On fluttering modes for aircraft wing model in subsonic air flow

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The paper deals with unstable aeroelastic modes for aircraft wing model in subsonic, incompressible, inviscid air flow. In recent author’s papers asymptotic, spectral and stability analysis of the model has been carried out. The model is governed by a system of two coupled integrodifferential equations and a two-parameter family of boundary conditions modelling action of self-straining actuators. The Laplace transform of the solution is given in terms of the ‘generalized resolvent operator’, which is a meromorphic operator-valued function of the spectral parameter $\lambda$, whose poles are called the aeroelastic modes. The residues at these poles are constructed from the corresponding mode shapes. The spectral characteristics of the model are asymptotically close to the ones of a simpler system, which is called the reduced model. For the reduced model, the following result is shown: for each value of subsonic speed, there exists a radius such that all aeroelastic modes located outside the circle of this radius centred at zero are stable. Unstable modes, whose number is always finite, can occur only inside this ‘circle of instability’. Explicit estimate of the ‘instability radius’ in terms of model parameters is given.

1. Introduction

We study the problem of stability of the aeroelastic modes for a high aspect ratio aircraft wing model in a subsonic air flow. As is well known, a flexible wing vibrating in an air flow may lose stability owing to flutter. We recall that flutter, which is known as a very dangerous aeroelastic development, is the onset, beyond some speed–altitude combinations, of unstable and destructive vibrations of a lifting surface in an air stream [1–16]. We consider the model of an aircraft wing in an incompressible inviscid subsonic air flow, whose mathematical formulation can
be found in references [1,5–9,11–13,16–19]. This is a two-dimensional strip model, which applies to bare wings of high aspect ratio [1,2,7,8,10]. The structure is modelled by a uniform cantilever beam, which bends and twists. In addition, the action of self-straining actuators is modelled using a currently accepted approach [3–5,20,21]. Probably the most important type of aircraft flutter results from coupling between bending and torsional motions of a wing with large aspect ratio. The model we consider has been designed to treat flutter caused by this type of coupling.

This work is a continuation of the research completed in references [11–19], where detailed asymptotic and spectral analysis of the model has been carried out. To keep the paper self-contained, we provide in §§3 and 4 a precise formulation of the model and all the necessary results from our previous works. In §2, we give a brief description of the abstract form of the model and introduce the ‘reduced model’ which, as we show, is appropriate for describing the fluttering modes. We also define the aeroelastic modes for both full and reduced models and give a preliminary statement of the main result of the paper. The result consists of the following. The set of the aeroelastic modes of the reduced model may contain only a finite number of unstable (fluttering) modes that belong to the ‘instability circle’ centred at the origin. An explicit estimate on the radius of this circle is obtained. Full statement of the main result and its proof is given in §6 (theorem 6.1).

### 2. Abstract form of the model and the main result

The model is governed by a system of two coupled partial integrodifferential equations with a two-parameter family of boundary conditions. The integral parts of the equations represent the forces and moments exerted to the wing owing to the air flow. The above-mentioned system with boundary conditions can be represented as a single-operator evolution–convolution equation in a Hilbert space $H$ of four-component vector-valued functions of one spatial variable. Namely it can be given in the form of

$$
\dot{\Psi}(t) = iL_{\beta\delta}\Psi(t) + \int_0^t F(t - \tau)\dot{\Psi}(\tau)\,d\tau, \quad \Psi|_{t=0} = \Psi_0.
$$

Here, $\Psi(t) \in H$ is the state of the system at time $t$, $L_{\beta\delta}$ is an unbounded non-selfadjoint operator in $H$ (a $4 \times 4$ matrix differential operator of fourth-order with the domain defined by the boundary conditions). This operator depends on two complex parameters $\beta$ and $\delta$ which enter the boundary conditions as control gains. $F(t) \equiv F(t; u)$ is a matrix-valued convolution kernel which depends on the speed $u$ of the air stream. The overdot represents the time derivative. $iL_{\beta\delta}$ is the dynamics generator for the structural part of the model, and the convolution integral represents the aerodynamic loads. Note equation (2.1) is not an evolution equation. It does not have a dynamics generator and does not define any semi-group in the standard sense.

By applying the Laplace transformation to both sides of equation (2.1), we obtain

$$
(\lambda I - iL_{\beta\delta} - \lambda \hat{F}(\lambda))\hat{\Psi}(\lambda) = (I - \hat{F}(\lambda))\Psi_0,
$$

where $I$ is the identity operator, and $\hat{F}(\lambda)$ is the Laplace transform of $F(t)$. At those points $\lambda$, where the operator $(\lambda I - iL_{\beta\delta} - \lambda \hat{F}(\lambda))$ has a bounded inverse, the solution of equation (2.2) can be given explicitly as

$$
\hat{\Psi}(\lambda) = (\lambda I - iL_{\beta\delta} - \lambda \hat{F}(\lambda))^{-1}(I - \hat{F}(\lambda))\Psi_0.
$$

To find the time–space representation of the solution, one has ‘to calculate’ the inverse Laplace transform of $\hat{\Psi}$. To carry out this step, it is necessary to investigate the ‘generalized resolvent operator’

$$
\mathcal{R}(\lambda) = (\lambda I - iL_{\beta\delta} - \lambda \hat{F}(\lambda))^{-1}.
$$

As shown in references [11–15,17–19], $\mathcal{R}(\lambda)$ is an operator-valued meromorphic function on the complex plane with a branch-cut along the negative real semi-axis. The poles of $\mathcal{R}(\lambda)$ are called the aeroelastic modes. (The branch-cut is associated with ‘the continuous spectrum’.)
In references [11–15,19], the author has derived explicit asymptotic formulae for the aeroelastic models. It turned out (see theorem 4.3) that the whole set of aeroelastic modes, which we denote \( \{ \lambda_n \} \), splits asymptotically into two distinct branches; each branch approaches its own vertical asymptote. Both branches have only two points of accumulation \( \pm \infty \). The aeroelastic modes depend on the speed \( u \) of the airstream: \( \lambda_n = \tilde{\lambda}_n(u) \). Flutter occurs if with \( u \) increasing at least one of the modes crosses the imaginary axis from the left into the right half-plane, i.e. \( R\tilde{\lambda}_n(u) > 0 \) for some \( n \).

If the speed of the air stream \( u = 0 \), then the convolution (aerodynamic load) term in (2.1) vanishes and (2.1) takes the form

\[
\Psi(t) = i\mathcal{L}_{\beta\delta}\Psi(t), \quad \Psi(0) = \Psi_0.
\]  

(2.5)

Equation (2.5) describes the ground vibrations of the wing in the absence of the airflow. \( \mathcal{L}_{\beta\delta} \) is an unbounded non-selfadjoint operator in \( \mathcal{H} \) with a compact resolvent. This operator has a discrete spectrum of complex eigenvalues \( \{ \lambda_n(0) \}_{n \in \mathbb{Z}} \), which is symmetric with respect to the imaginary axis. The numbers \( i\lambda_n(0) \) are the eigenmodes of the ground vibrations. Because the ground vibrations occur when \( u = 0 \), it is obvious that \( i\lambda_n(0) = \tilde{\lambda}_n(0) \). The operator \( \mathcal{L}_{\beta\delta} \) is dissipative \( \Im(\mathcal{L}_{\beta\delta} \Psi, \Psi)_{\mathcal{H}} \geq 0 \) for any \( \Psi \in \mathcal{D}(\mathcal{L}_{\beta\delta}) \) and, therefore, its spectrum is located in the upper half-plane: \( \Re \lambda_n(0) \geq 0 \) for all \( n \). By \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \), we denote the energy inner product in \( \mathcal{H} \) defined in (4.1). Accordingly, the ground-vibration modes \( i\lambda_n(0) \) belong to the left half-plane: \( \Re(i\lambda_n(0)) \leq 0 \), i.e. all these modes are stable.

It has been shown by the author in references [14,17] (see lemma 4.1) that the Laplace transform \( \hat{F}(\lambda) \) of the matrix kernel \( F(t) \) from (2.1) admits the following decomposition:

\[
\lambda \hat{F}(\lambda) = \mathbb{M} + \mathbb{N}(\lambda),
\]

(2.6)

where \( \mathbb{M} \) is a constant \( 4 \times 4 \) matrix independent of the spectral parameter \( \lambda \) and \( \mathbb{N}(\lambda) \) is asymptotically small: \( \| \mathbb{N}(\lambda) \| \to 0 \) as \( |\lambda| \to \infty \). Both terms in the right-hand side of (2.6) depend on the airstream speed \( u \). Denote by \( G(t) \) the inverse Laplace transform of \( \mathbb{N}(\lambda) \). Then, the main model equation (2.1) can be represented in the form

\[
\Psi(t) = (i\mathcal{L}_{\beta\delta} + \mathbb{M})\Psi(t) + \int_0^t G(t - \tau)\Psi(\tau) \, d\tau.
\]

(2.7)

Let us discard the convolution term in (2.7). Then, we obtain equation (2.8)

\[
\Psi(t) = \mathcal{K}_{\beta\delta}\Psi(t), \quad \text{where} \quad \mathcal{K}_{\beta\delta} = i\mathcal{L}_{\beta\delta} + \mathbb{M}.
\]

(2.8)

We call (2.8)—the equation of the ‘reduced model’. The dynamics generator of the reduced model is the operator \( \mathcal{K}_{\beta\delta} = i\mathcal{L}_{\beta\delta} - i\mathbb{M} \). We denote its spectrum by \( \{ \lambda_n \}_{n \in \mathbb{Z}'} \) and point out that the eigenvalues depend on \( u \): \( \lambda_n = \lambda_n(u) \). This spectrum, as well as the spectrum of \( i\mathcal{L}_{\beta\delta} \), is symmetric with respect to the real axis. That is why we use the index \( n \in \mathbb{Z}' = \mathbb{Z} \setminus \{ 0 \} \) numbering the eigenvalues. It has been shown by the author in reference [13] that the eigenmodes of the reduced model have the same asymptotic representation as \( \lambda_n \) and \( i\lambda_n(0) \). The differences in the asymptotic approximations of all three sets \( \{ \lambda_n \}_{n \in \mathbb{Z}'} \), \( \{ i\lambda_n(0) \}_{n \in \mathbb{Z}'} \) and \( \{ \lambda_n \}_{n \in \mathbb{Z}} \) are in the remainder terms. We note that, unlike \( \mathcal{L}_{\beta\delta} \), the operator \( \mathcal{L}_{\beta\delta} - i\mathbb{M} \) is no longer dissipative. So, it may have the eigenvalues in the lower half-plane. Accordingly, the dynamic generator \( \mathcal{K}_{\beta\delta} \) may have unstable (fluttering) eigenvalues in the right half-plane: \( \Re \lambda_n(u) > 0 \). In other words, as was pointed out in reference [13], the perturbation term \( \mathbb{M} \) is responsible for the appearance of the fluttering modes.

