral equations, which was suggested in Motygin (2006). The criterion will allow us to develop an algorithm for finding trapped modes for given geometries. Some numerical examples showing the effectiveness of the algorithm will be given and discussed. The criterion and the algorithm are also applicable and new for the water wave problem with surface tension neglected; in this case, our computations agree with the results of Porter (2002). The numerical investigation also reveals influence of the surface tension effect—the numerically found trapped modes disappear when surface tension increases. By using the scheme of Motygin & McIver (2003), we shall also prove a new uniqueness theorem that is formulated in the form of simple bounds for parameters corresponding to possible trapped modes. Examples of numerical application of the bounds for some geometries will be given.

The outline of the paper is as follows: in §2 we introduce notations and formulate the mathematical problem; in this section we also present the Green function. In §3 we derive Green's representation for a solution to the problem. Section 4 is devoted to boundary integral equations of the potential theory, which are equivalent to the boundary-value problem. By using the integral equations in §5, we derive a new uniqueness criterion. Results of numerical investigation based on the criterion are presented and discussed in §6. A theorem for uniqueness, which gives bounds for the parameters of the problem that correspond to possible trapped modes, is proved in §7.

2. Statement of the problem

We shall denote by W, B and F the domain occupied by the fluid, the interior of the bodies and the free surface of the fluid, respectively. A Cartesian coordinate system (x, y) will be used in which x is a horizontal coordinate and y is directed vertically upwards (figure 1). Under the imposed assumptions, the motion of the fluid is described by a velocity potential $\mathrm{Re}\{u(x, y)\mathrm{e}^{-\mathrm{i}\omega t}\}$, where ω is the angular

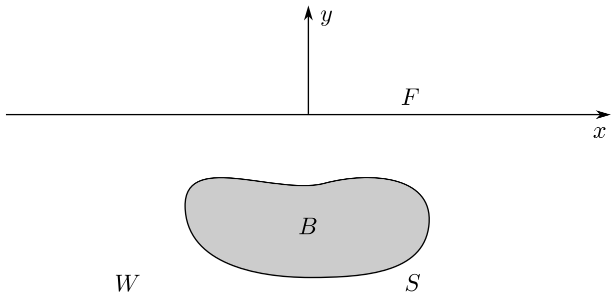

Figure 1. Notations.

frequency; $\nu = \omega^2/g$ is the wavenumber for free surface waves in the absence of surface tension; and g is the acceleration due to gravity. The potential u satisfies the following boundary-value problem (see Billingham & King 2000; Harter *et al.* 2007):

$$\nabla^2 u = 0 \quad \text{in } W = \mathbb{R}^2_- \setminus \bar{B}, \qquad \mathbb{R}^2_- = \{y < 0\}, \tag{2.1}$$

$$\partial_y u - \nu u - \beta \partial_x^2 \partial_y u = 0, \quad \text{on } F = \{y = 0\}, \tag{2.2}$$

$$\partial_n u = f, \quad \text{on } S = \partial B, \tag{2.3}$$

$$\partial_{|x|} u - \mathrm{i} k_0 u = o(1), \quad \text{as } |x| \to \infty, \tag{2.4}$$

$$\sup_W |\nabla u| < \infty. \tag{2.5}$$

Equation (2.2) is the linearized free surface condition including surface tension T through the parameter $\beta = T/(\rho g)$, where ρ is the fluid density. The function f in (2.3) is defined by the type of the oscillations (but its particular form is not important for our consideration). We shall suppose that $f \in C(S)$ and S belongs to the Hölder space $C^{1,\alpha}$, $0 < \alpha < 1$ (e.g. Gilbarg & Trudinger 1983, §4.1). Furthermore, k_0 in the radiation condition (2.4) is the (unique) real positive root of the dispersion relationship

$$\beta k_0^3 + k_0 - \nu = 0.$$

It will be convenient to use the dimensionless parameter $s = T k_0^2/(\rho g)$ that measures the relative importance of surface tension and gravity. Then, from the dispersion relationship, we have $k_0 = \nu/(1+s)$ and $\beta = s(1+s)^2 \nu^{-2}$.

Furthermore, we shall use Green's function $G(x, y, \xi, \eta)$ for the problem satisfying as a function of the first two arguments the conditions (2.2), (2.4) and (2.5) (where the supremum is taken over $\mathbb{R}^2_-$ with a vicinity of the point (ξ, η) excluded) and the condition

$$\nabla^2_{x,y} G(x, y, \xi, \eta) = -\delta(x - \xi)\delta(y - \eta), \tag{2.6}$$

where $y, \eta < 0$ and δ is Dirac's delta function.

Using the expressions given in Harter *et al.* (2007), we write

$$G(z,\zeta) = -\frac{1}{2\pi}\log\frac{r_-}{r_+} + \frac{1}{\pi(3s+1)}\sum_{\pm} g_{\pm}(z,\zeta)$$

$$+\frac{1+s}{\pi(3s+1)}\left[g_0(z,\zeta) + \mathrm{i}\pi\,\mathrm{e}^{k_0(y+\eta)}\cos k_0(x-\xi)\right]. \qquad (2.7)$$

Here, $r_{\pm} = \sqrt{(x-\xi)^2 + (y\pm\eta)^2}$, $z = x+\mathrm{i}y$, $\zeta = \xi + \mathrm{i}\eta$,

$$g_0(z,\zeta) = \int_0^{\infty} \frac{\mathrm{e}^{\mu(y+\eta)}\cos\mu(x-\xi)}{\mu - k_0}\,\mathrm{d}\mu,$$

where the integral is understood in the sense of a Cauchy principal value,

$$g_{\pm}(z,\zeta) = \mathrm{Re}\left\{A_{\pm}\int_0^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\mu(z-\bar{\zeta})}}{\mu - a_{\pm}}\,\mathrm{d}\mu\right\}$$

and

$$a_{\pm} = k_0\left(\pm\mathrm{i}\sqrt{\frac{3}{4}+\frac{1}{s}} - \frac{1}{2}\right), \qquad A_{\pm} = s \pm \frac{\mathrm{i}}{2\sqrt{\frac{3}{4}+\frac{1}{s}}}.$$

By using the exponential integrals Ei and E_1 (see Abramowitz & Stegun 1965, §5.1), we can also write

$$g_0(z,\zeta) = -\mathrm{Re}\left\{\mathrm{e}^{-\mathrm{i}k_0(z-\bar{\zeta})}\,\mathrm{Ei}(\mathrm{i}k_0(z-\bar{\zeta}))\right\}, \qquad (2.8)$$

$$g_{\pm}(z,\zeta) = \mathrm{Re}\left\{A_{\pm}\mathrm{e}^{-\mathrm{i}a_{\pm}(z-\bar{\zeta})}\,\mathrm{E}_1(-\mathrm{i}a_{\pm}(z-\bar{\zeta}))\right\}. \qquad (2.9)$$

