Thermal shock fracture mechanics analysis of a semi-infinite medium based on the dual-phase-lag heat conduction model

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The generalized lagging behaviour in solids is very important in understanding heat conduction in small-scale and high-rate heating. In this paper, an edge crack in a semi-infinite medium subjected to a heat shock on its surface is studied under the framework of the dual-phase-lag (DPL) heat conduction model. The transient thermal stress in the medium without crack is obtained first. This stress is used as the crack surface traction with an opposite sign to formulate the crack problem. Numerical results of thermal stress intensity factor are obtained as the functions of crack length and thermal shock time. Crack propagation predictions are conducted and results based on the DPL model and those based on the classical Fourier heat conduction model are compared. The thermal shock strength that the medium can sustain without catastrophic failure is established according to the maximum local stress criterion and the stress intensity factor criterion.

1. Introduction

The classical Fourier heat conduction law assumes that the speed of heat propagation in a body is infinite so that the body will be affected by the heat boundary condition and initial conditions at the instant of heating. However, the speed of heat propagation in a body is always finite [1]. Nowadays, it is extensively accepted that consideration of the non-Fourier effect of heat...
conduction is required for certain problems of practical interest in contemporary engineering. The first formulation of the non-Fourier heat conduction problem was presented by Cattaneo [2] and Vernotte [3] based on the local energy balance. The approach uses relaxation behaviour to describe pulsed heat transport at short time intervals and gives a hyperbolic wave theory of heat conduction (see [4–8] for a literature review)

\[ q_i(t + \tau_q) = -k \frac{\partial T}{\partial x_i} \]

(1.1)

and

\[ \rho c T \frac{\partial^2 T}{\partial t^2} + \rho c \frac{\partial T}{\partial t} = k \nabla^2 T + \tau_q \frac{\partial Q}{\partial t} + Q, \]

(1.2)

where \( \rho \) is the mass density, \( c \) the specific heat and \( Q \) is the internal heat generation rate per unit volume, \( k \) is the thermal conductivity, \( q_i \) are the components of the heat flux vector \( q \), \( \partial T/\partial x_i \) the temperature gradients and \( \tau_q \) is the thermal relaxation time, which is related to the collision frequency of the molecules within the energy carrier. The thermal relaxation time, \( \tau_q \), can be calculated using the thermal wave speed, \( C \), and the thermal diffusivity, \( k/\rho c \), as \( \tau_q \propto k/\rho c C^2 \). In particular, for metals at room temperature, \( k/\rho c \sim 10^{-4} \text{ m}^2 \text{ s}^{-1} \), \( \tau_q \sim 10^{-12} \text{ s} \) and \( C \sim 10^4 \text{ m} \text{ s}^{-1} \), the physical space and time scales corresponding to dimensionless \( x = 1 \) and \( t = 1 \) are in the order of 1 nm and 0.01 ps, respectively [1]. Since the proposal of the above hyperbolic wave theory, considerable effort has been devoted to the study of the related non-Fourier heat conduction problems. Analytical solutions for the hyperbolic heat conduction equation have been obtained for a few relatively simple problems [1,8]. Numerical methods for more complicated problems have been also developed, such as transient heat conduction in multilayer materials [9–11] and infinitely wide slab [12]. Miller & Haber [13] reviewed the modification of the Fourier conduction law and described its implementation of the Galerkin finite-element method within a discontinuous space–time that admits jumps in primary variables across element boundaries with an arbitrary orientation in space and time. Chen [14] proposed a hybrid Green’s function method to investigate the hyperbolic heat conduction problems. Wang et al. [15] proposed a coupled finite-element–finite difference scheme for the analysis of heat conduction based on the hyperbolic wave theory.