The main result of this paper deals with the eigenvalues \( \{ \lambda_n \} \) of the reduced model (2.8). This result consists of the following (see theorem 6.1 for the precise statement).

For each value of the airstream speed \( u \), there exists \( R(u) > 0 \) such that all eigenmodes of the reduced model (2.8) satisfying \( |\lambda_n| > R(u) \) are stable: \( \Re \lambda_n < 0 \). An explicit estimate for \( R(u) \) in terms of the model parameters is given. This result means that there exists a ‘circle of instability’
of radius $R(u)$ such that unstable (fluttering) eigenmodes may occur only inside this circle, and
the number of unstable modes is always finite.

To the best of our knowledge, no similar estimates obtained analytically for a physically
realistic wing model are available in the literature on aeroelasticity.

The paper is organized as follows. In §3, we present a precise description of the wing model
as a system of two coupled integrodifferential equations with a set of boundary conditions. In §4,
we give a reformulation of the original initial-boundary problem to the form of a single-operator
evolution–convolution equation (2.1). In particular, we introduce the ground-vibrations energy
functional $\mathcal{E}_0[\psi]$ naturally associated with equation (2.5). This functional defines the norm in the
state space $H$ (formulae (2.11)). In §5, we also recall the statements of the main spectral results
obtained in the author’s papers [11–19]. Paragraphs 5 and 6 contain the main results of the paper.
In §6, we derive a new non-local energy functional $\mathcal{E}[\psi; t]$ naturally associated with the equation
of the reduced model (2.8). The two energy functionals satisfy the inequality $\mathcal{E}_0[\psi(t)] \leq \mathcal{E}[\psi; t]$ for
any function $\psi = \psi(t) \in \mathcal{H}$. In §6, we prove the main result of the paper, theorem 6.1. The proof is
based on a special type estimate that we derive for the time derivative of the energy $\mathcal{E}(t) = \mathcal{E}[\psi; t]$ of
the reduced model ($\psi = \psi(t)$ is a solution of the reduced model equation (2.8)). More precisely,
we introduce a notion of a strongly stable mode. Namely if $(\lambda_k, \psi_k)$ is an eigenpair of the operator
$\mathcal{K}_{\beta \delta}$, then it is said that $\lambda_k$ is strongly stable if $\dot{\lambda}_k(t) < 0$, where $\dot{\lambda}_k(t)$ is the new energy functional
evaluated on the solution $e^{\lambda_k(t)}\psi_k$ of the reduced model equation (2.8). We show that if a mode $\lambda_k$
is strongly stable, then it is stable in the usual sense: $\Re \lambda_k < 0$. From the aforementioned estimate for
$\mathcal{E}(t)$, we derive that all modes such that $|\lambda_k| > R(u)$ are strongly stable and, therefore, are stable.


To give a mathematical formulation of the model, let us introduce the dynamic variables

$$X(x, t) = \begin{pmatrix} h(x, t) \\ a(x, t) \end{pmatrix}, \quad 0 \leq x \leq L < \infty, \quad t \geq 0,$$

where $h(x, t)$ is the bending at location $x$ and time moment $t$, and $a(x, t)$ is the torsion angle at $(x, t)$.
The model can be described by the following linear system [3,5,11–15,18,19]:

$$(M_s - M_a)\ddot{X}(x, t) + (D_s - uD_a)\dot{X}(x, t) + (K_s - u^2K_a)X(x, t) = \begin{bmatrix} f_1(x, t) \\ f_2(x, t) \end{bmatrix},$$

where the overdot is used for the derivative with respect to time. We use the subscripts ‘s’ and ‘a’ to distinguish structural and aerodynamic parameters. All $2 \times 2$ matrices in equation (3.2) are given by the formulae

$$M_s = \begin{bmatrix} m & S \\ S & I \end{bmatrix}, \quad M_a = \pi \rho \begin{bmatrix} -1 & a \\ a & -(a^2 + 1) \end{bmatrix}, \quad D_s - uD_a = -\pi \rho u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where $m$ is the density of the structure (mass per unit length), $S$ is the mass moment, $I$ is the
moment of inertia, $\rho$ is the density of air, $u$ is the speed of an air stream and $a$ is a structural
parameter such that $|a| < 1$.}

$$K_s = \begin{bmatrix} E \frac{d^4}{dx^4} & 0 \\ 0 & G \frac{d^2}{dx^2} \end{bmatrix}, \quad K_a = \pi \rho \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$
where $E$ is the bending stiffness and $G$ is the torsion stiffness. The right-hand side of system (3.2) (the aerodynamic forces and moments) can be represented as the following convolution integrals:

$$
f_1(x,t) = -2\pi \rho \int_0^t \left[ uC_2(t-\sigma) - \dot{C}_3(t-\sigma) \right] g(x,\sigma) \, d\sigma, \quad (3.5)$$

$$
f_2(x,t) = -2\pi \rho \int_0^t \left[ \frac{1}{2} C_1(t-\sigma) - auC_2(t-\sigma) + a\dot{C}_3(t-\sigma) + u\dot{C}_4(t-\sigma) + \frac{1}{2} C_5(t-\sigma) \right] g(x,\sigma) \, d\sigma \quad (3.6)$$

and

$$
g(x,t) = u\dot{\alpha}(x,t) + \ddot{h}(x,t) + (\frac{1}{2} - a)\dddot{\alpha}(x,t). \quad (3.7)$$

Note the kernels in both integrals (3.5) and (3.6) are independent on the spatial variable $x$. However, $f_1$ and $f_2$ depend on $x$, because the function $g$ from (3.7) does. The aerodynamic functions $C_i$, $i = 1, \ldots, 5$, are defined in the following ways [2,5-7]:

$$
\hat{C}_1(\lambda) \equiv \int_0^\infty e^{-\lambda t} C_1(t) \, dt = \frac{u}{\lambda} \frac{e^{-\lambda/u}}{K_0(\lambda/u) + K_1(\lambda/u)} \quad \forall \lambda > 0, \quad C_2(t) = \int_0^t C_1(\sigma) \, d\sigma, \quad (3.8)
$$

$$
\hat{C}_3(t) = \int_0^t C_1(t-\sigma)(u\sigma - \sqrt{u^2\sigma^2 + 2u\sigma}) \, d\sigma, \quad C_4(t) = C_2(t) + C_3(t),
$$

$$
and \quad C_5(t) = \int_0^t C_1(t-\sigma)((1 + u\sigma)\sqrt{u^2\sigma^2 + 2u\sigma} - (1 + u\sigma)C_4(t)) \, d\sigma.
$$

In formulae (3.8), $\hat{C}_1(\lambda)$ is the Laplace transform of the function $C_1(t)$; $K_0$ and $K_1$ are modified Bessel functions of the zero- and first-orders, respectively [22]. Formulae (3.8) can be found in reference [6]. As is known, the self-straining control actuator action can be modelled by the following boundary conditions [5,19-21,23]:

$$
Eh''(L, t) + \beta \ddot{h}(L, t) = 0, \quad h''(L, t) = 0 \quad (3.9)
$$

and

$$
Ga'(L, t) + \delta \dot{\alpha}(L, t) = 0, \quad \beta, \delta \in \mathbb{C}^+, \quad (3.10)
$$

where the prime denotes the derivative with respect to $x$. In equation (3.10), $\mathbb{C}^+$ denotes the open right half-plane of the complex plane; $\beta$ and $\delta$ are called the control parameters. At the left end, the structure is fixed, i.e. the following boundary conditions are imposed

$$
h(0,t) = h'(0,t) = \alpha(0,t) = 0. \quad (3.11)$$

Let the initial state of the system be given by

$$
h(x,0) = h_0(x), \quad \dot{h}(x,0) = h_1(x), \quad \alpha(x,0) = \alpha_0(x), \quad \dot{\alpha}(x,0) = \alpha_1(x). \quad (3.12)$$

We consider the problem defined by equation (3.2) and conditions (3.9)–(3.12) under the assumptions

$$
\det \begin{bmatrix} m & S \\ S & 1 \end{bmatrix} > 0, \quad 0 < u < \frac{\sqrt{\pi G}}{2L\sqrt{\rho}}, \quad (3.13)
$$

**Remark 3.1.** The second condition in (3.13) physically means that the flow speed must be below the ‘divergence’ or static instability speed for the system. Indeed, if we consider the static problem corresponding to system (3.2) and conditions (3.9)–(3.11), then we can check that the component $\alpha$ satisfies the following Sturm–Liouville problem:

$$
G\alpha''(x) + \pi \rho u^2 \alpha(x) = 0, \quad \alpha(0) = \alpha'(L) = 0.
$$

This problem has non-trivial solutions only for a countable set of speeds, $\{u_n\}_{n=1}^\infty$, given explicitly by the following formula: $\pi n^2 u_n^2 = ((2n - 1)/2L)n^2 G, \quad n = 1, 2, 3, \ldots$. The smallest value corresponding to $n = 1$ is exactly the aerelastic static divergence speed.
To introduce the energy of the system, we have to complete some preliminary steps. Let \( \tilde{C}_1(t) \) and \( \tilde{C}_2(t) \) be the kernels in the convolution integrals in (3.5) and (3.6), that is

\[
\tilde{C}_1(t) = -2\pi \rho (uC_2(t) - \tilde{C}_3(t))
\]

and

\[
\tilde{C}_2(t) = -2\pi \rho (\frac{1}{2}C_1(t) - auC_2(t) + a\tilde{C}_3(t) + uC_4(t) + \frac{1}{2}\tilde{C}_5(t)).
\]

Denoting \( M = M_s - M_a, \ D = D_s - uD_a, \ K = K_s - u^2K_a \), we rewrite equation (3.2) in the form

\[
M\ddot{X}(x, t) + D\dot{X}(x, t) + KX(x, t) = (\mathcal{F}\dot{X})(x, t), \quad t \geq 0,
\]

with the matrix integral operator \( \mathcal{F} \) being given by the formula

\[
\mathcal{F} = \begin{bmatrix}
\int_0^t \tilde{C}_1(t - \sigma) \frac{d}{d\sigma} \cdot d\sigma & \int_0^t \tilde{C}_1(t - \sigma) \left[ u + \left( \frac{1}{2} - a \right) \frac{d}{d\sigma} \cdot \right] d\sigma \\
\int_0^t \tilde{C}_2(t - \sigma) \frac{d}{d\sigma} \cdot d\sigma & \int_0^t \tilde{C}_2(t - \sigma) \left[ u + \left( \frac{1}{2} - a \right) \frac{d}{d\sigma} \cdot \right] d\sigma
\end{bmatrix}.
\]