3. Green's identity

In this section, we shall derive Green's representation for a solution to (2.1)–(2.5). First, we need to find the general solution to the problem in the empty half-space $\mathbb{R}^2_-$. This solution u satisfies the following set of conditions:

$$\nabla^2 u = 0, \quad \text{in } \mathbb{R}^2_-, \qquad (3.1)$$

$$\partial_y u - \nu u - \beta\partial_x^2\partial_y u = 0, \quad \text{on } F, \qquad (3.2)$$

$$\sup_{\mathbb{R}^2_-} |\nabla u| < \infty. \qquad (3.3)$$

Applying the Fourier transform in x, defined by

$$\tilde{u} = \mathcal{F}u = \frac{1}{2\pi}\int_{\mathbb{R}} u(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x,$$

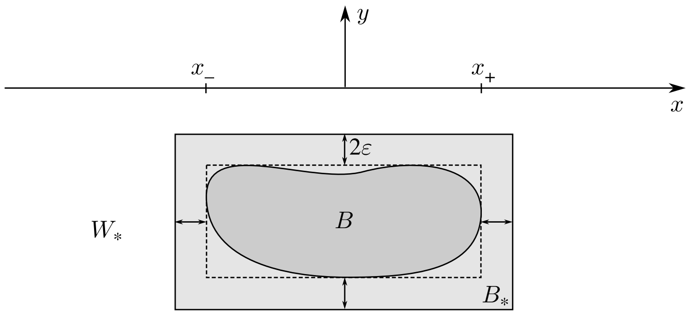

Figure 2. Auxiliary notations.

to (3.1) and (3.2), we arrive at

$$\partial_y^2\tilde{u} - k^2\tilde{u} = 0, \quad \text{when } y < 0, \tag{3.4}$$

$$\partial_y\tilde{u} - \nu\tilde{u} + \beta k^2\partial_y\tilde{u} = 0, \quad \text{when } y = 0. \tag{3.5}$$

The general form of a solution to (3.4) with an account of (3.3) is $\tilde{u} = c(k)\mathrm{e}^{|k|y}$. Substituting this solution into (3.5), we find

$$c(k)\mathrm{e}^{|k|y}(|k| - \nu + |k|^3\beta) = 0.$$

The expression in brackets has a simple zero at $|k| = k_0$, which means that

$$c(k) = c_+\delta(k - k_0) + c_-\delta(k + k_0),$$

where c_+ and c_- are the arbitrary coefficients.

Finally, applying the inverse Fourier transform to $\tilde{u} = c(k)\mathrm{e}^{|k|y}$ and taking into account that $\mathcal{F}^{-1}[\delta(k-a)] = \mathrm{e}^{\mathrm{i}ax}$, we arrive at the following general form of solution to the equations (3.1)–(3.3):

$$u = c_+\mathrm{e}^{k_0 y}\mathrm{e}^{\mathrm{i}k_0 x} + c_-\mathrm{e}^{k_0 y}\mathrm{e}^{-\mathrm{i}k_0 x}. \tag{3.6}$$

It is notable that the solution (3.6) satisfies (2.5) only if $c_+ = c_- = 0$.

We can now derive Green's representation for a solution u to the problem (2.1)–(2.5). We introduce a new potential

$$u_0(x, y) = [1 - \chi_{x_-,x_+}(x)\chi_{y_-,y_+}(y)]u(x, y),$$

where $\chi_{p,q}$ ($p, q \in \mathbb{R}$) is an infinitely smooth cut-off function

$$\chi_{p,q}(t) = \begin{cases} 1, & \text{for } p - \varepsilon \le t \le q + \varepsilon, \\ 0, & \text{for } t \le p - 2\varepsilon \text{ and } t \ge q + 2\varepsilon, \end{cases}$$

and $y_\pm = \pm\max\{\pm y\}$, $x_\pm = \pm\max\{\pm x\}$, where the maxima are taken over all $(x, y) \in S$. Furthermore, we choose $0 < \varepsilon < |y_+|/2$.

The function u_0 coincides with u in $W_* = W \setminus \bar{B}_*$, where (figure 2)

$$B_* = \{(x, y) : \pm x < \pm x_\pm + 2\varepsilon, \pm y < \pm y_\pm + 2\varepsilon\} \supset B.$$

The complex-valued function u_0 is defined for all $(x, y) \in \mathbb{R}^2_-$, belongs to $C^\infty(\mathbb{R}^2_-)$ and

$$-\nabla^2 u_0 = \sigma, \quad \operatorname{supp}\sigma \subset B_*, \quad \sigma \in C^\infty(\mathbb{R}^2_-). \tag{3.7}$$

Define now

$$u_1(x, y) = \int_{B_*} \sigma(x, y) G(x, y, \xi, \eta) \mathrm{d}\xi \, \mathrm{d}\eta.$$

By definition of Green's function, the potential u_1 satisfies (2.2), (2.4), (2.5) and (3.7). The difference $u_0 - u_1$ satisfies (3.1) and, thus, by the above results on the general form of solution in an empty half-plane $u_0 = u_1$ in $\mathbb{R}^2_-$.

Applying Green's formula over B_* when $(x, y) \in W_*$, we find from (3.7)

$$\begin{aligned} u_0(z) &= \int_{B_*} \left[u_0(\zeta) \nabla^2_\zeta G(z, \zeta) - G(z, \zeta) \nabla^2 u_0(\zeta) \right] \mathrm{d}\xi \, \mathrm{d}\eta \\ &= \int_{\partial B_*} [u_0(\zeta) \partial_{n(\zeta)} G(z, \zeta) - \partial_n u_0(\zeta) G(z, \zeta)] \mathrm{d}s_\zeta, \end{aligned}$$

where the normal on ∂B_* is defined to be outward with respect to B_* and we also used the fact that $\nabla^2_\zeta G(z, \zeta) = 0$ when $z \in W^*$, $\zeta \in B^*$ due to (2.6) and $G(z, \zeta) = G(\zeta, z)$.

Applying Green's identity over $B_* \backslash B$ for $z \in W_*$, we find

$$\begin{aligned} 0 &= \int_{B_* \backslash B} \left[G(z, \zeta) \nabla^2 u(\zeta) - u(\zeta) \nabla^2_\zeta G(z, \zeta) \right] \mathrm{d}\xi \, \mathrm{d}\eta \\ &= \left(\int_{\partial B_*} - \int_S \right) [\partial_n u(\zeta) G(z, \zeta) - u(\zeta) \partial_{n(\zeta)} G(z, \zeta)] \mathrm{d}s_\zeta. \end{aligned}$$

Combining the last two formulae, we have

$$u(z) = \int_S [u(\zeta) \partial_{n(\zeta)} G(z, \zeta) - \partial_n u(\zeta) G(z, \zeta)] \mathrm{d}s_\zeta, \tag{3.8}$$

for $z \in W_*$. However, the right-hand side defines an analytic function in W, and, since it coincides with the analytic function u in W_*, the latter representation is valid for any $z \in W$. It is also notable that, in view of the definition of Green's function, the expression in the right-hand side of (3.8) defines an analytic function in the wider than W domain $\{y < -y_+\} \backslash \bar{B}$.

