Significance of the actual nonlinear slope geometry for catastrophic failure in submarine landslides

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A simple approach to slope stability analysis of naturally occurring, mild nonlinear slopes is proposed through extension of shear band propagation (SBP) theory. An initial weak zone appears in the steepest part of the slope where the combined action of gravity and seismic loads overcomes the degraded peak shear resistance of the soil. If the length of this steepest part is larger than the critical length, the shear band will propagate into the quasi-stable parts of the slope, where the gravitational and seismically induced shear stresses are smaller than the peak but larger than the residual shear strength of the soil. Growth of a shear band is strongly dependent on the shape of the slope, seismic parameters and the strength of soil and less dependent on the slope inclination and the sensitivity of clay. For the slope surface with faster changing inclination, the criterion is more sensitive to the changes of the parameters. Accounting for the actual nonlinear slope geometry eliminates the main challenge of the SBP approach—determination of the length of the initial weak zone, because the slope geometry can be readily obtained from submarine site investigations. It also helps to identify conditions for the early arrest of the shear band, before failure in the sliding layer or a change in loading or excess pore water pressures occurs. The difference in the size of a landslide predicted by limiting equilibrium and SBP approaches can reach orders of magnitude, potentially providing an explanation for the immense dimensions of many observed submarine landslides that may be caused by local factors acting over a limited portion of the slope.
1. Introduction

(a) Submarine landslides

Gigantic catastrophic underwater landslides are among the most challenging natural hazards due to difficulties in understanding their mechanisms, the problem of future prediction in environments where only indirect observations are possible, and because of their potentially devastating consequences. In 1998, an earthquake of magnitude 7 triggered a landslide off the northern coast of Papua, New Guinea, causing a 15 m high tsunami wave, which destroyed a number of villages and claimed more than 2000 lives [1,2]. Evidence of large (up to $3 \times 4$ km) shallow underwater landslides was found in the continental slope off the southern coast of Israel [3–6]. These are believed to have caused tsunami and were triggered by M 7+ earthquakes of 1202 and 1759 [7]. Even more impressive submarine slides (up to $10 \times 14$ km) were discovered off the shore of California near Santa Barbara [8] and are also believed to be the cause for tsunami associated with M 7+ earthquakes in 1812 [9]. The enormous Storegga slides, situated on the continental slope off the western coast of Norway, are among the largest and best-studied submarine landslides in the world [10–12]. The first slide occurred about 30,000–50,000 years before present involving a volume of 3880 km$^3$, the average thickness of 114 m and a run-out distance of 350–400 km from the headwall. According to the geological data, this landslide generated a tsunami wave with a height of 10–12 m on the coast of Norway; on the Shetland Islands, it was higher than 20 m [13]. Many other examples exist in the geological record and are often observed on industry-acquired geophysical data [14–16]. The common features of the above landslides are that they are very long (10$^1$–10$^2$ km), relatively shallow (less than 10$^2$ m thick) and occur in very mild continental slopes (inclination less than 10$^{-1}$) built of normally consolidated sediments.

(b) Shear band propagation approach to slope stability

Many factors have been proposed to cause underwater landslides, with longer acting conditioning factors ‘priming’ the slope and short-term, transient factors providing the final trigger (see [17] for a discussion). Conventional limiting equilibrium (LE) analysis assumes that failure takes place simultaneously along the entire sliding surface where the shear stress exceeds the peak shear strength. Thus, back-analysis of palaeo-landslides implies that conditions along the entire failure surface must have approached the point of failure at the same time. Consequently, there has been a tendency to focus on exotic causes of failure in the published literature (e.g. basin-wide methane-hydrate decomposition or the presence of extensive weak layers) with considerable uncertainty remaining in the correlation of landslide activity with globally acting driving variables [18].

By contrast, the shear band propagation (SBP) approach [19–21] provides criteria allowing an initial shear band (where the shear stress exceeds the peak shear strength) to propagate into portions of the sliding surface where the shear stress is lower than the peak shear strength but exceeds the residual strength. As such, it is feasible for an initial weak zone to grow, meaning that only a portion of the failure surface needs to reach the critical conditions for failure and that local, readily occurring factors can offer a simpler explanation to the enormous dimensions and common occurrence of submarine landslides in nature.