The hyperbolic wave theory addresses the inertial effect in the short-time transient via a macroscopic approach. The discontinuity existing at the thermal wavefront results in the thermal shock formation and the thermal resonance phenomenon [16–20], which cannot be depicted by the classical diffusion Fourier heat conduction theory. The theory assumes a macroscopic behaviour averaged over many grains. When the small-scale structural effect becomes pronounced, as associated with rapidly shortening of the response time, the concept of macroscopic average may lose its physical support and the applicability of the thermal wave model becomes open to debate [21,22]. The non-equilibrium thermodynamic transition and microstructural effect become increasingly important for energy transport involving high-rate heating at small length scale [23–25]. Based on the first law of thermodynamics and the dual-phase-lag (DPL) constitutive relation of heat flux density, the DPL heat conduction equation has been developed [8,26,27]. Mathematically, if the volumetric heat source is ignored, the DPL model is illustrated by the following one-dimensional equations:

\[ q(x, t + t_q) = -k \frac{\partial T(x, t + \tau_T)}{\partial x} \]

(1.3)

and

\[ \frac{\partial^2 T}{\partial x^2} + \tau_T \frac{\partial^3 T}{\partial x^2 \partial t} = \frac{\tau_q^2}{2\kappa} \frac{\partial^3 T}{\partial t^3} + \frac{\tau_q}{\kappa} \frac{\partial^2 T}{\partial t^2} + \frac{1}{\kappa} \frac{\partial T}{\partial t}, \]

(1.4)

where \( t_q \) and \( \tau_T \) are the phase lags of the heat flux and temperature, respectively, which are two intrinsic thermal properties of the medium, \( k \) is the thermal conductivity of the material, \( \kappa = k/\rho c \)
is the thermal diffusivity \((m^2s^{-1})\), and where \(\rho\) is the mass density and \(c\) is the specific heat. If \(\tau_T\) and the term \(\tau_q^2/2\kappa \partial^3 T/\partial t^3\) are omitted, equation (1.4) reduces to the thermal wave model employing the single-phase-lag concept (the hyperbolic heat conduction theory). If \(\tau_T = \tau_q\) and the term \(\tau_q^2/2\kappa \partial^3 T/\partial t^3\) are omitted, equation (1.4) reduces to the classical diffusion model without the lag response (i.e. the classical Fourier heat conduction law). Thus, equation (1.4), based on the DPL concept, covers a wide scale of space and time for physical observations.

The physical meanings of the DPL model are shown by the experimental results [28]. Various heat transfer problems have been described with the DPL mode. Antaki [29,30] studied transient heat conduction in a semi-finite slab with surface flux and in micro-scale during short time. Tzou & Chen [31] investigated the thermal lag behaviour in amorphous. Al-Nimr & Al-Huniti [32] explored the transient thermal stresses in a thin plate induced by a rapid heating. Liu & Chang [33] investigated the transient heat conduction in an infinitely long solid cylinder for an exponentially decaying pulse boundary heat flux and for a short-pulse boundary heat flux. Ghazanfarian & Shomali [34] investigated the numerical simulation of non-Fourier transient heat transfer in a two-dimensional sub-100 nm metal-oxide-semiconductor field-effect transistor. Ho et al. [35] analysed the heat conduction problem in a two-layered structure. Hu & Chen [36] investigated the transient temperature field around a partially insulated crack in a semi-infinite medium.

Thermal shock fracture mechanics based on the classical Fourier heat conduction has been studied extensively due to its practical significance in engineering applications. Rizk [37,38] obtained the transient thermal stress intensity factors for an orthotropic semi-infinite plate and two bonded dissimilar materials. Clayton [39] developed an anisotropic ceramic crystals model to investigate the nonlinear electromechanical behaviour of silicon carbide subjected to high shock temperatures. The influence of the temperature dependence of the material properties on the thermal shock fracture of ceramics was investigated experimentally by Nishikawa et al. [40]. There are also some pioneering studies of thermal shock fracture mechanics based on the hyperbolic heat conduction theory of equations (1.1) and (1.2). For example, a transient thermal fracture problem corresponding to a semi-infinite medium with a surface crack was studied in [41], a penny-shaped crack in a piezoelectric material was studied in [42], transient thermal cracking associated with non-classical heat conduction in cylindrical coordinate system was studied in [43] and the thermal shock resistance of solids associated with hyperbolic heat conduction theory was investigated in [44]. To the authors’ knowledge, the thermal shock fracture problem and associated crack propagation problem was not studied on the basis of the DPL heat conduction model.