4. Integral equations of potential theory

Following the usual scheme of potential theory (e.g. Kuznetsov *et al.* 2002, §2.1), we shall seek solutions to the problem (2.1)–(2.5) in the form of a single-layer potential

$$u(z) = (V\mu)(z), \quad z \in W, \tag{4.1}$$

where

$$(V\mu)(z) = \int_S \mu(\zeta) G(z, \zeta) \mathrm{d}s_\zeta, \tag{4.2}$$

and μ is some unknown density belonging to $C(S)$. Properties of the single-layer potential are described in detail in Kuznetsov *et al.* (2002, §2.1.1.1), the arguments are valid in the present case in view of (2.7), where Green's function is written as a sum of $-(2\pi)^{-1} \log r_-$ and some function that is smooth in $z, \zeta \in \mathbb{R}^2_-$.

The potential (4.1) satisfies conditions (2.1), (2.2), (2.4) and (2.5) and, by the jump relationship for the normal derivative of a single-layer potential, the condition (2.3) leads us to the boundary integral equation

$$-\mu(z) + (T\mu)(z) = 2f(z), \quad z \in S, \tag{4.3}$$

where

$$(T\mu)(z) = 2\int_S \mu(\zeta)\partial_{n(z)} G(z,\zeta)\mathrm{d}s_\zeta.$$

Under the assumption $S \in C^{1,\alpha}$ the operator is compact in $L^2(S)$, which is the Hilbert space of square integrable functions with the scalar product

$$\langle v, w\rangle = \int_S v\bar{w}\,\mathrm{d}s.$$

In addition, the integral equation (4.3) is of Fredholm's type. We also emphasize that if equation (4.3) is solvable in $L^2(S)$ and $f \in C(S)$, then the solution μ belongs to $C(S)$.

The adjoint operator T^* appears in the integral equation of the direct method. To obtain the equation, we consider (3.8) and move the point z onto the contour S. By using the jump relationship for the double-layer potentials, we arrive at the equation

$$-u(z) + (\overline{T^*}u)(P) = 2(Vf)(P), \tag{4.4}$$

where

$$(\overline{T^*}u)(z) = 2\int_S u(\zeta)\partial_{n(\zeta)} G(\zeta, z)\mathrm{d}s_\zeta.$$

The arguments applied in Kuznetsov *et al.* (2002, §2.1) for the investigation of the integral equations for the water wave problem without surface tension effects can be repeated literally for the problem (2.1)–(2.5) and integral equations (4.3) and (4.4). In this way, we find that the problem (2.1)–(2.5) is uniquely solvable if and only if the homogeneous boundary integral equations on S,

$$-\mu + T\mu = 0, \qquad -u + \overline{T^*}u = 0, \tag{4.5}$$

have only the trivial solution in $L^2(S)$. Otherwise, the equations and the homogeneous problem have the same number of linearly independent solutions and solutions to the second equation (4.5) are traces of solutions to the homogeneous problem (2.1)–(2.5), which in their turn are given by $V\mu$ from solutions to the first equation (4.5).

We shall use the real and imaginary parts of the operator T^* defined by

$$(T^*_{\mathrm{r}}u)(z) = 2\int_S u(\zeta)\partial_{n(\zeta)}\,\mathrm{Re}\,G(\zeta, z)\mathrm{d}s_\zeta,$$

$$(T^*_{\mathrm{i}}u)(z) = -2\int_S u(\zeta)\partial_{n(\zeta)}\,\mathrm{Im}\,G(\zeta, z)\mathrm{d}s_\zeta.$$

It is easy to note that if a function $u: S \mapsto \mathbb{C}$ satisfies the equations

$$-u + T_{\mathrm{r}}^* u = 0, \tag{4.6}$$

$$T_{\mathrm{i}}^* u = 0, \tag{4.7}$$

then they also hold for Re u and Im u. At the same time, for any real-valued function v, we have $(T_{\mathrm{r}}^* - \mathrm{i}\, T_{\mathrm{i}}^*)v = \overline{T^*} v$. This means that each of the functions Re u and Im u satisfies the second equation (4.5) and it is also true for u.

Let us now show that any solution to the second equation (4.5) satisfies the system (4.6) and (4.7). For this, we shall need some asymptotic analysis for a solution to the problem (2.1)–(2.5). We use the Green identity (3.8) and asymptotic representation of Green's function as $|z| \to \infty$, which can readily be obtained from the formula (2.7) and known asymptotic representations of Ei and E_1 given in Abramowitz & Stegun (1965) and Motygin & McIver (2003; see also §7). This gives us the following asymptotics as $|z| \to \infty$ and $\pm x > 0$:

$$u(z) = C_{\pm} \mathrm{e}^{k_0 y} \, \mathrm{e}^{\mathrm{i}k_0 |x|} + \varphi_{\pm}(z), \tag{4.8}$$

where $C_{\pm}$ are some complex coefficients and $\partial_x^n \, \partial_y^m \varphi_{\pm} = O(|z|^{-1})$ as $|z| \to \infty$ for any $n, m \geq 0$.

Let u be a solution to the homogeneous problem (2.1)–(2.5). Consider a domain $W_a = W \cap \{ |x| < a \}$ and apply Green's formula over the domain for u and $\bar{u}$. We have

$$0 = \int_{W_a} [\bar{u} \nabla^2 u - u \nabla^2 \bar{u}] \mathrm{d}x \, \mathrm{d}y = \int_{\partial W_a} [u \partial_n \bar{u} - \bar{u} \partial_n u] \mathrm{d}s, \tag{4.9}$$

where the normal $\boldsymbol{n}$ is directed to the interior of W_a.

We write $\partial W_a = S \cup F_a \cup L_+ \cup L_-$, where $F_a = F \cap \{|x| < a\}$, $L_{\pm} = \{(x, y): x = \pm a, \, y < 0\}$. Taking into account that $\partial_n = -\partial_y$ and $u = \nu^{-1}(u_y - \beta u_{xxy})$ on F_a (by (2.2)), in view of (4.8), integrating by parts, we find

$$\begin{aligned}\int_{F_a} [u \partial_n \bar{u} - \bar{u} \partial_n u] \mathrm{d}x &= \frac{\beta}{\nu} \int_{F_a} [\bar{u}_y u_{xxy} - u_y \bar{u}_{xxy}] \mathrm{d}x = \frac{\beta}{\nu} [\bar{u}_y u_{xy} - u_y \bar{u}_{xy}]_{x=-a}^{x=a} \\ &= \frac{2\mathrm{i}\beta k_0^3}{\nu} (|C_+|^2 + |C_-|^2) + o(1), \quad \text{as } a \to \infty.\end{aligned}$$

Analogously, we have

$$\int_{L_{\pm}} [u \partial_n \bar{u} - \bar{u} \partial_n u] \mathrm{d}y = \pm \int_{-\infty}^{0} [\bar{u} u_x - u \bar{u}_x] \mathrm{d}y = \mathrm{i} |C_{\pm}|^2 + o(1), \quad \text{as } a \to \infty.$$

Finally, in the limit $a \to \infty$, we find from (4.9) $(1 + 2\beta\nu^{-1} k_0^3)(|C_+|^2 + |C_-|^2) = 0$ and conclude that a solution u to the homogeneous problem (2.1)–(2.5) decays to zero at infinity and it is the so-called trapped mode. The latter means that each of the functions Re u, Im u is a solution to the homogeneous problem, its trace on S is a solution to the second equation (4.5) and, thus, satisfies the system (4.6) and (4.7). Finally, we can conclude that the second equation (4.5) and the system (4.6) and (4.7) are equivalent, having the same set of solutions.