For the initial weak zone to propagate dynamically parallel to the slope surface, its length should exceed some critical length. The value of this critical length can be defined based on the static analysis using fracture mechanics or the energy balance (slip weakening) criteria [19–27]. The initial weak zone can be caused by various mechanisms, such as earthquake-triggered loading, shear strength degradation and liquefaction [6,28–32], methane hydrate decomposition [17,33–35], local fluid fluxes [21,36,37] and salt diapirism [38].

Once the shear band starts propagating catastrophically, additional strong dynamic weakening mechanisms, such as thermal pressurization of pore fluid (e.g. [39]) or thermal decomposition of
gouge material (e.g. [40]) can further lower the residual shear strength from its static level. These mechanisms have been long implicated in earthquake rupture on faults characterized by very small resolved shear stress (generally considered true for most major faults) and can also provide explanation for large submarine landslides on very shallow slopes.

(c) Challenges of application of the shear band propagation approach to slope stability analysis

So far the SBP approach has been mainly developed for a case of an infinite slope, an assumption which creates a significant limitation for the application to real-world landslides. Indeed, the main challenge for SBP application lies in the practical difficulties of determination of the length of the initial weak zone. For triggers such as methane hydrate decomposition and local fluid fluxes, it is at least conceptually clear that their effect is spatially limited [21,39]. Reliable and accurate quantification of the spatial scale of this effect is, however, extremely difficult. For seismic triggers, the stability analysis is much more complicated, because it is conceptually unclear why only a portion of an infinite slope would be weakened as a result of practically uniform seismic loading with soil properties barely changing in a plane parallel to the slope surface.

It seems that the only way to eliminate this contradiction is to account in the SBP approach for the true, variable, nonlinear slope geometry. Typically in submarine settings, slopes can be characterized by monotonically decreasing functions, with many slopes exhibiting an ‘S-shaped’ Gaussian profile [41], formed as a result, e.g. of tectonic regime, a drape covered scarp or a pile of debris from a previous slide. In all these cases, it is logical to assume that the initial weak zone would appear parallel to the slope in its steepest part, where the combined action of the gravity and seismic loads could overcome the degraded peak shear resistance of the soil. If the length of this steepest part is larger than the critical length, the shear band will propagate into the parts of the slope, where the gravitational and seismically induced shear stresses are smaller than the peak but larger than the residual shear strength of the soil. This extension automatically eliminates the main challenge of the SPB approach (i.e. determination of the length of the initial weak zone), because the slope geometry is, probably, the most reliable and readily available component of any submarine site investigation, with typical spatial resolutions of about 2 m readily being achieved in industry bathymetric surveys.

Another limitation of most of the existing SBP approaches is also due to the infinite slope assumption, which implies that once the shear band starts propagating, it will not arrest unless some external conditions change (e.g. reaching passive or active failure in the sliding layer, termination of the earthquake loading, dissipation of excess pore water pressures, etc.). This assumption may lead to significant overestimation of the final landslide dimensions. It appears that accounting for the true nonlinear slope geometry in the SBP approach can also help to overcome this limitation and identify conditions for the early arrest of the shear band before the occurrence of failure in the sliding layer, even if loading and pore water pressures remain constant during the SBP.

This paper (i) defines the shear stress ratio for a nonlinear slope geometry; (ii) formulates the energy balance (slip weakening) conditions for the SBP and arrest in a submarine slope with a nonlinear geometry, (iii) proposes a simple approach to slope stability analysis based on these criteria, and (iv) explores implications of this analysis for different slope shapes.

2. Shear stress ratio

(a) Slope zonation with respect to shear stress ratio

For a nonlinear slope geometry considered in figure 1 (a generalized S-shaped slope represents a broad range of possible monotonic slope shapes), the sliding surface is assumed to be located at the depth $h$ parallel to the slope surface. Strain softening shear behaviour of the soil is assumed on
the sliding surface (figure 2), with the undrained shear strength dropping from peak $\tau_p$ to residual $\tau_r$. Both strength values are assumed to be proportional to effective stresses. Within the sliding body, the soil behaviour is elasto-plastic (figure 3) with different deformation moduli ($E_1$ and $E_u$) and failure pressures ($p_p$ and $p_a$) in loading and unloading, respectively. Seismic slope stability analysis is performed using a pseudo-static approach with degradation of the shear strength as a result of the cyclic loading.