(3.17)

\( M, D \) and \( K \) are usually called the spatial operators and \( \mathcal{F} \) the time operator. Note system (3.16) is not singular, because \( \det M > 0 \). Indeed, if

\[
\tilde{m} = m + \pi \rho, \quad \tilde{S} = S - \pi a \rho, \quad \tilde{I} = I + \pi \rho (a^2 + \frac{1}{2}),
\]

then using (3.13) one obtains

\[
\Delta \equiv \det M = \tilde{m}^2 - \tilde{S}^2 = (ml - S^2) + [\pi \rho I + \pi \rho m(a^2 + \frac{1}{2}) + (\pi \rho)^2(a^2 + \frac{1}{2}) + 2\pi \rho aS] > 0.
\]

Rewriting system (3.16) component-wise, we have

\[
\begin{align*}
\tilde{m}\ddot{h}(x, t) + \tilde{S}\ddot{u}(x, t) + \pi \rho \dot{u}\ddot{u}(x, t) + E\dddot{m} = & \int_0^t \tilde{C}_1(t - \sigma) g(x, \sigma) d\sigma \\
\tilde{S}\ddot{h}(x, t) + \tilde{I}\dddot{u}(x, t) - \pi \rho \dot{u}\dddot{u}(x, t) - (G\dddot{\alpha} + \pi \rho u^2\dddot{\alpha}(x, t)) = & \int_0^t \tilde{C}_2(t - \sigma) g(x, \sigma) d\sigma
\end{align*}
\]

(3.20)

with \( g \) being defined in (3.7). By setting \( \tilde{C}_1 = \tilde{C}_2 = 0 \) and completing some standard steps for system (3.20), we obtain that the energy of vibrations can be introduced by the following formula (see [11,19]):

\[
\mathcal{E}_0(t) = \frac{1}{2} \left[ \int_0^L \left[ E|\dddot{h}(x, t)|^2 + G|\dddot{u}(x, t)|^2 + \tilde{m}|\ddot{h}(x, t)|^2 + \tilde{I}|\dddot{u}(x, t)|^2 \right] dx + \tilde{S}(\dddot{u}(x, t) \dddot{h}(x, t) + \dddot{u}(x, t) \dddot{h}(x, t)) - \pi \rho u^2|\alpha(x, t)|^2 \right].
\]

(3.21)

**Lemma 3.2 ([11,19])**. (i) If the problem parameters satisfy conditions (3.13), then the system energy \( \mathcal{E}_0(t) \) evaluated on smooth functions subjected to the boundary conditions (3.11) is always non-negative and is equal to zero if and only if \( h(x, t) = \alpha(x, t) = 0, x \in (0, L), t \geq 0 \). (ii) If \( \Re \beta \geq 0 \) and \( \Re \delta \geq 0 \), then the energy of the system governed by the differential part of (3.20) dissipates: \( \mathcal{E}_0(t) \leq 0 \) (the differential part means that the right-hand sides of (3.20) have been replaced with zeros).

4. Main spectral results from papers [11−14,17−19]

(a) Operator evolution—convolution form of the model

We consider the solution of the problem given by system (3.2) and conditions (3.9)–(3.11) in the energy space \( \mathcal{H} \), which is a Hilbert space of Cauchy data. The norm in \( \mathcal{H} \) is induced by the expression for the energy (3.21). Namely let \( \mathcal{H} \) be the set of four-component complex
vector-valued functions $\Psi = (h, \dot{h}, \alpha, \dot{\alpha})^T = (\psi_0, \psi_1, \psi_2, \psi_3)^T$, (where the superscript ‘T’ stands for transposition) obtained as a closure of smooth functions satisfying the boundary conditions

$$\psi_0(0) = \psi_0'(0) = \psi_2(0) = 0$$

in the following energy norm

$$\|\Psi\|^2_\mathcal{H} = \frac{1}{2} \int_0^L [E|\psi_0''(x)|^2 + G|\psi_2'(x)|^2 + \tilde{m}|\psi_1(x)|^2 + \tilde{I}|\psi_3(x)|^2 \right. \]

$$+ \tilde{S}(\psi_1(x)\psi_3(x) + \psi_1(x)\psi_3(x)) - \pi \rho u^2|\psi_2(x)|^2 \left. dx, \right) (4.1)$$

with $\tilde{m}$, $\tilde{S}$ and $\tilde{I}$ being defined in (3.18). To rewrite the original initial-boundary-value problem as a single equation in the space $\mathcal{H}$, we apply $2 \times 2$ matrix $M^{-1} = (1/\Delta)\left[ \begin{array}{cc} \tilde{m} & -\tilde{S} \\ -\tilde{S} & \tilde{I} \end{array} \right]$ to equation (3.16) and have

$$\dot{X}(x, t) + (M^{-1}D)\dot{X}(x, t) + (M^{-1}K)X(x, t) = (M^{-1}F)\dot{X}(x, t). \quad (4.2)$$

It can be verified directly that the problem defined by equation (4.2) and conditions (3.9)–(3.12) can be written in the form of the evolution–convolution equation (2.1) presented in §2. In that equation, $\mathcal{L}_{B\delta}$ is a non-selfadjoint matrix differential operator in $\mathcal{H}$, which is defined by the following matrix differential expression

$$\mathcal{L}_{B\delta} = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{E\tilde{I}}{\Delta} \frac{d^4}{dx^4} & -\frac{\pi \rho \tilde{S}}{\Delta} & -\frac{\tilde{S}}{\Delta} \left( G \frac{d^2}{dx^2} + \pi \rho u^2 \right) & -\frac{\rho \mu \tilde{I}}{\Delta} \\ 0 & 0 & 0 & 1 \\ \frac{E\tilde{S}}{\Delta} \frac{d^4}{dx^4} & \frac{\pi \rho \mu \tilde{m}}{\Delta} & \frac{\tilde{m}}{\Delta} \left( G \frac{d^2}{dx^2} + \pi \rho u^2 \right) & \frac{\rho \mu \tilde{S}}{\Delta} \end{bmatrix}, \quad (4.3)$$

on the domain

$$\mathcal{P}(\mathcal{L}_{B\delta}) = \{ \Psi \in \mathcal{H} : \psi_0 \in H^4(0, L), \psi_1 \in H^2(0, L), \psi_2 \in H^2(0, L), \psi_3 \in H^1(0, L); \psi_1(0) = \psi_1'(0) = \psi_3(0) = 0; \psi_0''(L) = 0; \psi_1''(L) = 0; E\psi_0''(L) + \beta \psi_1'(L) = 0, G\psi_2(L) + \delta \psi_3(L) = 0 \}, \quad (4.4)$$

where $H^i, i = 1, 2, 4$ are the standard Sobolev spaces [24].

The matrix kernel $F(t)$ of the convolution integral in (2.1) is given by the formula

$$F(t) = \frac{1}{\Delta} \begin{bmatrix} 1 & 0 & 0 & 0 \\ E\tilde{C}_1(t) - \tilde{S}\tilde{C}_2(t) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & u & \frac{1}{2} - a \\ 0 & 0 & 0 & 0 \\ 0 & 1 & u & \frac{1}{2} - a \end{bmatrix} \quad (4.5)$$

(b) The integral part of the model

We describe the structure of the convolution part of the model.

Lemma 4.1 ([12–15,19]). Let $\tilde{F}(\lambda)$ be the Laplace transform of the kernel (4.5). The following decomposition holds $\lambda \tilde{F}(\lambda) = \mathcal{M} + \mathcal{N}(\lambda)$ (see (2.6.) The constant matrix $\mathcal{M}$ is given by the formula

$$\mathcal{M} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A & uA & (\frac{1}{2} - a)A \\ 0 & 0 & 0 & 0 \\ 0 & B & uB & (\frac{1}{2} - a)B \end{bmatrix}, \quad (4.6)$$
where
\[ A = -\frac{\pi \rho u}{\Delta} \left[ \tilde{I} - \left( \frac{1}{2} - a \right) \tilde{S} \right], \quad B = \frac{\pi \rho u}{\Delta} \left[ \tilde{S} - \left( \frac{1}{2} - a \right) \tilde{m} \right]. \] (4.7)

The matrix-valued function \( \mathcal{M}(\lambda) \) is given by the formula

\[
\mathcal{M}(\lambda) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & A_1(\lambda) & uA_1(\lambda) & \left( \frac{1}{2} - a \right) A_1(\lambda) \\
0 & 0 & 0 & 0 \\
0 & B_1(\lambda) & uB_1(\lambda) & \left( \frac{1}{2} - a \right) B_1(\lambda)
\end{bmatrix},
\] (4.8)

where
\[
A_1(\lambda) = -\Gamma(\lambda) \left[ \tilde{I} + \left( \frac{1}{2} + a \right) \tilde{S} \right], \quad B_1(\lambda) = \Gamma(\lambda) \left[ \tilde{S} + \left( \frac{1}{2} + a \right) \tilde{m} \right]
\] (4.9)

\( T(z) \) is the Theodorsen function defined by the formula
\[
T(z) = \frac{K_1(z)}{K_0(z) + K_1(z)},
\] (4.10)

with \( K_0 \) and \( K_1 \) being the modified Bessel functions of the zero- and first-orders, respectively [22]. It is an analytical function on the complex plane with the branch-cut along the negative real semi-axis.

**Lemma 4.2 ([12–15,19]).** \( \mathcal{M} \) is a bounded linear operator in \( \mathcal{H} \). \( \mathcal{M}(\lambda) \) is an analytic matrix-valued function on the complex plane with the branch-cut along the negative real semi-axis. For each \( \lambda \), \( \mathcal{M}(\lambda) \) is a bounded operator in \( \mathcal{H} \) with the following estimate for its norm:
\[
\| \mathcal{M}(\lambda) \| \leq C(1 + |\lambda|)^{-1},
\] (4.11)

where \( C \) is an absolute constant, whose precise value is immaterial for us.