This paper investigates the thermal shock fracture mechanics of a semi-infinite medium based on the DPL heat conduction model. An edge crack which is perpendicular to the surface of the medium is considered. The temperature field and the associated thermal shock stress for the un-cracked medium are obtained by Laplace transform. The thermal stress intensity factor for the crack problem is obtained by a numerical integration formula. Variations of the thermal shock stress and thermal shock stress intensity factor with thermal shock time and crack length are displayed graphically. Unstable and stable crack growths are discussed. Comparison between the DPL heat model and the classical Fourier heat conduction model is conducted and the significant difference between the two models is identified.

2. Thermal stress for the un-cracked semi-medium

Consider the problem of a semi-infinite medium subjected to a transient temperature decrease. Initial, the medium temperature is \(T_0\). At \(t = 0\), the surface of the medium \((x = 0)\) is suddenly cooled to zero. This represents a typical quenching experiment for thermal shock of the medium. During cooling, the thermal stress becomes tensile on the surface of the semi-infinite medium and compressive in the interior of the medium. If the thermal stress is strong enough, a crack may initiate at the surface of the medium and propagate from the surface towards to the depth direction of the medium. The problem can be synthesized from the general solution for a surface
Figure 1. A surface crack in a semi-infinite medium subjected to a sudden temperature decrease at its top surface.

A crack of length \( c \) in the semi-infinite medium shown in Figure 1. As the crack plane is normal to the surface of the medium, it does not perturb the transient temperature distribution in this arrangement. Therefore, the determination of the temperature field and the associated thermal stress distribution for the medium with no crack would be quite straightforward. The solution of the problem is presented in non-dimensional form. This avoids identifying the range of parameters where the effect substantially becomes important. In particular, a characteristic length parameter \( l_0 \) of the medium is introduced according to

\[
l_0 = \sqrt{\frac{\kappa}{\tau q}}.
\]

The length scale and time scale of the problem are normalized using

\[
\tilde{\tau} = \frac{t}{\tau q},
\]

\[
R = \frac{\tau}{\tau q},
\]

\[
\bar{b} = \frac{b}{l_0},
\]

and

\[
\bar{x} = \frac{x}{l_0}.
\]

In this case, equation (1.4) can be re-written as

\[
\frac{1}{2} \frac{\partial^3 T}{\partial \bar{x}^3} + \frac{\partial^2 T}{\partial \bar{x}^2} + \frac{\partial T}{\partial \bar{t}} = \frac{\partial^2 T}{\partial \bar{x}^2 \partial \bar{t}} + R \frac{\partial^3 T}{\partial \bar{x}^2 \partial \bar{t}}. \tag{2.1}
\]

Supposing the surface heat exchange coefficient of the medium is infinite, which represents the severest thermal shock to the material. In this simple one-dimensional situation, the temperature field in the Laplace transform domain \( \tilde{T} \) can be easily obtained

\[
\tilde{T}(\bar{x}, p) = \frac{T_0}{p} \left[ 1 - \exp \left( -\sqrt{\frac{p(1 + p + 0.5p^2)}{1 + Rp}} \bar{x} \right) \right]. \tag{2.2}
\]

The Laplace inversion of equation (2.2) has to be evaluated numerically. The numerical inversion formula developed previously [1] is used

\[
T(\bar{x}, \bar{t}) = \frac{e^{\gamma \bar{t}}}{\bar{t}} \left[ \tilde{T}(\bar{x}, \gamma) + Re \sum_{n=1}^{N} (-1)^n \tilde{T}(\bar{x}, \gamma + i\pi n) \right]. \tag{2.3}
\]

The above equation is the Riemann sum approximation of the Fourier integral transformed from the Laplace inversion integral. The quantity \( \gamma \) is the real value in the Bromwich cut from \( \gamma - i\infty \) to \( \gamma + i\infty \). As pointed out by Tzou [1], for a faster convergence, the value of \( \gamma \) satisfies the relation \( \gamma \bar{t} = 4.7 \), where \( \bar{t} \) is the dimensional physical time. Other values of \( \gamma \) will lead to the same solution, but the number of terms needed in the summation of equation (2.3) will increase by orders of magnitude for convergence [1], especially for problems involving discontinuities at the sharp wavefront.