In the SBP approach, Palmer & Rice [19] defined the shear stress ratio for an infinite slope, which is extended here to account for the seismic loading, shear strength degradation and
background excess pore water pressures

\[ r = \frac{\tau_g + \tau_h - \tau_r}{\tau_p - \tau_r} = \frac{s(\tau_g + \tau_h)/\tau_p - 1}{s - 1} \]  

(2.1)

and allows for identification of unstable \((1 \leq r)\) slopes, quasi-stable \((1 < r \leq 0)\) slopes (where the SBP can take place under certain conditions) and stable \((r < 0)\) slopes. Here,

\[ s = \frac{\tau_p}{\tau_r} \]  

(2.2)

is the ratio between the peak and residual shear strength, referred to here as sensitivity;

\[ \tau_g = \gamma' h \sin \alpha \]  

(2.3)

is the gravitational shear stress;

\[ \tau_h = k_h \gamma' h \cos \alpha = C a_{\text{max}} \gamma' h \cos \alpha \]  

(2.4)

is the pseudo-static earthquake-induced shear stress, where the factor \(C\) is recommended for earth dams to be in a range of 0.5 [42]; for submarine slopes, Dimmock et al. [43] treated \(C\) as a probabilistic value with a standard deviation of 0.08 about a mean of 0.23 (specific values of \(C\) depend on the acceptable displacement of the slope);

\[ \tau_p = \delta_d k_s' = \delta_d k y'(1 - r_u) h \cos \alpha \]  

(2.5)

is the peak undrained shear strength, where \(k = 0.20–0.30\) is the undrained shear strength coefficient for normally consolidated sediments, \(\delta_d\) is the seismic shear strength degradation index and \(r_u = u_e/\gamma' h\) is the normalized excess pore water pressure at the depth \(h\). It is assumed that the excess pore water pressure grows linearly with depth, \(r_u = \text{const.}\), causing flow normal to the slope surface, which is a more conservative assumption than the upward flow). Equation (2.1) can be then rewritten as

\[ r = \frac{s \chi (\tan \alpha + k_h) - 1}{s - 1}, \quad \text{where} \quad \chi = \frac{1}{\delta_d k (1 - r_u)}. \]  

(2.6)

Two earthquake loading factors in equation (2.6) are \(\delta_d\), the degradation index, and \(k_s\), the pseudo-static coefficient. They can be related to the earthquake parameters: \(a_{\text{max}}\), the peak horizontal ground acceleration, and \(M\), the earthquake magnitude [44].

For a nonlinear slope geometry, the main difference with the conventional SBP approach is that the slope angle changes continuously and the shear stress ratio becomes a function of the coordinate \(x\)

\[ r(x) = \frac{s \chi (f'(x) + k_h) - 1}{s - 1}, \]  

(2.7)

where \(z = f(x)\) is the equation of the slope surface in figure 1.

The shear stress ratio (equation (2.7)) is a critical parameter for the SBP stability analysis of nonlinear submarine slopes. In contrast to the infinite slope case, the unstable \((1 \leq r)\), quasi-stable \((0 < r < 1)\) and stable \((r < 0)\) zones can be identified within the single nonlinear slope (figure 4).

It can be assumed that earthquake loading causes the gravitational and seismic driving forces to overcome the degraded peak shear strength at all the points of the sliding surface where the inclination of the surface exceeds \(\alpha_i\) (figure 1). This results in the shear stress ratio \(r \geq 1\) and formation of the initial shear band of length \(L_i\), where the shear strength drops to its residual value \(\tau_r\). This paper formulates conditions for catastrophic propagation of the initial shear band into the quasi-stable zone of the length \(L_q\) and \(0 \leq r < 1\), and subsequent arrest of the shear band within the stable zone of \(r < 0\).

(b) Shear band propagation in an infinite slope

In an infinite slope \((r = \text{const.})\), Puzrin & Germanovich [20] showed that the shear band propagates catastrophically (i.e. under existing external forces) once its length exceeds the critical
value of

$$L_i \geq L_{cr} = \frac{1}{r} \left( 1 + \frac{E_l}{E_u} \right) \sqrt{\frac{sX \cos \alpha}{s-1}} \frac{2E_u\delta}{\gamma'}, \quad (2.8)$$

where

$$\delta = \frac{\int_0^\delta (\tau - \tau_r) d\delta}{\tau_p - \tau_r}, \quad (2.9)$$

and $\delta_r$ is the relative displacement within the shear band at which the residual strength is reached (figure 2); $E_l$ and $E_u$ are deformation plane strain moduli in loading and unloading, respectively (figure 3). The following two sections explore the case of a nonlinear slope with a variable shear stress ratio $r \neq \text{const.}$