As was mentioned in §2, the model governed by the evolution equation (2.8) with the dynamics generator \( \mathcal{H}_{\beta_3} = i\mathcal{L}_{\beta_3} + \mathcal{M} \) is called the reduced model and will be our main object of interest.

(c) Spectral asymptotics

The following facts are among the results obtained by the author in references [11–15,17–19].

---

The convolution part of the problem does not ‘destroy’ the main characteristics of the discrete spectrum produced by the differential part. Namely the set of the aeroelastic modes is asymptotically close to the set of the eigenvalues of the operator \( i\mathcal{L}_{\beta_3} \). Recall that the operator \( i\mathcal{L}_{\beta_3} \) is the dynamics generator for the model describing wing vibrations when an aircraft is on the ground (not in-flight), and the convolution part is missing. In turn, identification of the ground-vibration frequencies is always the first step in any aeroelastic analysis, because it is an experimentally known fact that the ‘flutter frequency’ is typically very close to one of the lower ground-vibration frequencies.

The set of the generalized eigenvectors of the operator \( \mathcal{L}_{\beta_3} \) forms a Riesz basis in the state space \( \mathcal{H} \). The set of the mode shapes forms a Riesz basis in its closed linear span in \( \mathcal{H} \). This result is crucial for an unconditional convergence of a series with respect to the residues at the poles of \( \mathcal{R}(\lambda) \).
As has already been mentioned, an aeroelastic mode is a pole of the generalized resolvent (2.4). Equivalently, it is a value of \( \lambda \), for which the equation

\[
(\lambda I - iL_{\beta\delta} - \mathcal{M} - \gamma(\lambda))\Phi = 0
\]

(4.12)

has a non-trivial solution in the energy space \( \mathcal{H} \), and the corresponding solution, \( \Phi \), is the mode shape.

**Theorem 4.3** ([11–14,19]). (i) The set of the aeroelastic modes is countable and does not have accumulation points on the complex plane \( \mathbb{C} \). There can be only a finite number of multiple poles of a finite multiplicity each. For a multiple mode, in addition to the aeroelastic mode shape or shapes, there may be the associate mode shapes.

(ii) Let the boundary control parameter \( \delta \) satisfy the condition

\[
|\delta| \neq \sqrt{Gi}.
\]

(4.13)

The set of the aeroelastic modes splits into two asymptotically disjoint sets, which we call the \( \beta \)-branch and the \( \delta \)-branch. Let \( \{\tilde{\lambda}_n^\beta\}_{n \in \mathbb{Z}} \) be the \( \beta \)-branch aeroelastic modes. The following asymptotic approximation holds as \(|n| \to \infty|

\[
\tilde{\lambda}_n^\beta = i(\text{sgn} \, n) \frac{\pi^2}{L^2} \sqrt{\frac{E\Gamma}{\Delta}} \left(|n| - \frac{1}{4}\right)^2 + \kappa_n(\omega), \quad \omega = |\delta|^{-1} + |\beta|^{-1}.
\]

(4.14)

The complex-valued sequence \( \{\kappa_n(\omega)\}_{n \in \mathbb{Z}} \) is bounded above in the following sense:

\[
\sup_{n \in \mathbb{Z}} |\kappa_n(\omega)| = C(\omega), \quad C(\omega) \to 0 \quad \text{as} \quad \omega \to 0.
\]

Let \( \{\tilde{\lambda}_n^\delta\}_{n \in \mathbb{Z}} \) be the \( \delta \)-branch aeroelastic modes. The following asymptotic approximation holds as \(|n| \to \infty|

\[
\tilde{\lambda}_n^\delta = i \frac{\pi n}{L\sqrt{1/G}} - \frac{1}{2L\sqrt{1/G}} \ln \frac{\delta + \sqrt{G}i}{\delta - \sqrt{G}i} + O\left(\frac{1}{\sqrt{|n|}}\right).
\]

(4.15)

Here, ‘ln’ means the principal value of logarithm. If both \( \beta \) and \( \delta \) stay away from zero, i.e. \( |\beta| \geq \beta_0 > 0 \) and \( |\delta| \geq \delta_0 > 0 \), then the estimate \( O(|n|^{-1/2}) \) is uniform with respect to the boundary parameters.

The following statement is concerned with the spectral results for the dynamics generator \( \mathcal{K}_{\beta\delta} = iL_{\beta\delta} + \mathcal{M} \) (see (2.8)) of the reduced model.

**Theorem 4.4** ([13,14]). The operator \( \mathcal{K}_{\beta\delta} = iL_{\beta\delta} + \mathcal{M} \) has a two-branch discrete spectrum. Let these branches be denoted by \( \{\lambda_n^\beta\}_{n \in \mathbb{Z}} \) and \( \{\lambda_n^\delta\}_{n \in \mathbb{Z}} \) and called the \( \beta \)-branch and the \( \delta \)-branch, respectively.

The asymptotic distribution of the eigenvalues of \( \mathcal{K}_{\beta\delta} \) coincides with the asymptotic distribution of the eigenvalues of the operator \( iL_{\beta\delta} \) which, in turn, coincides with the asymptotic distribution of the aeroelastic modes given by (4.14) and (4.15). The set of generalized eigenvectors of the operator \( \mathcal{K}_{\beta\delta} \) forms a Riesz basis in \( \mathcal{H} \).

It can be readily seen from theorems 4.3 and 4.4 that the leading terms in the asymptotic representations for the aeroelastic modes and for the eigenvalues of \( \mathcal{K}_{\beta\delta} \) do not depend on the air stream speed \( u \). However, the remainder terms in these asymptotic representations may depend on \( u \). Theorem 4.5 deals with the dependence of the remainder terms upon \( u \). The result has not been proved in author’s previous works. However, it is crucial for the description of the stability region (see §6).

**Theorem 4.5.** Let the second condition from (3.13) and (4.13) be satisfied. Then, the asymptotic approximations (4.14) and (4.15) as well as the asymptotic approximations for the eigenvalues of \( \mathcal{K}_{\beta\delta} \) are uniform with respect to the speed of an air stream, \( u \).

The proof of this statement is a generalization of the proof of the spectral asymptotics from references [11,13]. Because it is quite lengthy and technical, we have presented the proof in the electronic supplementary material (see theorem 2 of the electronic supplementary material).
Remark 4.6. As shown in reference [14], the set of the aeroelastic modes is asymptotically close to the set of the eigenvalues of the matrix differential operator \((iL_\beta)\) defined in (4.3) and (4.4). An important property of \(L_\beta\) is that it is a dissipative operator in \(\mathcal{H}\). Let \(\{\lambda_n^{\beta}\}_{n \in \mathbb{Z}}\) \(\bigcup\{\tilde{\lambda}^n\}_{n \in \mathbb{Z}}\) be the notation for the two-branch spectrum of \(iL_\beta\). An immediate consequence of the dissipativity is that the spectrum of the operator \((iL_\beta)\) is located in the closed left half-plane of the complex plane. We recall (see [11,18]) that the set of the aeroelastic modes is confined to the closed left half-plane of the complex plane. We recall (see [11,18]) that the set of the aeroelastic modes is confined to the closed left half-plane of the complex plane. 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We recall (see [11,18]) that the set of the aeroelastic modes is confined to the closed left half-plane of the complex plane. We recall (see [11,18]) that the set of the aeroelastic modes is confined to the closed left half-plane of the complex plane.

This means that any solution of the evolution problem (2.5) can be given in the form of an expansion with respect to the generalized eigenvectors of \(L_\beta\)

\[
\Psi(x,t) = \sum_{n \in \mathbb{Z}} (\Psi_0, \psi^n)_{\mathcal{H}} e^{i\lambda_n^\beta t} \psi^n(x) + \sum_{n \in \mathbb{Z}} (\Psi_0, \psi^n)_{\mathcal{H}} \gamma_n e^{i\tilde{\lambda}_n^\beta t} \phi^n(x). \tag{4.16}
\]

Using (4.16) and the fact that \(\Re \lambda_n^\beta \leq 0\) and \(\Re \tilde{\lambda}_n^\beta \leq 0\), one obtains the following estimate for the solution:

\[
\|\Psi(x,t)\|_{\mathcal{H}}^2 \leq C e^{-2\gamma t} \|\Psi_0\|_{\mathcal{H}}^2, \text{ where } \gamma = \inf\{|\Re \lambda_n^\beta|, |\Re \tilde{\lambda}_n^\beta|, n \in \mathbb{Z}'\}. \tag{4.17}
\]

with some absolute constant \(C\). Estimate (4.17) shows that the dissipativity of the operator \(iL_\beta\) implies that the ground-vibration dynamics (2.5), whose generator is \(iL_\beta\), is exponentially stable. Even though the set of the aeroelastic modes is asymptotically close to the eigenvalues of the operator \((iL_\beta)\), the fact that the set of the aeroelastic modes is confined to the closed left half-plane of the complex plane is not valid, in general. So for the expansion with respect to the mode shapes, the estimate similar to (4.17) is not valid any more owing to the existence of unstable aeroelastic modes.

5. Non-local energy functional for the reduced model

For the rest of the paper, we study the evolution equation (2.8), for which the operator \(\mathcal{H}_\beta\) is the dynamics generator, i.e.

\[
\Psi(x,t) = (\mathcal{H}_\beta \Psi)(x,t), \quad \Psi(x,0) = \Psi_0(x), \tag{5.1}
\]

where \(\Psi(x,t) = (h(x,t), \dot{h}(x,t), \alpha(x,t), \dot{\alpha}(x,t))\) and the dynamical variables \(h, \alpha\) defined in (3.1) satisfy the boundary conditions (3.9)–(3.11), i.e. the operator \(\mathcal{H}_\beta\) has the same domain as \(L_\beta\) (see (4.4)).

The goal of this section is to introduce the energy functional naturally associated with equation (5.1) (see formula (5.11)) and to prove that this functional is positively definite (lemma 5.2). The energy functional we present here plays a crucial role in the proof of our main result in §6. The functional is non-local in the time variable and can be called a memory-type energy. The system governed by equation (5.1) is not conservative and, accordingly, the energy we introduce is not conserved on the solutions of equation (5.1). The formula (5.11), which defines our energy functional, is quite complicated and is not intuitively obvious. Thus, instead of just giving an unmotivated definition of the energy, we present below a heuristic derivation of the formula and give a physical interpretation of each term in it.