The temperature field is one-dimensional along the \( x \)-direction (towards to the interior of the medium). Owing to symmetry and semi-infinity of the medium, the medium is fully constrained in the \( z \)-direction and the shear strain vanishes everywhere inside the medium. As a result, the normal strain \( \varepsilon_{zz} = 0 \) and the shear stresses \( \sigma_{xz} = 0 \) and \( \sigma_{xy} = 0 \), where \( y \) is the direction perpendicular to the \((x, z)\) plane. From the equilibrium equation of elasticity along the \( x \)-direction, \( \sigma_{xx,x} + \sigma_{xy,x} + \sigma_{xz,x} = 0 \), we know that \( \sigma_{xx} = 0 \). This suggests that \( \sigma_{xx} \) must be a constant. As the semi-infinite medium is free of surface traction on its surface \( x = 0 \), the constant stress \( \sigma_{xx} \) is zero everywhere inside the medium. Such a fact has also been demonstrated by Chang & Wang [41]. As the medium is subjected to a uniform thermal initial condition and a uniform boundary
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x
–5 0 1 5
s
zz/
σ


dual-phase-lag model,

\( R = 100 \)
dual-phase-lag model,

\( R = 50 \)
single-phase-lag model

Figure 2. Thermal stress as a function of normalized position for \( \bar{\tau} = 1 \). (Online version in colour.)

condition on the entire surface of the medium, \( x \) and \( t \) are the only independent variables for the medium without crack.

Now using \( \sigma_{xx} = 0 \) and \( \varepsilon_{zz} = 0 \), there is no difficulty to determine the stress \( \sigma_{zz} \) from the constitutive equation of the material as

\[
\sigma_{zz}(\bar{x}, \bar{\tau}) = -\bar{\lambda}_{33} \left[ T(\bar{x}, \bar{\tau}) - T_0 \right],
\]

where \( \bar{\lambda}_{33} \) is a material constant describing the relationship between the thermal stress and temperature change. For isotropic materials, \( \bar{\lambda}_{33} = E\alpha \) if the problem is plane stress (\( \sigma_{yy} = 0 \)) and \( \bar{\lambda}_{33} = E\alpha/(1 - v) \) if the problem is plane strain (\( \varepsilon_{yy} = 0 \)), in which \( E, \alpha \) and \( v \) are, respectively, Young’s modulus, the coefficient of thermal expansion and Poisson’s ratio.

In the following demonstration, the symbol \( \sigma_0 = \bar{\lambda}_{33} T_0 \) will be used to normalize the thermal stress \( \sigma_{zz}(\bar{x}, \bar{\tau}) \). Figure 2 shows the thermal stress distributions for \( R = 100 \), \( R = 50 \) and \( \bar{\tau} = 1 \). The classical Fourier heat conduction problem satisfies the parabolic diffusion equation, and the heat energy propagates instantaneously everywhere in the medium. As expected, the thermal stress predicted by the classical Fourier heat conduction exists everywhere inside the medium. However, the DPL model predicts a sharp stress wavefront at \( \bar{x} = \sqrt{2R\bar{\tau}} \), whereas the single-phase-lag model predicts a sharp stress wavefront at \( \bar{x} = \bar{\tau} \). These are the most distinguishable effects from the lag behaviour of heat transport in the medium. Owing to the second and third derivatives of the temperature with respect to time, equation (2.1) shows significant wave behaviour. As a result, the thermal stress associated with the DPL model is not a smooth function of the space and time. In addition, the DPL model induces a much larger heat-affected zone in the medium and much higher thermal stress level in the heat-affected zone. Variations of stress without crack with time and depth below the surface of the medium are shown in figures 3 and 4. Clearly, the DPL curves have sharp jumps (or drops) at the thermal stress wavefront. The location of thermal stress wavefront, to be reiterated, is at \( \bar{x} = \sqrt{2R\bar{\tau}} \) under various values of \( R \). This means that, at the same time, the thermal stress penetration depth into the medium increases with the ratio of \( R \) (such a fact can also be observed from figure 2). The position \( \bar{x} \) begins to feel the thermal stress at \( \bar{\tau} = \bar{x}/\sqrt{2R} \). The results also show that for any given time \( \bar{\tau} \), the region \( \bar{x} \in (0, \sqrt{2R\bar{\tau}}) \) inside the medium has thermal stress. Outside the region, that is \( \bar{x} \in (\sqrt{2R\bar{\tau}}, \infty) \), the thermal stress is always zero. Contrast to the DPL model, the classical Fourier heat conduction law predicts the thermal stress in the entire region of the medium. However, the stress level predicted by the classical Fourier model is much smaller than that predicted by the DPL model in the heat-affected zone at the same time and the same position.
3. Thermal stress intensity factor at the crack tip