3. Mechanics of the shear band propagation in a nonlinear slope

The SBP approach treats slope failure as an evolution of growing slip surfaces and provides a rational quantitative explanation of the mechanisms associated with enormous catastrophic landslide failures. Submarine landslides are naturally evolving processes, going through the stages of pre-failure conditioning, followed by local failure triggering, resulting in shear band initiation and propagation, which in turn cause the global slope failure leading to a post-failure evolution. In this paper, we consider the stage of the initiation of the catastrophic SBP, with the sliding layer above the shear band yielding (with different moduli under loading/unloading) but not yet failing. Subsequent stages—dynamic SBP leading to the global failure—have been treated in [45].

(a) Equilibrium equations

Consider the problem of the SBP at the depth $h$ parallel to the slope surface in a curved slope (figure 5). The principal normal stresses $\sigma_\alpha$, assumed to be acting in the sliding layer on the planes normal to the sliding surface (figure 6), will be taken as average values of these stresses over the thickness $h$ of the layer:

$$\sigma_\alpha(x) = \frac{1}{h} \int_{R-h}^R \sigma_\alpha(x, r) dr. \quad (3.1)$$

The origin $x = 0$ is chosen at the section of the sliding layer where the principal normal stresses have not changed as a result of the shear band formation and remained equal to the initial lateral earth pressure $\sigma_\alpha(0) = p_0$. 

**Figure 4.** Slope zonation with respect to the shear stress ratio (courtesy of David Rushton, Fugro).
Figure 5. SBP in a curved slope. (Online version in colour.)

Figure 6. Equilibrium in the sliding layer. (Online version in colour.)

Consider an incremental element $dl \times h$ of the sliding layer (figure 6), limited by the two radii of the slope curvature $R$ and $R + dR$, where

$$R(x) = \frac{(1 + f'^2)^{3/2}}{|f''|}.$$  \hfill (3.2)

Equilibrium of moments acting on the element with respect to the point $O$ of the intersection between the two radii can be expressed (neglecting second-order terms) as

$$d\sigma_a h \left( R - \frac{h}{2} \right) + \sigma_a h dR = -TR + (W \sin \alpha + k_W W \cos \alpha) \left( R - \frac{h}{2} \right),$$  \hfill (3.3)

where

$$T = \tau_r dl, \quad W = \gamma' h dl \quad \text{and} \quad d\sigma_a = \frac{\partial \sigma_a}{\partial l} dl.$$  \hfill (3.4)
are the shear resistance on the sliding surface; the effective weight of the element and the increment of the average normal stress, respectively.

Submarine slopes are relatively mild, long and shallow, with the curvature radii being much larger than the thickness of the sliding layer and the slope inclination and curvature changing very slowly \((R \gg h, R' \ll 1)\), so that

\[ |f'(x)| \ll 1, \quad |f''(x)| \ll h^{-1} \quad \text{and} \quad |f'''(x)| \ll h^{-2}, \quad (3.5) \]

and equations (3.2)–(3.5) can be resolved to produce the differential equation for the average normal stress \(\sigma_\alpha\):

\[ \frac{\partial \sigma_\alpha}{\partial l} = \frac{\tau_r h}{h} - \gamma'(\tan \alpha + k_h) \cos \alpha. \quad (3.6) \]

Substituting \(\tan \alpha(x) = f'(x)\), where \(z = f(x)\) is the equation of the slope surface, together with \(dx = dl \cos \alpha\) and equations (2.2) and (2.5) into equation (3.6), the differential equation of equilibrium is given by

\[ \frac{\partial \sigma_\alpha}{\partial x} = \gamma' \left( \frac{1}{s\chi} - k_h \right) - \gamma' f'(x); \quad \chi = \frac{1}{\delta qk(1 - ru)}. \quad (3.7) \]

Upon integration together with the boundary conditions at the origin \(x = 0\) taken at the section of the sliding layer where the principal normal stresses have not changed as a result of the shear band formation

\[ f(0) = 0 \quad \text{and} \quad \sigma_\alpha(0) = p_0. \quad (3.8) \]