Let us apply the matrix

\[
\mathcal{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \tilde{m} & 0 & \tilde{S} \\ 0 & 0 & 1 & 0 \\ 0 & \tilde{S} & 0 & i \end{bmatrix}
\]
to both sides of equation (5.1). Taking into account formulae (4.3) and (4.6) by direct calculations, we obtain the following equation

\[
\begin{bmatrix}
  \dot{h}(x, t) \\
  \widetilde{m}\dot{h}(x, t) + \widetilde{S}\dot{\alpha}(x, t) \\
  \dot{\alpha}(x, t) \\
  \dot{S}\dot{\alpha}(x, t) + \tilde{I}\dot{\alpha}(x, t)
\end{bmatrix}
= 
\begin{bmatrix}
  0 & 1 & 0 & 0 \\
  -E \frac{d^4}{dx^4} & 0 & 0 & -\pi\rho u \\
  0 & 0 & 0 & 1 \\
  0 & \pi\rho u & G \frac{d^2}{dx^2} + \pi\rho u^2 & 0
\end{bmatrix}
\begin{bmatrix}
  h(x, t) \\
  \dot{h}(x, t) \\
  \alpha(x, t) \\
  \dot{\alpha}(x, t)
\end{bmatrix}
\]

This system written component-wise yields

\[
\begin{align*}
\widetilde{m}\dot{h}(x, t) + \widetilde{S}\dot{\alpha}(x, t) + E\dot{h}'''(x, t) + \pi\rho u\dot{h}(x, t) \\
&+ \pi\rho u \left(\frac{3}{2} - a\right)\dot{\alpha}(x, t) + \pi\rho u^2\alpha(x, t) = 0
\end{align*}
\]

and

\[
\begin{align*}
\dot{S}\dot{h}(x, t) + \tilde{I}\dot{\alpha}(x, t) - G\dot{\alpha}''(x, t) - \pi\rho u \left(\frac{1}{2} + a\right)\dot{h}(x, t) \\
&+ \pi\rho u \left(\frac{1}{2} - a\right)^2\dot{\alpha}(x, t) - \pi\rho u^2 \left(\frac{1}{2} + a\right)\alpha(x, t) = 0.
\end{align*}
\]

Note system (5.3) contains all terms from the left-hand side of system (3.20) combined with the conditions (3.9)–(3.11). Let us take the first equation of system (5.3) and multiply it by \(\dot{h}\), then take the equation complex conjugated to the first equation from (5.3) and multiply it by \(\dot{h}\). Summing up these two equations we obtain

\[
\widetilde{m}\frac{d}{dt}|h(x, t)|^2 + \widetilde{S}[\dot{\alpha}\dot{h} + \ddot{\alpha}\ddot{h}](x, t) + E[\dot{h}'''\dot{h} + \ddot{h}'''\ddot{h}](x, t)
\]

\[
+ 2\pi\rho u|h(x, t)|^2 + \pi\rho u \left(\frac{3}{2} - a\right)[\alpha\dot{h} + \ddot{\alpha}\ddot{h}](x, t) + \pi\rho u^2[\alpha\dot{h} + \ddot{\alpha}\ddot{h}](x, t) = 0. \tag{5.4}
\]

Now, we take the second equation of system (5.3) and multiply it by \(\dot{\alpha}\), then we take the equation, which is complex conjugate to the second equation of (5.3), and multiply it by \(\dot{\alpha}\). Summing up these two equations, we obtain

\[
\tilde{S}[\dot{h}\dddot{\alpha} + \dddot{h}\dddot{\alpha}](x, t) + \tilde{I}\frac{d}{dt}|\dot{\alpha}(x, t)|^2 - G[\alpha''\dddot{\alpha} + \dddot{\alpha}''\dddot{\alpha}](x, t) - \pi\rho u \left(\frac{1}{2} + a\right)
\]

\[
\times [\dot{h}\dddot{\alpha} + \dddot{h}\dddot{\alpha}](x, t) + 2\pi\rho u \left(\frac{1}{2} - a\right)^2|\dot{\alpha}(x, t)|^2 - \pi\rho u^2 \left(\frac{1}{2} + a\right)\frac{d}{dt}|\alpha(x, t)|^2 = 0. \tag{5.5}
\]

Combining equations (5.4) and (5.5), we obtain the following relation

\[
\frac{d}{dt} \left\{ \widetilde{m}|h(x, t)|^2 + \tilde{I}|\dot{\alpha}(x, t)|^2 - \pi\rho u^2 \left(\frac{1}{2} + a\right)|\alpha(x, t)|^2 + \tilde{S}[\dot{\alpha}(x, t)\dot{h}(x, t) + \dddot{\alpha}(x, t)\dddot{h}(x, t)]
\right.
\]

\[
+ E[\dot{h}'''(x, t)\dot{h}(x, t) + \dddot{h}'''(x, t)\dddot{h}(x, t)] - G[\alpha''(x, t)\dddot{\alpha}(x, t) + \dddot{\alpha}''(x, t)\dddot{\alpha}(x, t)]
\]

\[
+ 2\pi\rho u|f(x, t)| = 0,
\]

where

\[
f(x, t) = b[\alpha(x, t)\dot{h}(x, t) + \ddot{\alpha}(x, t)\dddot{h}(x, t)] + |\dot{h}(x, t)|^2 + b^2|\dot{\alpha}(x, t)|^2 + \frac{u}{2}[\alpha(x, t)\dot{\alpha}(x, t)\dot{h}(x, t) + \ddot{\alpha}(x, t)\dddot{\alpha}(x, t)]
\]

\[
b = \frac{1}{2} - a. \tag{5.7}
\]
It can be easily seen that $J$ can be written in the form

$$J(x, t) = \int \left[ h(x, t) + b\dot{\alpha}(x, t) + \frac{u}{2} \alpha(x, t) \right]^2 + \left[ b^2 |\dot{\alpha}(x, t)|^2 - |b\dot{\alpha}(x, t) + \frac{u}{2} \alpha(x, t)|^2 \right]. \tag{5.8}$$

Indeed,

$$J(x, t) = \int \left[ h + b\dot{\alpha} + \frac{u}{2} \alpha \right]^2 + b^2 |\dot{\alpha}(x, t)|^2 - \left[ b\dot{\alpha} + \frac{u}{2} \alpha \right]^2 = \left[ \hat{h} \left( \dot{\alpha} + \frac{u}{2} \alpha \right) + \hat{h} \left( \dot{\alpha} + \frac{u}{2} \alpha \right) \right] + \left| \hat{h}(x, t) + b\dot{\alpha}(x, t) \right|^2 + b^2 |\dot{\alpha}(x, t)|^2 = |\hat{h}(x, t)|^2 + b|\hat{h}(x, t) + b\dot{\alpha}(x, t)|^2 + \frac{u}{2} \left[ \hat{h} \alpha \right] (x, t).$$

Owing to representation (5.8), we can rewrite equation (5.6) in the form

$$\frac{d}{dt} \left\{ \int \left[ \tilde{m}|\dot{h}(x, t)|^2 + \tilde{1}|\dot{\alpha}(x, t)|^2 - \pi \rho u\left( \frac{1}{2} + a \right) |\alpha(x, t)|^2 + \tilde{S}|\dot{\alpha}(x, t)|^2 \right\} + \int \left[ \frac{d}{dt} \left[ \tilde{m}|\dot{h}(x, t)|^2 + \tilde{1}|\dot{\alpha}(x, t)|^2 + \tilde{S}|\dot{\alpha}(x, t)|^2 \right] \right\} = 0. \tag{5.9}$$

Now, we use a heuristic consideration (quite traditional) to derive a non-local energy functional associated with this equation. Let us discard the right-hand side of equation (5.9) and assume that we consider a cantilever beam model, i.e. the right end is free: $h''(L, t) = h'''(L, t) = \alpha(L, t) = 0$. Integrating both sides of equation (5.9) and using the clamped-free boundary conditions, we obtain equation (5.10)

$$\frac{d}{dt} \left\{ \int_0^L \left[ E|h''(x, t)|^2 + G|\alpha'(x, t)|^2 + \tilde{m}|h(x, t)|^2 + \tilde{1}|\dot{\alpha}(x, t)|^2 + \tilde{S}|\dot{\alpha}(x, t)|^2 \right] \right\} + \int_0^L \left[ \frac{d}{dt} \left[ E|h''(x, t)|^2 + G|\alpha'(x, t)|^2 + \tilde{m}|h(x, t)|^2 + \tilde{1}|\dot{\alpha}(x, t)|^2 + \tilde{S}|\dot{\alpha}(x, t)|^2 \right] \right\} = 0. \tag{5.10}$$

This equation means that the following functional:

$$\mathcal{E}(t) = \int_0^L \left[ E|h''(x, t)|^2 + G|\alpha'(x, t)|^2 + \tilde{m}|h(x, t)|^2 + \tilde{1}|\dot{\alpha}(x, t)|^2 + \tilde{S}|\dot{\alpha}(x, t)|^2 \right] \right\} + \int_0^L \left[ \frac{d}{dt} \left[ E|h''(x, t)|^2 + G|\alpha'(x, t)|^2 + \tilde{m}|h(x, t)|^2 + \tilde{1}|\dot{\alpha}(x, t)|^2 + \tilde{S}|\dot{\alpha}(x, t)|^2 \right] \right\} = 0. \tag{5.11}$$

is not changing in time, being evaluated on the functions $h, \alpha$ satisfying equation (5.9) with the right-hand side replaced with zero and subjected to the clamped-free boundary conditions.

**Definition 5.1.** The functional defined in (5.11) is called the **non-local energy functional** associated with the problem given by operator equation (5.1) or, equivalently, by system (5.2) on the functions satisfying the boundary conditions (3.9)–(3.11).

The following interpretation of the different terms of equation (5.11) can be given. Namely

$$\int_0^L E|h''(x, t)|^2 \, dx$$

represents the potential energy of a structure owing to the bending displacement; $\int_0^L G|\alpha'(x, t)|^2 \, dx$—the potential energy owing to the torsional displacement; $\int_0^L \tilde{m}|h(x, t)|^2 \, dx$—the kinetic energy owing to the bending motion; $\int_0^L \tilde{1}|\dot{\alpha}(x, t)|^2 \, dx$—the kinetic energy owing to the torsion motion; $\int_0^L \tilde{S}|\dot{\alpha}(x, t)|^2 \, dx$—the kinetic
energy owing to coupling between bending and torsion; \( \pi \rho u \int_0^L \int_0^L |\dot{h}(x, t)|^2 + (\pi/2) \alpha(x, t)^2 \, dx \) — the energy of vibrations owing to flow–structure interaction.