As described above, under a cold shock event, the thermal stress is tensile and has the largest value at the surface of the medium. A sufficiently strong tensile stress may cause the unstable propagation of the pre-existing crack on the medium. In solving the crack problem, the stress without crack is solved first and the opposite value of the stress is used as the crack surface traction. Recall figure 1 for crack geometry, the corresponding thermal stress intensity factor can be evaluated from the following numerical integral [45]:

$$K_1(c, t) = 2 \sqrt{\frac{c}{\pi}} \int_0^c \frac{[1 + F(x/c)][-\sigma_{zz}(x, \tilde{t})]}{\sqrt{c^2 - x^2}} \, dx,$$

(3.1)
Figure 5. Thermal stress intensity factor as a function of normalized crack length at different time instants ($R = 100$, the broken lines are for classical Fourier heat conduction). (Online version in colour.)

where

$$f\left(\frac{x}{c}\right) = \left(1 - \frac{x}{c}\right) \left[0.2945 - 0.3912\left(\frac{x}{c}\right)^2 + 0.7685\left(\frac{x}{c}\right)^4 - 0.9942\left(\frac{x}{c}\right)^6 + 0.5094\left(\frac{x}{c}\right)^8\right].$$  (3.2)

The symbol $K_0 = \lambda_{33} T_0 \sqrt{l_0}$ will be used to normalize the thermal stress intensity factor $K_1(c, t)$. Variations of thermal stress intensity factors $K_1$ with time and crack length are shown in figures 5 and 6 for $R = 100$. For comparison, some sample results for the classical Fourier model are also given. As expected, curves based on the classical Fourier heat conduction law are smooth functions of time and crack length. However, curves based on the DPL model are non-smooth (but continuous) for various values of crack length and thermal shock time. This is due to the fact that the thermal stresses corresponding to equation (2.1) are not smooth functions of space and time. In figure 5, it can be seen that at any selected time, the thermal stress intensity factor increases when the crack length increases to the maximum value when the crack length is such that the crack tip is at the thermal stress wavefront (i.e. $\bar{c} = \sqrt{2R\bar{t}}$). Beyond this value, the thermal stress intensity factor decreases when the crack length increases. It can also be observed from figure 5 that for each specified time, the thermal stress intensity factors approach zero when the crack size is large enough. That is due to the fact that the thermal stress level approaches zero at sufficiently large crack sizes. Figure 6 shows that for each given crack length $\bar{c}$, the stress intensity factor increases with time to a peak value when the thermal shock time reaches $\bar{t} = \bar{c}/\sqrt{2R}$. Beyond this value, the thermal stress intensity factor displays a sharp drop and continuously decreases to a minimum value, then increases slowly with time. By contrast, the classical Fourier heat conduction law predicts a monotonously increasing function of thermal stress intensity factor with time. Comparison of the classical Fourier law and the DPL model suggests that the thermal stress intensity factors obtained from the DPL model are much higher than those from the classical Fourier heat conduction law. This means that for the same crack length, the thermal stress intensity factor corresponding to the DPL model achieves the fracture toughness of the medium much faster than the classical Fourier law.