Equation (3.7) provides the distribution of the average normal stress \(\sigma_\alpha(x)\) along the curved slope during the earthquake:

\[ \sigma_\alpha(x) = p_0 + \gamma' \left( \frac{1}{s\chi} - k_h \right) x - \gamma' f(x). \quad (3.9) \]

(b) Constitutive equations

Assuming that in an intact slope, the average normal stress would be equal to \(\sigma_\alpha = p_0 = \text{const.}\), the elasto-plastic constitutive relationship between the average normal stress \(\sigma_\alpha\) in the layer along the shear band and the average linear strain \(\varepsilon_\alpha\) parallel to the slope is given by the linear elastic perfectly plastic formula (figure 3):

\[ \sigma_\alpha = E_u \varepsilon_\alpha + p_0, \quad \text{for} \ p_0 \leq \sigma_\alpha \leq p_u \quad (3.10) \]

and

\[ \sigma_\alpha = E_l \varepsilon_\alpha + p_0, \quad \text{for} \ p_0 \leq \sigma_\alpha \leq p_p, \quad (3.11) \]

where \(E_l\) and \(E_u\) are the loading and unloading plane strain moduli, respectively. These constitutive equations, when combined with equation (3.9), produce distribution of the average linear strain in the sliding layer along the slope above the shear band (figure 7):

\[ \varepsilon_\alpha(x) = \frac{\gamma'}{E_u} \left[ \left( \frac{1}{s\chi} - k_h \right) x - f(x) \right], \quad \text{for} \ x \geq 0 \quad (3.12) \]

and

\[ \varepsilon_\alpha(x) = \frac{\gamma'}{E_l} \left[ \left( \frac{1}{s\chi} - k_h \right) x - f(x) \right], \quad \text{for} \ x \leq 0. \quad (3.13) \]

(c) Geometry and boundary conditions

For mild submarine slopes satisfying equations (3.5), with the sliding layer subjected to undrained constant volume deformation, average linear strains parallel to the slope (figure 6) can be
Figure 7. Distribution of linear strains in the sliding layer.

approximated as
\[ \varepsilon_\alpha = \frac{1}{R} \frac{\partial u_\alpha}{\partial \alpha} + \frac{u_r}{R} \approx \frac{\partial u_\alpha}{\partial l}, \]  
so that the tangential displacements in the sliding layer are given by
\[ u_\alpha(x) = \int_{l(x_1)}^{l(x)} \varepsilon_\alpha(l) dl \quad \text{and} \quad l(x) = \int_0^x (1 + f'^2)^{1/2} dx. \] 

Boundary conditions at the ends of the shear band prescribe zero displacements
\[ u_\alpha(x_1) = u_\alpha(x_2) = 0, \]  
where the first condition is satisfied by equation (3.16) automatically, while the second one can be approximated as
\[ u_\alpha(x_2) = \int_{l(x_1)}^{l(x)} \varepsilon_\alpha(l) dl \approx \frac{1}{2} [\varepsilon_\alpha(x_2) l_2 - \varepsilon_\alpha(x_1) l_1] = 0, \]  
leading to the following relationship (figure 6):
\[ \frac{\varepsilon_\alpha(x_1)}{\varepsilon_\alpha(x_2)} \approx \frac{l_2}{l_1} \approx \frac{x_2}{x_1}, \]  
which can significantly simplify solution of the SBP problem. For mild slopes satisfying conditions (3.5), approximation (3.17) is rather accurate, which can be demonstrated by using expressions (3.12) and (3.13) and the fact that at the ends of the initial shear band
\[ r(x_1) = r(x_2) = 1. \]

For the case of the straight infinite slope, equality (3.18) becomes exact.

Displacements below the shear band are neglected; therefore, tangential displacement \( u_\alpha(x) \) also represents the relative slide \( \delta(x) \) in the shear band parallel to the sliding surface.

(d) Incremental propagation of the shear band

The energy balance criterion for SBP investigates the changes produced in the body by the shear band propagating by increment \( \Delta L = dx_1 + dx_2 \). As a result of this incremental propagation, the sliding layer experiences deformations at both ends of the shear band leading to the displacement
parallel to the slope
\[ \Delta u_\alpha = \varepsilon_\alpha(x_1)dl_1 = -\varepsilon_\alpha(x_2)dl_2. \] (3.20)

This displacement is assumed to be constant along the slope, while the horizontal and vertical components are dependent on the local slope inclination
\[ u_x(x) = \Delta u_\alpha \cos \alpha(x) \] (3.21)
and
\[ u_z(x) = u_x(x) \tan \alpha(x) = \Delta u_\alpha \tan \alpha(x) \] (3.22).