5. Criterion for uniqueness

We start with a symmetrization, suggested in Motygin (2006), for the integral equation (4.6) and the adjoint one

$$(I - T_{\mathrm{r}})\mu = 0. \tag{5.1}$$

Applying the operator $I - T_{\mathrm{r}}^*$ to the last equation and $I - T_{\mathrm{r}}$ to (4.6), we arrive at

$$-\mu + \mathfrak{T}\mu = 0, \qquad \mathfrak{T} = T_{\mathrm{r}} + T_{\mathrm{r}}^* - T_{\mathrm{r}}^* T_{\mathrm{r}}, \tag{5.2}$$

$$-u + \mathfrak{T}' u = 0, \qquad \mathfrak{T}' = T_{\mathrm{r}} + T_{\mathrm{r}}^* - T_{\mathrm{r}} T_{\mathrm{r}}^*, \tag{5.3}$$

and, obviously, solutions to (5.1) and (4.6) satisfy (5.2) and (5.3), respectively. It can also be observed that solutions to (5.2) and (5.3) satisfy (5.1) and (4.6), respectively. First of all (5.2) can be written as $(I - T_{\mathrm{r}}^*)(I - T_{\mathrm{r}})\mu = 0$ and either $(I - T_{\mathrm{r}})\mu = 0$ or $(I - T_{\mathrm{r}})\mu = v \neq 0$ and $(I - T_{\mathrm{r}}^*)v = 0$. By Fredholm's alternative, the subspace $\mathrm{Ker}(I - T_{\mathrm{r}}^*)$ is orthogonal to $\mathrm{Im}(I - T_{\mathrm{r}})$. Hence $\langle v, v\rangle = 0$ because v belongs to both subspaces. Analogously, we can consider (5.3) and (4.6).

It is important to note that $\mathfrak{T}$ and $\mathfrak{T}'$ are compact and, unlike T_{r}, self-adjoint operators with real eigenvalues $\lambda_i \in \sigma(\mathfrak{T})$ and $\lambda_i' \in \sigma(\mathfrak{T}')$. It can be observed that $\langle (I - T_{\mathrm{r}}^*)(I - T_{\mathrm{r}})v, v\rangle = \langle (I - T_{\mathrm{r}})v, (I - T_{\mathrm{r}})v\rangle \geq 0$. Thus, $\langle \mathfrak{T}v, v\rangle \leq \langle v, v\rangle$ and all eigenvalues of the operator $\mathfrak{T}$ are submitted to the inequality $\lambda_i \leq 1$. Analogously, $\lambda_i' \leq 1$. Furthermore, we shall use the notation $\lambda_1 = \max\{\lambda_i\}$. It follows from the above that (4.6) has only the trivial solution if and only if $\lambda_1 < 1$, and non-trivial solutions exist only when $\lambda_1 = 1$.

The above arguments were first outlined in the short note by Motygin (2006), where they were also used for numerical investigation of the uniqueness property for the water wave problem in the case when the surface tension was not taken into account. Further in this section, we shall make use of the formalism introduced above and apply more detailed analysis for the operators $\mathfrak{T}$ and $\mathfrak{T}'$. This will allow us to derive a new criterion of uniqueness for the problem (2.1)–(2.5).

Let $\lambda_1 \neq 1$. It can be observed that in this case the operators $\mathfrak{T}$ and $\mathfrak{T}'$ have the same system of eigenvalues $\{\lambda_i\}$ and dimensions N_i and N_i' of eigenspaces $\mathcal{E}_i$ and $\mathcal{E}_i'$ corresponding to any eigenvalue λ_i are equal. Consider an eigenvalue λ_i of the operator $\mathfrak{T}$. From the definition of the eigenvalue and taking into account the definition of $\mathfrak{T}$, for $\mu_i \in \mathcal{E}_i$, we have

$$(I - T_{\mathrm{r}}^*)(I - T_{\mathrm{r}})\mu_i = (1 - \lambda_i)\mu_i.$$

Applying the operator $I - T_{\mathrm{r}}$ from the left to the latter equality, we arrive at

$$(I - T_{\mathrm{r}})(I - T_{\mathrm{r}}^*)(I - T_{\mathrm{r}})\mu_i = (1 - \lambda_i)(I - T_{\mathrm{r}})\mu_i,$$

or $(I - \mathfrak{T}')u_i = (1 - \lambda_i)u_i$, where $u_i = (I - T_{\mathrm{r}})\mu_i$. The latter means that λ_i is an eigenvalue of $\mathfrak{T}'$ and $(I - T_{\mathrm{r}})\mathcal{E}_i \subset \mathcal{E}_i'$. Since $I - T_{\mathrm{r}}$ is bijective on $L^2(S)$, this means that $N_i \leq N_i'$. Applying now $I - T_{\mathrm{r}}^*$ to $\mathcal{E}_i'$, we find $(I - T_{\mathrm{r}}^*)\mathcal{E}_i' \subset \mathcal{E}_i$ and $N_i' \leq N_i$ and, finally, $N_i = N_i'$.

Let us denote by $\mu_1^{(i)}$ and $u_1^{(i)}$ $(i = 1, \ldots, N_1)$ the eigenfunctions of $\mathfrak{T}$ and $\mathfrak{T}'$ corresponding to λ_1, where N_1 is the multiplicity of λ_1. We can prove that if for some i and for all $j = 1, \ldots, N_1$

$$\left\langle (I - T_{\rm r}^*)u_1^{(i)}, \mu_1^{(j)} \right\rangle = 0, \tag{5.4}$$

then $\lambda_1 = 1$ and $(I - T_{\rm r}^*)u_1^{(i)} = 0$. Suppose the contrary, i.e. (5.4) holds and $\lambda_1 \neq 1$. Then, the mapping $I - T_{\rm r}^*$ is a bijection between span$\{u_1^{(1)}, \ldots, u_1^{(N_1)}\}$ and span$\{\mu_1^{(1)}, \ldots, \mu_1^{(N_1)}\}$. This means that we can write

$$(I - T_{\rm r}^*)u_1^{(i)} = \sum_{k=1}^{N_1} c_k \mu_1^{(k)}.$$

Then, from (5.4), we obtain

$$\sum_{k=1}^{N_1} c_k \left\langle \mu_1^{(k)}, \mu_1^{(j)} \right\rangle = 0, \quad j = 1, \ldots, N_1.$$

Since $\mu_1^{(i)}$ are linearly independent, from the last system of linear equation with Gram matrix, it follows that $c_k = 0$ $(k = 1, \ldots, N_1)$ and $(I - T_{\rm r}^*)u_1^{(i)} = 0$, which contradicts the assumption that $\lambda_1 \neq 1$. Repeating the arguments literally, we can also replace (5.4) by the conditions $\langle (I - T_{\rm r})\mu_1^{(i)}, u_1^{(j)} \rangle = 0$.