Figure 7 depicts the time-history of the thermal stress intensity factor for considering the effect of the ratio of $\tau_T$ to $\tau_q$ at $\bar{t} = 1, 3$. Because of complication of the problem, the thermal stress intensity factor cannot be obtained in a closed-form and can only be evaluated from the numerical integration of equation (3.1). The location of the peak stress intensity factor, however, can be obtained as $\bar{c} = \sqrt{2R\bar{t}}$. At the same time, the penetration depth of the stress intensity factor...
into the medium increases with the ratio of $R$ in a square-root sense. For the responses at longer time instants, evidenced by the distributions of $R = 10$ and $\bar{t} = 3$, the thermal stress intensity factor levels off when the crack tip approaches the thermal stress wavefront. Such unique behaviour is pertinent to the DPL effect and is not found in classical thermo-elasticity theory.

4. Thermal crack growth analysis

From the results given in §3, the overall trend of the thermal stress intensity factor is that it increases with the thermal shock time $t$. Therefore, once the applied stress intensity factor $K_1$ reaches the fracture toughness of the medium $K_c$ at a certain time instant, the pre-existing crack may grow in length. In order to investigate the crack growth behaviour, variation of the
Figure 8. Crack growth for various values of the applied thermal stress intensity factor $K_0$ ($R = 100$, the broken lines are for classical Fourier heat conduction). (Online version in colour.)

stress intensity factor with the crack length for various values of time is plotted in figure 8. For comparison, results for both the DPL model and the classical Fourier model are given. Basically, at any given thermal shock time, the thermal stress intensity factor ascends from zero, to a maximum value, and then descend while the crack length increases. Therefore, an initial crack of length $c_0$ will start to propagate when $K_1$ reaches the fracture toughness of medium $K_c$. The crack growth is instantaneous until $K_1$ attains fracture toughness $K_c$ of the medium again at a new crack length $c_1$. Subsequently, owing to the continuous increase of the thermal stress intensity factor with thermal shock time, the crack will continue to grow in a stable manner. The phenomenon can also be found in figures 9 and 10, which display the crack growth trajectories for applied thermal stress intensity factors $K_0 = 0.45K_c$ and $K_0 = 0.41K_c$, respectively. For each specified thermal shock temperature load, there is a time $\bar{t}$ for which the crack begins to grow instantaneously from $c_0$ to $c_1$, then steadily propagates with thermal shock time. Specifically, for the applied thermal stress intensity factors $K_0 = 0.45K_c$ and at time $\bar{t} = 10$,

1. if the classical Fourier model is used, an initial crack of length $c_0 = 2.3l_0$ will grow unstably to $c_1 = 5.1l_0$ and
2. if the DPL mode is used, an initial crack of length $c_0 = 1.6l_0$ will grow unstably to $c_1 = 47.3l_0$.

The results reveal that the semi-infinite medium exhibits a crack growth resistance behaviour. The DPL model predicts a shorter initial crack length for which the unstable crack growth takes place. This also suggests that for a given initial crack length and the same thermal shock temperature load, the crack unstably propagates earlier for the DPL model than for the classical Fourier model. In addition, unstable crack propagation length is much larger for the DPL model than that of the classical Fourier model. Comparison of figures 9 and 10 detects that the lower the fracture toughness of the medium $K_c$, the earlier the crack starts to propagate. The values of the crack propagation length $c_1$ depend on the applied thermal shock temperature load, the initial crack length $c_0$ and the fracture toughness of the medium.

5. Thermal shock resistance of the medium

Thermal shock resistance must be evaluated properly for the design of heat-resistant materials for high-temperature applications. A core problem in selecting a material against thermal shock
Figure 9. Crack growth trajectory for the values of applied thermal stress intensity factors $K_0 = 0.45K_c$ ($R = 100$, the subscripts Dhl and Fou denote, respectively, the DPL model and the classical Fourier model).

Figure 10. The same as figure 9 but for $K_0 = 0.41K_c$.

is the identification of appropriate material selection criteria. Basically, stress-based failure and fracture-based failure are two major criteria to evaluate material performance [46].