**Lemma 5.2.** Let conditions (3.13) be satisfied. Then, for each \( t \geq 0 \), the functional \( \mathcal{E}(t) \), defined by (5.11) on smooth functions \( h \) and \( \alpha \) satisfying conditions (3.11), is positively definite, i.e. \( \mathcal{E}(t) \geq 0 \) and \( \mathcal{E}'(t) = 0 \) if and only if \( h(x, t) = \alpha(x, t) = 0, x \in (0, L), t \geq 0 \).

**Proof.** To prove that \( \mathcal{E}(t) > 0 \) for non-trivial \( (h, \alpha) \), it suffices to show that for \( (h, \alpha) \neq (0, 0) \)

\[
\begin{align*}
(i) & \quad E|\dot{h}(x, t)|^2 + \tilde{m}|\dot{h}(x, t)|^2 + \tilde{I}\dot{\alpha}(x, t)|^2 + \tilde{S}(|\dot{\alpha}(x, t)|^2 + \tilde{\hat{h}}(x, t) + \tilde{\alpha}(x, t)|\dot{h}(x, t)|) \geq 0 \\
(ii) & \quad \int_0^L \left[ G|\alpha'(x, t)|^2 - \pi \rho u^2 \left( \frac{1}{2} + a \right) |\alpha(x, t)|^2 \right] \, dx > 0.
\end{align*}
\]

It can be readily estimated that

\[
E|\dot{h}(x, t)|^2 + \tilde{m}|\dot{h}(x, t)|^2 + \tilde{I}\dot{\alpha}(x, t)|^2 + \tilde{S}(|\dot{\alpha}(x, t)|^2 + \tilde{\hat{h}}(x, t) + \tilde{\alpha}(x, t)|\dot{h}(x, t)|)
\geq E|\dot{h}(x, t)|^2 + (\sqrt{\tilde{m}}|\dot{h}(x, t)| - \sqrt{\tilde{I}}\dot{\alpha}(x, t)|^2 + 2(\sqrt{\tilde{m}}|\alpha| - \tilde{S})|\dot{\alpha}(x, t)|\dot{h}(x, t)|.
\]

Using formulae (3.18) for \( \tilde{m}, \tilde{I} \) and \( \tilde{S} \), we have that \( \Delta = \tilde{m} - \tilde{S}^2 > \tilde{m} - \tilde{S}^2 > 0 \), and therefore

\[
(\tilde{m} - \tilde{S})|\dot{\alpha}(x, t)\dot{h}(x, t)| = \frac{\Delta}{\sqrt{\tilde{m}} - \tilde{S}} |\dot{\alpha}(x, t)| |\dot{h}(x, t)| \geq 0.
\]

Substitution (5.14) into (5.13) and yields estimate (5.12) (i). Now we prove estimate (5.12) (ii). Owing to the boundary condition \( \alpha(0, t) = 0 \), we obtain that \( \|\alpha\|^2_{L^2(0, L)} \leq L^2/2 \|\alpha'\|^2_{L^2(0, L)} \) by one-dimensional Poincare inequality. Now, we estimate the integral from (5.12) (ii) as follows

\[
\int_0^L \left[ G|\alpha'(x, t)|^2 - \pi \rho u^2 \left( \frac{1}{2} + a \right) |\alpha(x, t)|^2 \right] \, dx \geq \frac{2}{L^2} G \|\alpha\|^2_{L^2(0, L)} - \pi \rho u^2 \left( \frac{1}{2} + a \right) \|\alpha\|^2_{L^2(0, L)}.
\]

For \( -1 \leq a \leq 1 \), we have \( (2/L^2) G - \pi \rho u^2 \left( \frac{1}{2} + a \right) \geq (2/L^2) G - \frac{3}{2} \pi \rho u^2 \). Because \( u \) satisfies the second estimate from (3.13), the positivity of the difference \( (2/L^2) G - \pi \rho u^2 \left( \frac{1}{2} + a \right) \) is obvious.

Finally, we prove that \( \mathcal{E}(t) = 0 \) implies that \( h(x, t) = \alpha(x, t) = 0, x \in (0, L), t \geq 0 \). Using (5.13), we obtain that the equation \( \mathcal{E}'(t) = 0 \) yields

\[
(i) \quad \dot{h}(x, t) = 0, \quad (ii) \quad \sqrt{\tilde{m}}|\dot{h}(x, t)| - \sqrt{\tilde{I}}|\dot{\alpha}(x, t)| = 0 \quad \text{and} \quad (iii) \quad \dot{h}(x, t)\dot{\alpha}(x, t) = 0.
\]

Combining (ii) and (iii) from (5.16), we obtain that \( \dot{h}(x, t) = 0 \) and \( \dot{\alpha}(x, t) = 0 \), which means that \( h(x, t) = \tilde{h}(x) \), \( \alpha(x, t) = \tilde{\alpha}(x) \). From (i) of (5.16), we obtain that \( \tilde{h}(x) = ax + b \) and to satisfy the boundary conditions at \( x = 0 \), \( \tilde{h}(0) = \tilde{h}'(0) = 0 \), we obtain \( \tilde{h}(x) = 0 \).

Evaluating \( \mathcal{E}'(t) \) from (5.11) on \( h = 0 \) and \( \alpha(x, t) = \tilde{\alpha}(x) \), we obtain that

\[
\mathcal{E}(t) = \frac{1}{2} \int_0^L \left[ G|\tilde{\alpha}'(x)|^2 - \pi \rho u^2 \left( \frac{1}{2} + a \right) |\tilde{\alpha}(x)|^2 + 2\pi \rho u t \left( \frac{h}{2} \right)^2 |\tilde{\alpha}(x)|^2 \right] \, dx = 0.
\]

It is clear that \( \mathcal{E}(t) \) from (5.17) is a linear function of \( t \) and it can be equal to zero only if \( \|\tilde{\alpha}\|^2_{L^2(0, L)} = 0 \), which yields \( \tilde{\alpha} = 0 \).

The lemma is completely shown.

---

### 6. Proof of main result: circle of instability

Here, we prove the main result of the paper. Our main object of interest is the dynamics generator \( \mathcal{X}_{\beta\delta} \) of the evolution problem (5.1). Let us denote the spectrum of \( \mathcal{X}_{\beta\delta} \) by \( \{\lambda_n\}_{n \in \mathbb{Z}} \). According to theorems 4.3–4.5, this spectrum splits asymptotically into two branches, the \( \beta \)-branch and the \( \delta \)-branch. For this reason, it will be convenient at some point below to use an alternative notation and denote the spectrum by \( \{\lambda_n^\beta\}_{n \in \mathbb{Z}} \cup \{\lambda_n^\delta\}_{n \in \mathbb{Z}} \). Now, we formulate the main result.
Theorem 6.1. For each value \( u \) of the air stream speed satisfying the second condition of (3.13), there exists \( R(u) > 0 \) such that the following statement holds. If an eigenmode \( \lambda_n \) satisfies \( |\lambda_n| > R(u) \), then this eigenmode is stable, i.e. \( \Re \lambda_n < 0 \). Formula (6.1) provides an estimate on \( R(u) \):

\[
R(u) = C \sqrt{\frac{\rho}{G \delta}} u^{3/2},
\]

where \( C > 0 \) is an absolute constant.

Corollary 6.2. For each \( u \), all the unstable eigenmodes are located inside the ‘circle of instability’ \( |\lambda| = R(u) \). The number of these eigenmodes is always finite.

To prove theorem 6.1, we have to complete some preliminary steps.

For each eigenvalue \( \lambda_n \), equation (5.1) has a solution given explicitly in the form

\[
\psi_n(x, t) = e^{\lambda n t} (h_n(x), \lambda_n h_n(x), \alpha_n(x), \lambda_n \alpha_n(x))^T.
\]

Obviously, \((h_n(x), \lambda_n h_n(x), \alpha_n(x), \lambda_n \alpha_n(x))^T\) is an eigenvector of \( \mathcal{K}_{\beta \delta} \). It can be easily seen that the pair of functions \((h_n(x) , \alpha_n(x))^T\) satisfies the system of two coupled ordinary differential equations

\[
\begin{align*}
\bar{m} \lambda^2 h(x) + \bar{S} \lambda^2 \alpha(x) + Eh''(x) + \pi \rho \mu h(x) \\
+ \pi \rho u \left(\frac{3}{2} - a\right) \lambda \alpha(x) + \pi \rho u^2 \alpha(x) &= 0, \\
\end{align*}
\]

and

\[
\begin{align*}
\bar{S} \lambda^2 h(x) + \bar{I} \lambda^2 \alpha(x) - Gh''(x) - \pi \rho u \left(\frac{1}{2} + a\right) \lambda \alpha(x) \\
+ \pi \rho u b^2 \lambda \alpha(x) - \pi \rho u^2 \left(\frac{1}{2} + a\right) \alpha(x) &= 0,
\end{align*}
\]

and the boundary conditions

\[
\begin{align*}
h(0) = h'(0) = \alpha(0) = 0, & \quad h''(L) = 0 \\
and & \quad Eh''(L) + \beta \lambda h'(L) = 0, \quad G \lambda' (L) + \delta \lambda \alpha(L) = 0
\end{align*}
\]

Let \( \Phi_n(x) = (h_n(x), \alpha_n(x))^T \). Formulae (6.3) and (6.4) mean that \( \lambda_n \) and \( \Phi_n \) are an eigenvalue and the corresponding eigenvector of the following non-selfadjoint quadratic operator pencil [25]:

\[
\lambda^2 \begin{bmatrix}
\bar{m} & \bar{S} \\
\bar{S} & \bar{I}
\end{bmatrix} \Phi(x) + \lambda \pi \rho u \begin{bmatrix}
1 & \left(\frac{3}{2} - a\right) \\
\left(\frac{1}{2} + a\right) & \left(\frac{1}{2} - a\right)^2
\end{bmatrix} \Phi(x) \\
\end{bmatrix} \\
+ \begin{bmatrix}
E \frac{d^4}{dx^4} & \Pi \rho u^2 \\
0 & -G \frac{d^2}{dx^2} + \pi \rho u^2 \left(\frac{1}{2} + a\right)
\end{bmatrix} \Phi(x) = 0,
\]

and

\[
\begin{bmatrix}
E \frac{d^2}{dx^2} & 0 \\
0 & G \frac{d}{dx}
\end{bmatrix} \Phi \bigg|_{x=L} + \lambda \begin{bmatrix}
\beta \frac{d}{dx} & 0 \\
0 & \delta
\end{bmatrix} \Phi \bigg|_{x=L} = 0
\]

Polynomial pencil (6.5) has been studied in reference [18]. Its spectrum coincides with the spectrum of the operator \( \mathcal{K}_{\beta \delta} \). At this point, we switch to an alternative notation \( \lambda_n \in \mathbb{Z} \cup \{0\} \) for the spectrum, as was explained above, and denote by \( \{\Phi_n^\beta\}_n \in \mathbb{Z} \cup \{0\} \) the corresponding eigenvectors of the pencil. In reference [18], we have obtained the asymptotic
approximation for the components of the vectors \( \Phi_n^\alpha \) and \( \Phi_n^\beta \) as \(|n| \to \infty\). In what follows, we need asymptotic approximation as \(|n| \to \infty\) only for the \( \alpha \)– component of \( \Phi_n^\alpha \) and \( \Phi_n^\beta \), i.e. for \( \alpha_n^\alpha \) and \( \alpha_n^\beta \). We reproduce the necessary results from reference [18].