Summing up the above arguments, we arrive at the following assertion.

Theorem 5.1. *The homogeneous problem* (2.1)–(2.5) *has non-trivial solutions if and only if some eigenfunction* $u_1^{(i)}$ *of the operator* $\mathfrak{T}'$ *satisfies the conditions* (5.4) *and* (4.7).

In conclusion of this section, we note that for a real-valued function u in view of (2.7)

$$-\frac{1 + 3s}{1 + s}(T_{\rm i}^* u)(z) = 2\mathrm{e}^{k_0 y} \cos k_0 x \,\mathrm{Re}\{\Theta(u)\} + 2\mathrm{e}^{k_0 y} \sin k_0 x \,\mathrm{Im}\{\Theta(u)\},$$

where

$$\Theta(u) := \int_S u(\zeta)\partial_n(\mathrm{e}^{k_0 \eta}\,\mathrm{e}^{\mathrm{i}k_0 \xi})\,\mathrm{d}s_\zeta.$$

Therefore, a function u satisfies (4.7) if it satisfies the condition $\Theta(u) = 0$. This means that, under the assumption that λ_1 is simple, for any geometry the three conditions

$$\Psi := \langle (I - T_{\rm r}^*)u_1, \mu_1 \rangle = 0, \qquad \mathrm{Re}\{\Theta(u_1)\} = 0 \quad \text{and} \quad \mathrm{Im}\{\Theta(u_1)\} = 0 \tag{5.5}$$

together guarantee the existence of a trapped mode.

6. Numerical algorithm for finding trapped modes

In this section, we shall apply the criterion (5.5) to find trapped modes numerically and to demonstrate the effects of surface tension. We shall consider the case when the geometry S is symmetric with respect to the y-axis, which simplifies finding solutions to (5.5); when λ_1 is simple, the function u_1 is either even or odd in x and one of the last two conditions (5.5) is fulfilled automatically.

Thus, we shall verify numerically that non-trivial solutions to the pair of equations (5.5) exist. To do this, we shall plot the two curves corresponding to the equations in the plane of two parameters (here it will be ν and one of the geometrical parameters); when the curves are found to intersect, rather than touch, this gives a robust numerical method for establishing the existence of trapped modes as the intersection cannot be lost as a result of numerical error. A similar scheme has been used by a number of authors, for example Porter (2002).

We should note some difficulty in numerical investigation based on (5.5), which is connected with ambiguity in the definitions of Ψ and Θ. Basically, even under the normalization $\|u_1\| = \|\mu_1\| = 1$, when, for instance,

$$\Psi(\nu) = \left\langle (I - T_{\mathrm{r}}^{*}) \frac{u_1}{\|u_1\|}, \frac{\mu_1}{\|\mu_1\|} \right\rangle, \tag{6.1}$$

each of the functionals Ψ and Θ is defined only up to the sign. Nevertheless, it can be shown that it is possible to define Ψ and Θ as piecewise analytic in ν or s functions and this allows us to find zeros of the functionals convincingly.

First of all, we recall that according to Kato (1976, ch. II, §1, ch. VII, §3), eigenvalues λ_i of the self-adjoint operator $\mathfrak{T}$, which depend analytically on a parameter ν (or s), are continuous functions of the parameter. They also depend on ν (or s) analytically with the exception of some isolated values, where algebraic type splitting of the curves happens; between these points, multiplicity of the eigenvalue does not change.

Let λ_1 be a simple eigenvalue for $\nu \in (\nu', \nu'')$. We denote by $P(\nu)$ and $P'(\nu)$ the projectors to the eigenspaces, corresponding to $\lambda_1(\nu)$ for $\mathfrak{T}$ and $\mathfrak{T}'$, respectively. According to Kato (1976, ch. VII, §3), these operators $P(\nu)$ and $P'(\nu)$ depend analytically on $\nu \subset (\nu', \nu'')$. Consider some finite interval $\gamma = [\nu_0, \nu_1] \subset (\nu', \nu'')$, fix some point $\nu_* \in \gamma$ and define

$$\mu_1(\nu) = P(\nu)\mu_1(\nu_*), \qquad u_1(\nu) = P'(\nu) u_1(\nu_*). \tag{6.2}$$

Since $\mu_1(\nu_*)$, $u_1(\nu_*) \neq 0$, the denominators of $\mu_1/\|\mu_1\|$, $u_1/\|u_1\|$ differ from zero in $V_{\varepsilon_*}(\nu_*)$—some open ε_*-neighbourhood of ν_*. This means that the function $\Psi(\nu)$, which is defined with the help of (6.1) and (6.2), is analytic in $V_{\varepsilon_*}(\nu_*)$. Since γ is finite, we can define $\varepsilon_0 = \min\{\varepsilon_*(\nu_*) : \nu_* \in \gamma\} > 0$ and introduce a finite system of overlapping intervals $V_{\varepsilon_0}(\nu_i)$, such that $\gamma \subset \cup_{i=1}^{m} V_{\varepsilon_0}(\nu_i)$, where $\nu_i < \nu_{i+1}$, $|\nu_i - \nu_{i+1}| < 2\varepsilon_0$, $i = 1, 2, \dots, m-1$. In each of the intervals $V_{\varepsilon_0}(\nu_i)$, the function $\Psi(\nu)$ has an analytic representation by (6.1) and (6.2) defined up to the sign. Choosing the signs to match the representations in $V_{\varepsilon_0}(\nu_i) \cap V_{\varepsilon_0}(\nu_{i+1})$ $(i = 1, 2, \dots, m-1)$, we obtain an analytic function on the whole interval γ.

Furthermore, we shall present and discuss the results of computations based on (5.5) with the above remarks taken into account. First, we consider a configuration consisting of two equal ellipses with horizontal semi-axis a and vertical semi-axis b, with centres at the depth d and distance between centres $2l$. Computations are carried out for $b/a = 0.08$ and $d/a = 0.25$ (this geometry was considered in the work of Porter (2002) for the water wave problem with surface tension neglected). We use a simple collocation scheme for the approximation of the integral operator T_{r}. Each of the ellipses is split into $N_p/2$ arcs uniformly in $\theta \in [0, 2\pi]$ in the representation $x(\theta) = \pm l + a\cos\theta$, $y(\theta) = -d + b\sin\theta$.