4. Energy balance criterion for a catastrophic shear band propagation

(a) Energy balance (slip weakening) approach

The SBP criterion for the problem in figure 5 will be derived directly from the energy balance (slip weakening) approach, following [19]. The main assumptions made in this analysis are that under condition of \( L_i \gg h \) and in the absence of dilation in the shear band, most of the energy transfer during SBP will take place due to

- external work made by gravitational forces on downslope movements of the layer above the shear band;
- internal work made by the normal stress acting parallel to the slope surface on deformations of the layer caused by changes in these stresses; and
- plastic work dissipated on the shear band.

We shall consider the case, where the end zone of the shear band is small compared with the thickness of the sliding layer. The energy balance criterion for the SBP requires that the energy surplus produced in the body by incremental propagation of the shear band \( \Delta L = dx_1 + dx_2 \) should exceed the work required for this incremental propagation. Mathematically, the energy balance criterion can be expressed as the following inequality:
\[ \Delta W_e - \Delta W_i - \Delta D_L \geq \Delta D_{\alpha r}, \] (4.1)
where the corresponding energy components are defined and calculated below.

The increment of the external work done in our case by gravitational and seismic forces on downslope movements of the sliding layer is given by
\[ \Delta W_e = \int_{x_1}^{x_2} (\gamma'hu_z(x) + k_h\gamma'hu_x(x))dl. \] (4.2)

Using equations \( dx = dl \cos \alpha \), and (3.21) and (3.22), the increment of the external work (4.2) is calculated as
\[ \Delta W_e = ((f(x_2) - f(x_1)) + k_h(x_2 - x_1))\gamma'hu_\alpha. \] (4.3)

The internal work is assumed to be done by the average principal stresses on the average principal strains in the layer caused by changes in these stresses. This assumption results in the following expression for the increment of internal work:
\[ \Delta W_i = hdl_1 \int_0^{\varepsilon_\alpha(x_1)} \sigma_\alpha(\varepsilon_\alpha) d\varepsilon_\alpha + hdl_2 \int_0^{\varepsilon_\alpha(x_2)} \sigma_\alpha(\varepsilon_\alpha) d\varepsilon_\alpha. \] (4.4)

Substituting constitutive relationships (3.10) and (3.11) and making use of condition (3.20), the increment of the internal work becomes
\[ \Delta W_i = \frac{1}{2} h(E_l \varepsilon_\alpha(x_1) - E_u \varepsilon_\alpha(x_2)) \Delta u_\alpha. \] (4.5)
with substitution of expressions for strains (3.12) and (3.13) producing
\[
\Delta W_i = \frac{1}{2} \left( (f(x_2) - f(x_1)) + \left( k_h - \frac{1}{s\chi} \right) (x_2 - x_1) \right) y' h \Delta u_\alpha. \tag{4.6}
\]

Next, the increment of the plastic work dissipated on the shear band, which is required to overcome the residual shear resistance along the band is defined as
\[
\Delta D_L = \int_{x_1}^{x_2} \tau_\alpha \Delta u_\alpha \, dl, \tag{4.7}
\]
which after using \( dx = dl \cos \alpha \) together with equations (2.2) and (2.5) reduces to
\[
\Delta D_L = (x_2 - x_1) \frac{y' h \Delta u_\alpha}{s\chi}. \tag{4.8}
\]

Finally, the increment of the plastic work dissipated in the shear band during its propagation, which is required to overcome the shear resistance in excess of residual in the end zones of the band, is given by
\[
\Delta D_\omega \approx dl_1 \int_0^{\delta_1} (\tau(\delta) - \tau_r) d\delta + dl_2 \int_0^{\delta_2} (\tau(\delta) - \tau_r) d\delta, \tag{4.9}
\]
which after substitution of expression (2.9) becomes
\[
\Delta D_\omega \approx (dl_1 + dl_2)(\tau_p - \tau_r) \delta \tag{4.10}
\]
and after using equations (2.2) and (2.5) reduces to
\[
\Delta D_\omega \approx (dl_1 + dl_2)(s - 1) \frac{y' h \cos \alpha}{s\chi} \delta. \tag{4.11}
\]
Expressions (4.3), (4.6), (4.7) and (4.12), after being substituted into inequality (4.1), yield the sufficient SBP condition
\[
\frac{1}{2} \left( (f(x_2) - f(x_1)) + \left( k_h - \frac{1}{s\chi} \right) (x_2 - x_1) \right) y' h \Delta u_\alpha \geq (dl_1 + dl_2)(s - 1) \frac{y' h \cos \alpha}{s\chi} \delta. \tag{4.12}
\]