**Theorem 6.3 ([18]).** (i) For \( n > 0 \) and \( x \in [0, L] \), the asymptotic representation for the \( \alpha \)– component of the \( \delta \)– branch eigenfunction can be given by formula (6.6):

\[
\alpha_n^\alpha(x) = \sin \left( \pi n + \frac{1}{2} \ln \frac{\delta + \sqrt{\delta^2 - 4\gamma L}}{\delta - \sqrt{\delta^2 - 4\gamma L}} \frac{L}{x} \right) + O \left( \frac{1}{\sqrt{n}} \right), \quad n \to \infty.
\]

For \( n < 0 \) and \( x \in (0, L) \), the following relation holds: \( \alpha_n^\alpha(x) = \bar{\alpha}_{|n|}^\delta(x) \).

(ii) For \( n > 0 \) and \( x \in [0, L] \), the asymptotic representation for the \( \alpha \)– component of the \( \beta \)– branch eigenfunction can be given by formula (6.7):

\[
\alpha_n^\beta(x) = \sin \left( \frac{dm^2 x}{L} \right) + V_\beta [e^{i \pi mx/L} + i e^{-i \pi mx/L} + (1 - i) e^{-i \pi mx/L}] + O \left( \frac{1}{n} \right), \quad n \to \infty.
\]

where \( m = n - \frac{1}{4}, \delta = \pi^2 L^{-1}(\tilde{m}^2 - \tilde{S}^2)^{-1}\tilde{E}^{1/2}G^{-1/2} \), and

\[
V_\beta = -\frac{1}{4}[K_+ + K_- e^{-2 i \rho^2 \delta}], \quad K_ \pm = \delta^{-1} \sqrt{\gamma} \alpha \pm 1. \quad (6.8)
\]

For \( n < 0 \) and \( x \in [0, L] \), the following relation holds: \( \alpha_n^\beta(x) = \bar{\alpha}_{|n|}^\beta(x) \). All estimates \( O() \) in formulae (6.6) and (6.7) are uniform with respect to \( x \in [0, L] \).

Having the necessary information on the spectrum and the eigenfunctions of the operator \( \mathcal{K}_{\beta \delta} \), we are in a position to introduce definition (6.4).

**Definition 6.4.** An eigenvalue \( \lambda_n \) of the operator \( \mathcal{K}_{\beta \delta} \) is called a strongly stable mode if the energy (5.11) evaluated on the corresponding solution (6.2) of the evolution problem (5.1) is a strictly decreasing function of \( t \): \( \dot{\varepsilon}_n(t) < 0 \). Here, \( \varepsilon_n(t) \) is the notation for the energy (5.11) evaluated on \( \Psi_n(x, t) \) from (6.2).

**Proposition 6.5.** If \( \lambda_n = r_n + i \omega_n \) is a strongly stable mode, then it is stable in the usual sense, i.e. \( r_n = \Re \lambda_n < 0 \). The inverse statement is not true in general.

**Proof.** Substitute (6.2) into (5.11) to obtain

\[
\varepsilon_n(t) = \frac{1}{2} e^{-2\pi u} \int_0^L \left\{ |E h_n''(x)|^2 + G |\alpha_n'(x)|^2 - \pi \rho u^2 \left( \frac{1}{2} + i \right) |\alpha_n(x)|^2 \\
+ (|\alpha_n(x)|^2 - \bar{\rho} |\alpha_n(x)|^2) \right\} dx \\
+ 2 \pi \rho u \int_0^L \left( \lambda_n h_n(x) + b_n \alpha_n(x) + \frac{u}{2 \alpha_n(x)} \right)^2 dx \int_0^t e^{2\pi u} dr. \quad (6.9)
\]

From (6.9), it follows that for any \( n \in \mathbb{Z} \), we have

\[
\varepsilon_n(t) = A_n e^{-2\pi u} + B_n e^{2\pi u} - \frac{1}{2r_n}, \quad A_n > 0, \quad B_n > 0, \quad \text{if} \quad r_n \neq 0
\]

\text{and} \quad \varepsilon_n(t) = A_n + B_n t, \quad \text{if} \quad r_n = 0. \quad (6.10)

Obviously, both functions of (6.10) increase for \( r_n \geq 0 \). Therefore, because \( \varepsilon_n(t) \) decreases, we have \( r_n < 0 \).

The proposition is shown. \( \blacksquare \)

We need one technical result given below. In the sequel, if two positive sequences \( \{a_n\}_{n \in \mathbb{Z}} \) and \( \{b_n\}_{n \in \mathbb{Z}} \) are connected by relations \( C_1 a_n \leq b_n \leq C_2 a_n \) with two absolute constants \( C_1 \) and \( C_2 \), then we write \( a_n \asymp b_n \).
Lemma 6.6.

(i) For the set of the $\alpha$-components of the $\delta$-branch eigenfunctions the following relation holds

$$\|\alpha_n^\delta\|_{L^2(0,L)} \asymp |\alpha_n^\delta(L)|, \quad n \in \mathbb{Z}'. \tag{6.11}$$

(ii) For the set of the $\alpha$-components of the $\beta$-branch eigenfunctions the following relations hold

$$|\alpha_n^\beta(L)| \asymp 1, \quad \|\alpha_n^\beta\|_{L^2(0,L)} \lesssim C|\alpha_n^\beta(L)|, \quad n \in \mathbb{Z}', \tag{6.12}$$

with $C$ being some absolute constant.

Proof. Let $n > 0$. Based on (5.6), we obtain the following estimates for $|\alpha_n^\delta(L)|$:

$$|\alpha_n^\delta(L)| = \frac{\sin \left( \pi n + \frac{i}{2} \ln \frac{\delta + \sqrt{G \delta}}{\delta - \sqrt{G \delta}} \right) + O \left( \frac{1}{\sqrt{n}} \right) \sinh \left( \frac{1}{2} \ln \frac{\delta + \sqrt{G \delta}}{\delta - \sqrt{G \delta}} \right) + O \left( \frac{1}{\sqrt{n}} \right). \tag{6.13}$$

If $\delta = \delta_R + i\delta_I$ with $\delta_R > 0$, then $\ln((\delta + \sqrt{G \delta})/(\delta - \sqrt{G \delta})) = A - iB$, where

$$A = \frac{1}{2} \ln \left| \frac{\delta_R + \sqrt{G \delta} + i\delta_I}{\delta_R - \sqrt{G \delta} + i\delta_I} \right|^2 = \frac{1}{2} \ln \left| \frac{\delta_R + \sqrt{G \delta} + i\delta_I}{\delta_R - \sqrt{G \delta} + i\delta_I} \right|^2 = \frac{1}{2} \ln |\delta|^2 + 2\delta_R \sqrt{G \delta} + G \delta,$$

and

$$B = \text{Arg} \left( \frac{\delta_R + \sqrt{G \delta} + i\delta_I}{\delta_R - \sqrt{G \delta} + i\delta_I} \right) = \frac{1}{2} \ln \left| \frac{|\delta_R + \sqrt{G \delta} + i\delta_I|^2}{|\delta_R - \sqrt{G \delta} + i\delta_I|^2} \right| = -\tan^{-1} \frac{2\delta_I \sqrt{G \delta}}{|\delta|^2 - G \delta}. \tag{6.15}$$

It can be easily seen from (6.14) and (6.15) that when $|\delta| \neq G \delta$, both $A$ and $B$ are well defined and $|A| > 0$. It means that $\sinh(A/2 + i(B/2)) > 0$. Thus, (5.7) implies that there exist two absolute constants, $D_1$ and $D_2$, such that

$$0 < D_1 \leq |\alpha_n^\delta(L)| \leq D_2 < \infty. \tag{6.16}$$

It remains to be shown that the sequence $\{\alpha_n^\delta(x)\}_{n \in \mathbb{Z}'}$ is almost normalized in $L^2(0,L)$, i.e. there exist two absolute constants, $D_3$ and $D_4$, such that

$$0 < D_3 \leq \|\alpha_n^\delta\|_{L^2(0,L)} \leq D_4 < \infty. \tag{6.17}$$

Combining (6.16) and (6.17) yields (6.11). To prove (6.17), we use (5.6) and estimate $\|\alpha_n^\delta\|_{L^2(0,L)}$ as follows

$$\|\alpha_n^\delta\|_{L^2(0,L)} = \int_0^L \left| \sin \left( \pi n + \frac{i}{2} \ln \frac{\delta + \sqrt{G \delta}}{\delta - \sqrt{G \delta}} \right) \right|^2 \, d\tau + O \left( \frac{1}{\sqrt{n}} \right)$$

$$= \int_0^L \left| \sin \left( \pi n + \frac{B}{2} + i \frac{A}{2} \right) \right|^2 \, d\tau + O \left( \frac{1}{\sqrt{n}} \right) \tag{6.18}.$$}

Taking into account that $|\sin(x + iy)|^2 = \frac{1}{2} \cosh 2y - \cos 2x$, we evaluate the integral (6.18) to have

$$\|\alpha_n^\delta\|_{L^2(0,L)}^2 = \frac{L}{2} \int_0^1 \cos(2\pi n - B) \, d\tau + \frac{L}{2} \int_0^1 \cosh(A \tau) \, d\tau + O \left( \frac{1}{\sqrt{n}} \right)$$

$$= \frac{L \sinh A}{2A} + O \left( \frac{1}{\sqrt{n}} \right) = \frac{2L \sinh((1/2) \ln(|\delta|^2 + 2\delta_R \sqrt{G \delta} + G \delta) / (|\delta|^2 - 2\delta_R \sqrt{G \delta} + G \delta))}{\ln(|\delta|^2 + 2\delta_R \sqrt{G \delta} + G \delta) / (|\delta|^2 - 2\delta_R \sqrt{G \delta} + G \delta))}$$

$$+ O \left( \frac{1}{\sqrt{n}} \right), \tag{6.19}$$

which obviously implies (6.17).
Note expression (6.19) makes sense as $\delta_R \to \infty$. Indeed,

$$
\lim_{\delta_R \to \infty} \frac{2\sinh((1/2)\ln((\delta^2 + 2\delta_R \sqrt{G} + G)/((\delta^2 - 2\delta_R \sqrt{G} + G))))}{\ln((\delta^2 + 2\delta_R \sqrt{G} + G)/((\delta^2 - 2\delta_R \sqrt{G} + G)))} = \lim_{\delta_R \to \infty} \frac{\delta_R \sqrt{G} - G}{|\delta^2 - 2\delta_R \sqrt{G} + G|} = 1.
$$

Statement (i) of the lemma is shown.