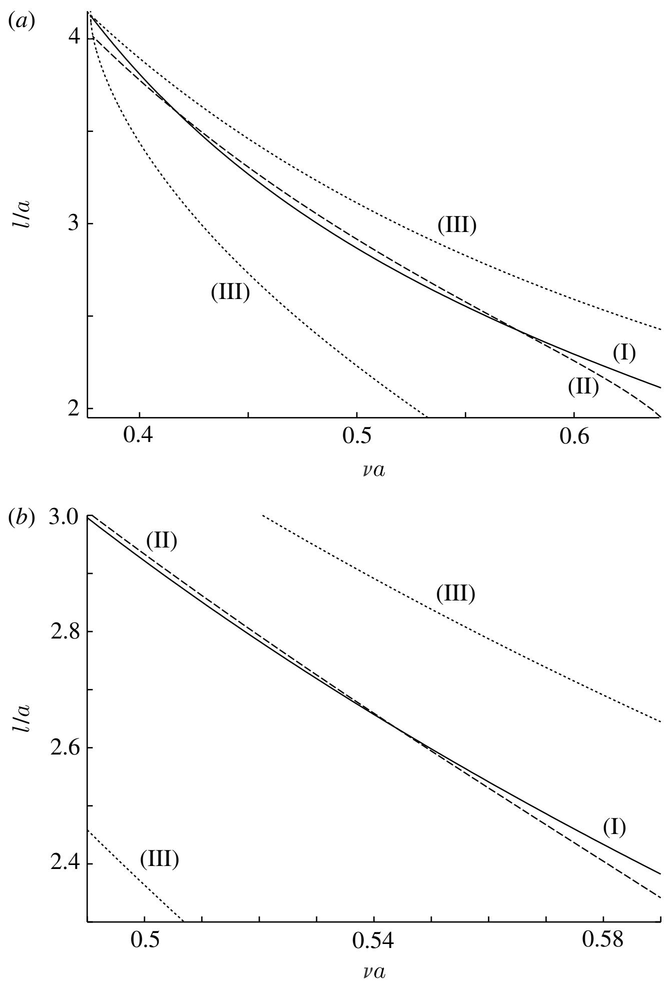

Figure 3. Curves $\Psi=0$ (solid line, I), $\Theta=0$ (dashed line, II) and points of u_1 parity change (dotted line, III) for two ellipses and (*a*) $s=0.002$ and (*b*) $s=0.006$.

In figure 3*a*,*b*, we show the behaviour of the curves $\Psi=0$ (I) and $\Theta=0$ (II) in the plane (νa, l/a) for $s=0.002$ and 0.006, respectively. We checked numerically that $\lambda_1-\lambda_2$ is separated from zero at the points of intersection of the curves (I) and (II). It is also very likely that the simplicity of λ_1 takes place for all values of parameters (νa, l/a), shown in figure 3, except for the values belonging to the curves (III), where multiplicity of λ_1 is found to be equal to 2 (or 3, at the point of intersection of the curves (III) in figure 3*a*). These points reveal themselves as points of parity change for the function u_1. The curves in figure 3*b* were found with $N_{\rm p}=240$; the computations near the point of intersection of curves (III) in figure 3*a* are more demanding to the accuracy and $N_{\rm p}$ was increased to 360.

It can be observed that the computations are quite strongly influenced by surface tension effects. With an increase of s, the intersection of the curves $\Psi=0$ and $\Theta=0$ becomes less distinct and the intersection disappears at some value of s.

Table 1. Location of trapped modes for different values of the surface tension parameter s.

	ellipses		rounded rectangles	
s	$\nu a\approx$	$l/a\approx$	$\nu a\approx$	$l/a\approx$
0	0.5842	2.348	0.6104	2.057
0.001	0.5798	2.381	0.6100	2.070
0.002	0.5749	2.417	0.6096	2.083
0.003	0.5693	2.457	0.6090	2.096
0.004	0.5628	2.504	0.6084	2.110
0.005	0.5548	2.559	0.6076	2.124
0.006	0.5444	2.630	0.6068	2.138

We do not aim to define the limit value with high accuracy, but it is found that for $s=0.0075$ the curves (I) and (II) shown in figure 3 are separated one from another.

Similar computations were done for a configuration consisting of two equal rectangles with rounded corners of radius c. Each of the rectangles has width $2a$ and height $2b$, its centre is located on the depth d and the distance between centres is $2l$. Computations were carried out for $b/a=0.08$, $d/a=0.25$ and $c/a=0.02$. Results for the computation of trapped modes parameters for some values of the parameter s for the two geometries (ellipses and rectangles) are given in table 1. We note that the values obtained for two ellipses and $s=0$ can be compared and they agree with the results of Porter (2002). It is also worth pointing out that the locations of the trapped modes in the $(\nu a, l/a)$ parameter space are less sensitive to changes in s for the rounded rectangles.

We note that although trapped modes are found only for small values of the parameter s, the values are, however, sufficiently large from a physical point of view. Consider, for example, the pair water–air, where the coefficient of surface tension T is approximately equal to $0.072\ \mathrm{N\,m^{-1}}$ and $\rho=1000\ \mathrm{kg\,m^{-3}}$. Solving the equation $s(1+s^2)=T(\nu a)^2/(a^2\rho g)$ for $s=0.006$ and the corresponding value $\nu a\approx 0.544$ (table 1), we find $a\approx 19$ mm and smaller values of s correspond to bigger values of the ellipse length.

7. Uniqueness theorem

In this section, we shall prove a uniqueness theorem for the problem (2.1)–(2.5) using the approach suggested by Motygin & McIver (2003). We also emphasize that from arguments given by Kuznetsov *et al.* (2002, §2.1)—in §4, we discussed their applicability to the problem in question—it follows that a uniqueness theorem guarantees solvability of the boundary-value problem where the solution has the form (4.1).

Consider Green's identity (3.8) for a hypothetical trapped mode u, which is a non-trivial solution to the homogeneous problem (2.1)–(2.5). We recall that without loss of generality we can suppose that u is real-valued and, hence, we have

$$u(z) = \int_S u(\zeta)\partial_{n(\zeta)} G_{\mathrm{r}}(z,\zeta)\,\mathrm{d}s_\zeta, \quad z\in W\cup F,$$

where G_{r} is the real part of Green's function. From the last formula, we conclude that

$$|u(z)| \leq \max_S |u| \int_S |\partial_{n(\zeta)} G_{\mathrm{r}}(z,\zeta)| \, \mathrm{d}s_\zeta, \tag{7.1}$$

and, hence,

$$\sup_F |u| \leq \max_S |u| \sup_{z\in F} \left\{ \int_S |\partial_{n(\zeta)} G_{\mathrm{r}}(z,\zeta)| \, \mathrm{d}s_\zeta \right\}. \tag{7.2}$$

Let us now make use of maximum principle to compare $\sup_F |u|$ and $\max_S |u|$. The strong maximum principle of Hopf (e.g. Gilbarg & Trudinger 1983, theorem 3.5) guarantees that a non-constant potential u satisfying (2.1) cannot attain its maximum or minimum value at an internal point of any subset of W. This assertion is valid for the whole unbounded domain W; for this, it is sufficient to note that u decays to zero at infinity (see §4).