(b) Energy balance shear band propagation criterion for a nonlinear slope

Introducing expression for the average shear stress ratio
\[
\bar{\tau} = \frac{s\chi ((f(x_2) - f(x_1))/(x_2 - x_1)) + k_h}{s - 1}, \tag{4.13}
\]
the energy balance criterion (4.12) can be transformed into a more compact form
\[
\bar{\tau} \Delta u_\alpha (x_2 - x_1) \geq 2(dl_1 + dl_2) \delta \cos \alpha, \tag{4.14}
\]
Linear strains at the tips of the shear band are given by equations (3.12) and (3.13)
\[
\varepsilon_\alpha(x_1) = -\frac{y'}{E_1} \frac{s - 1}{s\chi} \bar{r}_1 x_1 \quad \text{and} \quad \varepsilon_\alpha(x_2) = -\frac{y'}{E_u} \frac{s - 1}{s\chi} \bar{r}_2 x_2, \tag{4.15}
\]
where
\[
\bar{r}_1 = \frac{\delta\chi (f(x_1)/x_1) + k_h}{s - 1} \quad \text{and} \quad \bar{r}_2 = \frac{s\chi (f(x_2)/x_2) + k_h}{s - 1}. \tag{4.16}
\]
From equation (3.20) we obtain
\[
\Delta u_\alpha = -\varepsilon_\alpha(x_2) dl_2 = \frac{s - 1}{s\chi} \frac{y'}{E_u} \bar{r}_2 x_2 dl_2. \tag{4.17}
\]
Using equation (4.17), the energy balance criterion (4.14) transforms
\[
\bar{r}_2 (x_2 - x_1)^2 \geq \left( 1 - \frac{x_1}{x_2} \right) \left( 1 + \frac{dl_1}{dl_2} \right) \frac{s\chi \cos \alpha}{s - 1} \frac{2E_u \delta}{y'}. \tag{4.18}
\]
Next, from equations (4.15) we obtain

$$\frac{\varepsilon_a(x_1)}{\varepsilon_a(x_2)} = \frac{E_u \tilde{r}_1 x_1}{E_1 \tilde{r}_2 x_2},$$

(4.19)

Which in combination with equations (3.18) and (3.19) results in

$$\frac{d l_1}{d l_2} = \frac{x_1}{x_2} = \sqrt{\frac{E_1 \tilde{r}_2}{E_u \tilde{r}_1}},$$

(4.20)

and after substitution into (4.18) gives the energy balance criterion for the SBP in a curved slope:

$$L_i = x_2 - x_1 \geq L_{cr} = \frac{1}{\sqrt{r_2}} \left( 1 + \sqrt{\frac{E_1 \tilde{r}_2}{E_u \tilde{r}_1}} \right) \sqrt{\frac{s \chi \cos \alpha_1 2E_u \delta}{s - 1 - \gamma'}}.$$

(4.21)

### (c) Simplified energy balance criterion

For an infinite slope

$$\tilde{r}_1 = \tilde{r}_2 = \bar{r},$$

(4.22)

and the energy balance criterion (4.21) degenerates into the Puzrin & Germanovich [20] energy balance criterion for the SBP in an infinite slope (2.8). Moreover, the energy balance criterion for an infinite slope (2.8) can be used as a simplified conservative estimate of the critical length of the shear band in a curved slope by adopting the average value for the shear stress ratio \( r = \bar{r}. \) Indeed, from expressions (4.13) and (4.16), we obtain

$$\bar{r} = \frac{x_2 \tilde{r}_2 - x_1 \tilde{r}_1}{x_2 - x_1},$$

(4.23)

which can be used together with expression (4.20) to prove the following inequality:

$$\frac{1}{\sqrt{r_2}} \left( 1 + \sqrt{\frac{E_1 \tilde{r}_2}{E_u \tilde{r}_1}} \right) = \frac{1}{r} \left( 1 + \sqrt{\frac{E_1 \tilde{r}_1}{E_u \tilde{r}_2}} \right) \left( 1 + \sqrt{\frac{E_1 \tilde{r}_2}{E_u \tilde{r}_1}} \right) \geq \frac{1}{r} \left( 1 + \sqrt{\frac{E_l}{E_u}} \right),$$