Now, we turn to the $\beta$-branch and show that $|\alpha_n^{\beta}(L)| \approx 1$, i.e. this sequence is bounded above and below uniformly with respect to $n$. The fact that this sequence is bounded above follows immediately from formulae (6.7) and (6.8). Let us show the estimate from below: $|\alpha_n^{\beta}(L)| \geq C$. We have for $n > 0$

$$
\alpha_n^{\beta}(L) = \sin(dm^2) + V_\beta[-e^{i\pi m} + i e^{-i\pi m}] + O(n^{-1})
$$

$$
= \sin(dm^2) - (\frac{i}{4})[K_+ + K_- e^{-2i\alpha m}] e^{i\alpha m} [-e^{2i\alpha m} + i] + O(n^{-1})
$$

$$
= \sin(dm^2) - (\frac{i}{4})[K_+ + e^{i\alpha m} + K_- e^{-i\alpha m}] 2i + O(n^{-1})
$$

$$
= 2 \sin(dm^2) + i\delta^{-1} GR \cos(dm^2) + O(n^{-1}).
$$

Equation (6.20) obviously means that $|\alpha_n^{\beta}(L)|^2 \approx 4 \sin^2(dm^2) + \delta^{-2} GR \cos^2(dm^2)$. Because $\sin^2(dm^2) + \delta^{-2} GR \cos^2(dm^2) \approx 1$, we obtain the desired estimate from below: $|\alpha_n^{\beta}(L)| \approx 1$, which yields the estimates of statement (ii).

The lemma is shown.

Now, we are in a position to prove the main result of the paper.

**Proof of theorem 6.1.** We start with the formula for $\dot{\alpha}(t)$ that follows from (5.11)

$$
\dot{\alpha}(t) = \frac{1}{2} \int_0^L \left[ E[h'' h'' + h'' h'''](x, t) + G[\tilde{\alpha}' \tilde{\alpha}' + \tilde{\alpha} \tilde{\alpha}'](x, t) + \tilde{m}[h h + h h](x, t) \right.
$$

$$
+ \tilde{\alpha} \tilde{h} + \tilde{h} \tilde{\alpha}](x, t) + \tilde{S}[\tilde{\alpha} h + \tilde{h} \alpha + \tilde{\alpha} \tilde{h}](x, t) + \tilde{a} \tilde{\alpha} + \tilde{\alpha} \tilde{a}](x, t) + \tilde{S}[\tilde{\alpha} h + \tilde{h} \alpha + \tilde{\alpha} \tilde{h}](x, t) + 2 \pi \rho u |\tilde{h}(x, t) + \tilde{b} \alpha(x, t) + \tilde{a} \alpha(x, t)|^2 - \pi \rho u^2 \left( \frac{1}{2} + a \right) \frac{d}{dt} |\alpha(x, t)|^2 \right] dx.
$$

Integrating by parts twice in the first two integrals of (6.21) and using the boundary conditions (3.9)–(3.11), we obtain

$$
\int_0^L E[h'' h'' + h'' h'''](x, t) dx = E[h''(L, t)h''(L, t) + h''(L, t)\tilde{h}'(L, t)] + E \int_0^L [\tilde{h}'' h + h'' \tilde{h}](x, t) dx
$$

$$
= -2E \Re \beta \dot{h}'(L, t)^2 + E \int_0^L [\tilde{h}'' h + h'' \tilde{h}](x, t) dx.
$$

Similarly, we obtain

$$
\int_0^L G[\tilde{\alpha}' \tilde{\alpha}' + \tilde{\alpha} \tilde{\alpha}'](x, t) dx = -2G \Re \delta \dot{\alpha}(L, t)^2 - G \int_0^L [\tilde{\alpha}'' \alpha + \tilde{\alpha} \alpha''](x, t) dx.
$$

With (6.22) and (6.23), formula (6.21) can be modified to

$$
\dot{\alpha}(t) = \frac{1}{2} \int_0^L \left[ E[h'' h'' + h'' h'''](x, t) - G[\alpha'' \alpha + \alpha'' \alpha](x, t) + \tilde{m} \frac{d}{dt} |\tilde{h}(x, t)|^2 + \tilde{S} \frac{d}{dt} |\tilde{h}(x, t)|^2 - \pi \rho u^2 \left( \frac{1}{2} + a \right) \frac{d}{dt} |\alpha(x, t)|^2 \right] dx
$$

$$
+ \tilde{S} \frac{d}{dt} [\tilde{\alpha} \tilde{h} + \tilde{h} \alpha] - \pi \rho u^2 \left( \frac{1}{2} + a \right) \frac{d}{dt} |\alpha(x, t)|^2 + 2 \pi \rho u \left| h + b \alpha + \frac{u}{2} \alpha \right|^2 (x, t) \right] dx.
$$

$$
- 2E \Re \beta |\dot{h}'(L, t)|^2 + G \Re \delta |\dot{\alpha}(L, t)|^2.
$$

(6.24)
Because $h$ and $\alpha$ satisfy equation (5.6), the representation (6.24) for $\dot{\delta}(t)$ can be simplified as follows

$$
- \dot{\delta}(t) = 2\pi \rho u \int_0^L \left[ b^2 |\dot{\omega}(x, t)|^2 - \left| b\dot{\omega}(x, t) + \frac{u}{2}\alpha(x, t) \right|^2 \right] \, dx + E\beta \dot{h}^2(L, t) + G\delta |\dot{\alpha}(L, t)|^2. \tag{6.25}
$$

Now, we evaluate $\dot{\delta}(t)$ on $h(x, t) = e^{\lambda t} h_n(x)$ and $\alpha(x, t) = e^{\lambda t} \alpha_n(x)$. Recall that according to definition 6.4 the result is denoted by $\dot{\delta}_n(t)$. Let $Q_n(t)$ be

$$
Q_n(t) = |\lambda_n|^2 e^{2\alpha t} \left[ E\beta |h'_n(L)|^2 + G\delta |\alpha_n(L)|^2 \right], \quad \text{where } r_n = \Re \lambda_n. \tag{6.26}
$$

Rewriting (6.25) for the above $h$ and $\alpha$ and using (6.26), we obtain

$$
\dot{\delta}_n(t) = -2\pi \rho u e^{2\alpha t} |\lambda_n|^2 \int_0^L \left[ b^2 |\dot{\omega}_n(x, t)|^2 - \left| b\dot{\omega}_n(x, t) + \frac{u}{2}\alpha_n(x, t) \right|^2 \right] \, dx - Q_n(t)
$$

$$
= 2\pi \rho u e^{2\alpha t} |\lambda_n|^2 \left( \frac{u^2}{4|\lambda_n|^2} + 2br_n \right) \int_0^L |\alpha_n(x)|^2 \, dx - Q_n(t)
$$

$$
= e^{2\alpha t} \left\{ \pi \rho u^2 \left( \frac{u}{2} + 2br_n \right) \| \alpha_n \|^2_{L^2(0, L)} - |\lambda_n|^2 \left[ E\beta |h'_n(L)|^2 + G\delta |\alpha_n(L)|^2 \right] \right\}. \tag{6.27}
$$

Because $\Re \beta > 0$, equation (6.27) yields

$$
\dot{\delta}_n(t) \leq e^{2\alpha t} \left\{ \pi \rho u^2 \left( \frac{u}{2} + 2br_n \right) \| \alpha_n \|^2_{L^2(0, L)} - G\delta |\lambda_n|^2 |\alpha_n(L)|^2 \right\}. \tag{6.28}
$$

Now, we use lemma 6.6. Note, to prove this lemma, we have to consider the $\beta$-branch and the $\delta$-branch of the spectrum separately. Now, when the result is proven, we can return to the original notation for the spectrum: $|\lambda_n|_{n \in \mathbb{Z}}$. Owing to (6.11) and (6.12), estimate (6.28) can be modified to

$$
\frac{\dot{\delta}_n(t)}{\| \lambda_n \|^2} \leq e^{2\alpha t} \left\{ C_1 \pi \rho u^2 \left( \frac{u}{2|\lambda_n|^2} + \frac{2br_n}{|\lambda_n|^2} \right) - G\delta \right\} |\alpha_n(L)|^2, \tag{6.29}
$$

with $C_1$ being some absolute constant. Obviously, for a given $u$, one can find a positive integer $N$ such that for all $n$ with $|n| > N$, the expression in the curly brackets in (6.29) is negative, because $\Re \delta > 0$. It means that for such $n$, the energy decreases with time.

Using the spectral asymptotics (4.14) and (4.15), and the second inequality from (3.13), one can see that there exists $N$ independent on $u$ such that for $|n| > N$, the following estimate holds

$$
C_1 \pi \rho u^2 \frac{2br_n}{|\lambda_n|^2} - G\delta \leq - \frac{G\delta}{2}, \tag{6.30}
$$

with $C_1$ being independent on $u$ (lemma 6.6). It is clear that for such $n$, we have

$$
C_1 \pi \rho u^2 \left( \frac{u}{2|\lambda_n|^2} + \frac{2br_k}{|\lambda_n|^2} \right) - G\delta \leq C_2 \pi \rho u^3 \frac{2}{|\lambda_n|^2} - \frac{G\delta}{2}. \tag{6.31}
$$

Therefore, if $|\lambda_n| > R(u)$, where $R(u)$ is defined in (6.1) with $C = \sqrt{C_1 \pi}$, then $\dot{\delta}_n(t) < 0$. It means that the corresponding mode is strongly stable.

The proof is complete.

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References