Furthermore, we recall that the wetted surface of bodies S is assumed to be sufficiently smooth, belonging to the Hölder class $C^{1,\alpha}$, $0<\alpha<1$, so we can make use of Giraud's theorem (e.g. Miranda 1970, p. 6). Consider a bounded domain $W_0 \subset W$, such that $S \subset \partial W_0$. Let $\min_{W_0} u \leq 0$ ($\max_{W_0} u \geq 0$) and u takes this minimum (maximum) value at points M_i on ∂W_0; $\boldsymbol{n}_i$ is the normal vector at M_i directed into W_0. The theorem guarantees that at every point M_i for any vector $\boldsymbol{m}$, such that the scalar product $\boldsymbol{n}_i \cdot \boldsymbol{m}$ is strictly positive, there exists a positive constant L, such that for $P = M_i + t\boldsymbol{m}$, $P \in W_0$, and sufficiently small $t>0$

$$u(P) - u(M_i) > Lt \quad (< -Lt). \tag{7.3}$$

In view of the homogeneous Neumann condition on S, this means that $\max_{W_0} |u|$ cannot be attained on S. Finally, we can conclude that either $u \equiv 0$ in W or

$$\max_S |u| < \sup_F |u|. \tag{7.4}$$

Combining (7.2) and (7.4), we come to the following sufficient condition for uniqueness of solution to the problem (2.1)–(2.5).

Theorem 7.1. *If for a contour S and values of parameters ν, s the following inequality holds*:

$$\sup_{z\in F} \left\{ \int_S |\partial_{n(\zeta)} G_{\mathrm{r}}(\nu, s; z,\zeta)| \, \mathrm{d}s_\zeta \right\} \leq 1, \tag{7.5}$$

then the homogeneous problem (2.1)–(2.5) *has only the trivial solution.*

The condition (7.5) can be improved by using auxiliary potentials, e.g. waveless singular potentials or multipoles located inside the bodies. Here, we shall not discuss this possibility; more details can be found in Motygin & McIver (2003).

We can generalize the statement of the last theorem to contours S belonging to C^1 by using the lemma proved by Nadirashvili (1985). The latter assertion states that regardless of the fact that for such domains a derivative in a non-tangent direction at an extremum point can be equal to zero, in any vicinity of the extremum, there is a point where all such derivatives are non-zero.

It can be also shown that theorem 7.1 is valid for some classes of contours with corners and cusps protruding to the domain occupied by fluid. We shall denote these points at S by P_i and assume that S consists of arcs of class C^3 (including the arcs' endpoints). If the local finiteness of energy is assumed for a solution to the homogeneous problem (2.1)–(2.4)

$$\int_E |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y < \infty,$$

where E is a compact subset of $\overline{W}$, then the results of Wigley (1970) guarantee that the solution is classical, i.e. $u \in C^2(\overline{W} \setminus \{P_i\}) \cap C(\overline{W})$.

The maximum of $|u|$ in W cannot be located at a regular point of S; as we have seen above, this would mean that $\partial_n u < 0$ or $\partial_n u > 0$, which contradicts the homogeneous condition (2.3). Suppose that the maximum $|u|$ is located at $M \in \{P_i\}$, let us choose the sign of u in such a way that it is the minimum. Let $\alpha \in (\pi, 2\pi]$ be the angle between one-sided tangents to S at M. We can place a sector with opening α_0 and with the origin at the point M so that the sector belongs to W. For solution u, the main assertion of a paper by Oddson (1968) guarantees that in some neighbourhood of M

$$u(x,\, y) \geq u(M) + c\rho^{\pi/\alpha_0} \cos(\pi\theta/\alpha_0), \tag{7.6}$$

where $(\rho,\, \theta)$ are polar coordinates, such that $\theta \in [-\alpha_0/2, \alpha_0/2]$ inside the sector, and c is a coefficient, such that $c > 0$.

By using lemma 10 in Wigley (1970), it is not difficult to obtain the following asymptotics as $\rho \to 0$:

$$u(\rho, \theta) = u(M) + d\rho^{\pi/\alpha} \sin(\pi\theta'/\alpha) + O(\rho),$$

where d is a coefficient and $\theta' = \pm\alpha/2$ on the one-sided tangents at M. It can be seen that there is a contradiction between the last formula and (7.6) (consider, for example, the direction $\theta' = 0$). Hence, the maximum $|u|$ in W cannot be attained at $M \in \{P_i\}$ and we can conclude that (7.4) (and, thus, theorem 7.1) holds for bodies with the corners and cusps directed to the fluid.

Furthermore, we shall show that the sufficient condition for uniqueness (7.5) is effective for finding bounds for parameters of a problem corresponding to possible trapped modes. Namely, we shall derive an estimate in the form $|\nabla_{\xi,\eta} G(z,\, \zeta)| \leq g(\eta)$ for $z \in F$. Obviously, uniqueness is then guaranteed by (7.5) if

$$\int_S g(\eta; \nu)\mathrm{d}s_\zeta \leq 1. \tag{7.7}$$

Now, differentiation of the real part G_r of Green's function G given in equation (2.7) with g_0 and $g_\pm$ expressed in terms of the exponential integrals through equations (2.8) and (2.9) yields, for $z \in F$,

$$\begin{aligned}\frac{\partial G_\mathrm{r}}{\partial \xi} + \mathrm{i}\frac{\partial G_\mathrm{r}}{\partial \eta} = \frac{1}{\pi(z - \bar{\zeta})} + \frac{\mathrm{i}}{\pi(3s+1)} \Bigg\{ &- \nu \mathrm{e}^{-\mathrm{i}k_0(z-\bar{\zeta})}\, \mathrm{Ei}(\mathrm{i}k_0(z - \bar{\zeta})) \\ &+ \sum_{\pm} A_\pm a_\pm \mathrm{e}^{-\mathrm{i}a_\pm(z-\bar{\zeta})}\, \mathrm{E}_1(-\mathrm{i}a_\pm(z - \bar{\zeta})) \Bigg\}.\end{aligned} \tag{7.8}$$

Following Motygin & McIver (2003), the exponential integrals are represented by their asymptotic expansions by writing, for $|\arg(-z)| < \pi$,

$$\mathrm{Ei}(z) = \pi\mathrm{i}\,\mathrm{sgn}(\mathrm{Im}\, z) - \mathrm{E}_1(-z), \qquad \mathrm{E}_1(-z) = -\mathrm{e}^z \left[\sum_{n=1}^{N} \frac{(n-1)!}{z^n} + R_N(z) \right].$$

An integral form of the remainder

$$R_N(z) = -\frac{N!}{\mathrm{i}^N}\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\tau}\,\mathrm{d}\tau}{(\tau - \mathrm{i}z)^{N+1}}, \quad \mathrm{Im}\ z \geq 0 \tag{7.9}$$

(a similar form holds for Im $z<0$) is given in Motygin & McIver (2003, eqn (A7)) and two alternative bounds on $R_N(z)$ are obtained under the assumption that Re $z>0$. In fact, it is straightforward to show that the same bounds apply for Re $z<0$ and, furthermore, the coefficients in the bounds may be improved slightly to obtain

$$|R_N(z)| \leq \frac{[(N+1)\sqrt{2}-N](N-1)!}{|z|^N} \quad \text{and} \quad |R_N(z)| \leq \frac{[(N+2)\sqrt{2}-N]N!}{|z|^{N+1}}. \tag{7.10}$$

(In the improved procedure, the range of integration in (7.9) is split at a value τ_0, say. For Re $z>0$, Im $z>0$, the bound $|\tau - \mathrm{i}z| \geq 2^{-1/2}(\tau + \mathrm{Re}\ z + \mathrm{Im}\ z)$ given after eqn (A4) in Motygin & McIver (2003) is used for the upper part of the range of integration, and $|\tau - \mathrm{i}z| \geq |z|$ is used for the lower part. The value τ_0 is then chosen to minimize the bound. A similar procedure applies for Re $z<0$ and/or Im $z<0$.)