According to the stress-based failure criterion, the thermal shock resistance of the material is the temperature for which the maximum thermal stress in the medium attains the strength of the medium $\sigma_b$. From the above analysis, it can be seen that the maximum thermal stress $\sigma_{33}T_0$ appears on the surface of the medium at the beginning of the thermal shock. As a result, the maximum temperature drop sustainable by the medium $\Delta T_c$ follows:

$$\Delta T_c = \Delta T_{c,\text{stress}} = \frac{\sigma_b}{\lambda_{33}}. \quad (5.1)$$

The sustainable temperature drop $\Delta T_c$ of the medium predicted by the stress-based criterion has no dependence on the length of the pre-existing crack.

According to the fracture-based failure criterion, the thermal shock resistance of the material is the temperature for which the maximum thermal stress intensity factor $K_{\max}(c^*, t^*)$ for a dominant crack in the medium attains the toughness of the medium $K_c$ at a crack length $c^*$ and at time $t^*$. In general, as the maximum stress intensity factor at the crack tip of the medium depends on...
the crack length, the maximum temperature drop, which is sustainable by the medium $\Delta T_c$ and predicted by the fracture-based criterion, also depends on the length of the pre-existing crack. However, from the above analysis, as the thermal shock time increases to infinity, the thermal stress intensity factor of the pre-existing crack is maximum and the medium is under a uniform tensile stress $\lambda_{33} T_0$. As the surface crack for a medium under uniform tensile stress $\lambda_{33} T_0$ is known to be $K_{\text{max}} = 1.122 \lambda_{33} T_0 \sqrt{\pi c}$ [47], the thermal shock resistance of the medium dominated by the toughness-based criterion is

$$\Delta T_c = \Delta T_c^{\text{fracture}} = \frac{K_c}{1.122 \lambda_{33} \sqrt{\pi c}}.$$ 

Clearly, $\Delta T_c$ predicted by the fracture-based criterion has a strong dependence on the pre-existing crack length. As the stress intensity factor is zero for the un-cracked medium (that is, $c = 0$), $\Delta T_c$ based on the fracture-based failure criterion will be infinity at $c = 0$. $\Delta T_c^{\text{fracture}}$ decreases monotonously with crack length. The failure of the medium will be dominated by the stress-based criterion for small crack length. A transient crack length $c_t$ exists for which $\Delta T_c$ is the same for the fracture-based criterion and the stress-based criterion. This length can be obtained by equalling equations (5.1) and (5.2) to give

$$c_t = 0.253 \left( \frac{K_c}{\sigma_b} \right)^2.$$ 

Figure 11 depicts the result of $\Delta T_c$ as a function of crack size. The thermal shock resistance of the medium with a pre-existing crack larger than $c_t$ will be dominated by the fracture-based criterion. If the length of the pre-existing crack is smaller than $c_t$, the thermal shock resistance of the medium will be dominated by the stress-based criterion. Thus, the real thermal shock resistance is solid line ABC in Figure 11. It should be mentioned that, although the crack growth trajectory for the DPL model is considerably different from the classical Fourier model, the thermal shock resistance predictions of the medium for the two models remain the same.

### 6. Conclusion

This research studied the surface fracture of a semi-infinite medium using the DPL model. The analytical solutions for the temperature field and the thermal stress without crack were evaluated...
by Laplace transform method. The inverse value of the thermal stress of the medium without crack was used as the crack face traction to obtain the transient thermal stress intensity factor through a numerical integration scheme. The following conclusions can be drawn:

(1) It was discovered that the DPL heat conduction model predicts a much larger heat-affected zone than the classical Fourier model. The thermal stress and thermal stress intensity levels based on the DHL model are much higher than those based on the classical Fourier model.

(2) Crack propagation analysis was conducted and comparison between the DHL model and the Fourier heat conduction law was conducted. The initial crack starts to propagate unstably for the DHL model much earlier than for the classical Fourier model for specified initial crack length and fracture toughness of the medium.

(3) Meanwhile, it was discovered that the crack length \( c_1 \) after unstable crack growth depends on the applied thermal shock temperature, the fracture toughness of the medium and the initial crack length \( c_0 \). \( c_1 \) predicted by the DHL model is much larger than that predicted by the classical Fourier heat conduction model.

(4) All results were presented in non-dimensional forms. Therefore, no specified material properties are required for the analysis. The model developed in this study can be applied to any thermoelastic materials.

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**References**