With the above representation of the exponential integrals, equation (7.8) gives

$$\frac{\partial G_\mathrm{r}}{\partial \xi} + \mathrm{i}\frac{\partial G_\mathrm{r}}{\partial \eta} = \frac{1}{\pi(3s+1)}\Bigg[\pi\nu \mathrm{e}^{-\mathrm{i}k_0(z-\bar{\zeta})}\,\mathrm{sgn}[\mathrm{Im}(x-\xi)] - \mathrm{i}\nu R_1(\mathrm{i}k_0(z-\bar{\zeta})) - \mathrm{i}\sum_{\pm} A_\pm a_\pm R_1(\mathrm{i}a_\pm(z-\bar{\zeta}))\Bigg], \quad z \in F$$

and, hence, by the bounds given in (7.10),

$$g(\eta;\nu) = \frac{1}{\pi(3s+1)}\left[\pi\nu \mathrm{e}^{k_0\eta} + \nu \min\left\{\frac{2\sqrt{2}-1}{|k_0\eta|}, \frac{3\sqrt{2}-1}{(k_0\eta)^2}\right\} + 2|A_+a_+|\min\left\{\frac{2\sqrt{2}-1}{|a_+\eta|}, \frac{3\sqrt{2}-1}{|a_+\eta|^2}\right\}\right] \tag{7.11}$$

(note that $|A_-|=|A_+|$ and $|a_-|=|a_+|$). The boundary of the region of uniqueness defined by the inequality (7.7) is shown in figure 4 for two submerged circles of radius a, and for two submerged ellipses with horizontal semi-axis a and vertical semi-axis b, such that $b/a=0.08$. The region of uniqueness is displayed in terms of the submergence depth d/a and the frequency parameter νa; the bounds are independent of the spacing between the obstacles. It is shown in the figure how the boundary of the region varies with surface tension, and it can be seen that significant deviations from the gravity-wave case, $s=0$, are obtained only for physically large values of the surface tension. Uniqueness is guaranteed for sufficiently large values of both the depth d and the frequency parameter ν. Trapped modes for the elliptical geometry are discussed in §6.

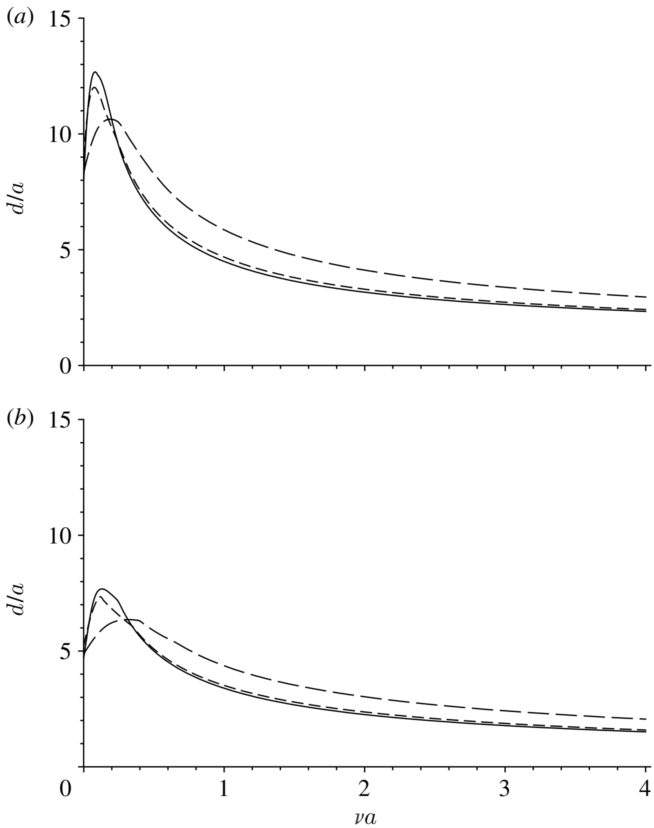

Figure 4. Bounds given by equations (7.7) and (7.11) for (*a*) two circles and (*b*) two ellipses with $d/a = 0.08$, all submerged to a depth d: $s = 0$ (solid curve), $s = 0.2$ (short dashed curve) and $s = 1$ (long dashed curve). The uniqueness domain is located above the curves.

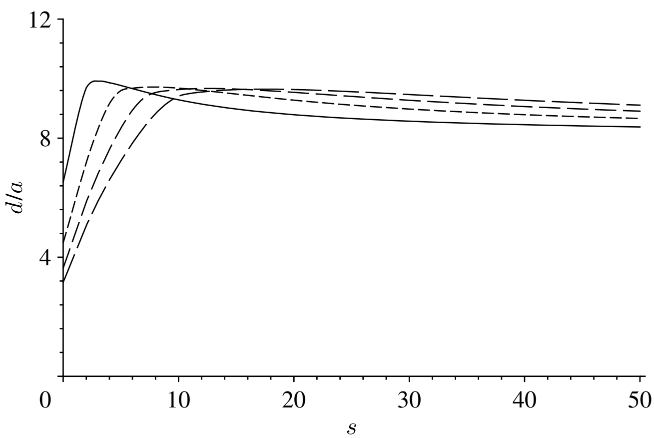

Figure 5. Bounds given by equations (7.7) and (7.11) for two circles submerged to a depth d: $\nu a = 0.5$ (solid curve), $\nu a = 1$ (short dashed curve), $\nu a = 1.5$ (medium dashed curve) and $\nu a = 2$ (long dashed curve). The uniqueness domain is located above the curves.

A different view of the bounds for two circles is shown in figure 5, where the boundary of the domain for which uniqueness is guaranteed is given in terms of the depth of submergence as a function of the surface tension parameter s for various values of νa. There are significant variations in the boundary only for s less than approximately 10 and, for all ν, the curves approach

$$d/a = \sqrt{1 + 16(2\sqrt{2} - 1)^2} \approx 7.38, \quad \text{as } s \to \infty.$$

8. Conclusion

In this paper, we have studied the uniqueness question for the linear problem of wave–body interaction with account taken of surface tension effects. The subject of uniqueness has been extensively investigated in recent years, but surface tension effects have been neglected. The effects have only been included recently when a number of examples of non-uniqueness were constructed with help of the inverse procedure (Harter *et al*. 2007, 2008). Furthermore, uniqueness theorems for the problem in question were not known. In the present paper, we derive a criterion of uniqueness which is formulated in a form convenient for finding trapped modes for given submerged bodies numerically. The numerical algorithm is applied to pairs of ellipses and rounded rectangles and numerical results are used to demonstrate the effects of surface tension—in particular, it is shown that the numerically found trapped modes disappear when the surface tension is increased. In addition, we apply the approach by Motygin & McIver (2003) and prove a uniqueness theorem that provides simple bounds for the possible non-uniqueness parameters.

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