(4.24)

which after substitution into equation (4.21) produces an estimate for the critical length of the shear band in a curved slope

$$L_i \geq L_{cr} = \frac{1}{r} \left( 1 + \sqrt{\frac{E_1}{E_u}} \right) \sqrt{\frac{s \chi \cos \alpha_1 2E_u \delta}{s - 1 - \gamma'}}.$$

(4.25)

and

$$\bar{r} = \frac{s \chi (f(x_2) - f(x_1)/(x_2 - x_1)) + k_h) - 1}{s - 1}.$$

(4.26)

In elastic soil (with \( E_u = E_1 \)), estimate (4.25) is identical to the exact solution for the critical length for an arbitrary skew-symmetric S-shaped slope

$$f^+(x) = -f^-(x),$$

(4.27)

where \( f^-(x) \) and \( f^+(x) \) are the slope surface functions for negative and positive values of \( x \), respectively. Indeed, in this case, condition (4.22) is satisfied automatically, leading to the equality in expression (4.24).

In elastic–plastic soil (with \( E_u \neq E_1 \)), estimate (4.25) is identical to the exact solution for the critical length only for the special S-shaped slope geometry

$$f^+(x) = -\sqrt{\frac{E_u}{E_1}} f^-(x),$$

(4.28)

with condition (4.22) again satisfied automatically, leading to equality in expression (4.24).
For elastic–plastic soil behaviour and a general slope shape, the simplified energy balance criterion (4.25), based on averaging the slope of the initial shear band (4.26), represents a reasonably conservative estimate for the critical length of the shear band. A considerable advantage of the simplified criterion over the general criterion (4.21) is that the former does not depend on the position of the origin of coordinates and can be evaluated simply by calculating the slope between the ends of the initial shear band, regardless of the location of these ends with respect to the origin.

5. Simple shear band propagation approach to stability analysis of nonlinear slopes

Landslide risk analysis of large underwater basins requires determination of probabilities of failure for multiple seabed locations. These are derived by means of probabilistic slope stability analysis applying stochastic procedures to GIS distributions of deterministic safety factors calculated using simple slope stability models (e.g. [43]). Such models consider the LE of an infinite slope, where the slope failure takes place simultaneously along the entire slope length once the driving forces exceed the peak shear strength, and do not account for progressive and catastrophic failure. This section proposes an alternative simple approach to calculating deterministic safety factors for seismically loaded submarine slopes with a nonlinear geometry, based on the slope zonation with respect to the shear stress ratio and the energy balance (slip weakening) conditions for the SBP defined in the previous sections.

(a) Limiting equilibrium approach

Consider a non-symmetric curved slope surface (figure 1), with the origin of the coordinate system is chosen at the point of the zero curvature with the maximum inclination angle $\alpha_0$. The following two slope inclinations can be defined:

$$\tan \alpha_i = \frac{1}{\chi} - k_h$$

(5.1)

and

$$\tan \alpha_q = \frac{1}{s\chi} - k_h,$$

(5.2)

corresponding to the unity and zero values of the gravitational shear stress ratio $r$ in equation (2.6), respectively.

In order to derive local safety factors for a particular location on the sliding surface of a mild, shallow, long nonlinear submarine slope, the practical LE approach, e.g. [43], treats this location as a part of an infinite slope with the same inclination. Within the LE approach, the failure takes place simultaneously along the entire length of the slope once the driving forces exceed the peak shear strength. Hence, for the slope in figure 1, this approach would produce the following local safety factors with respect to the peak strength:

— for the slope inclination $f'(x) \geq \tan \alpha_i$, from equations (2.3)–(2.7) and (5.1),

$$FS(x) = \frac{\tau_p}{\tau_g + \tau_h} = \frac{1}{\chi f'(x) + k_h} \leq 1,$$

(5.3)

i.e. the slope in the interval $x_i - L_i \leq x \leq x_i$ in figure 1 is unstable;

— for the slope inclination $f'(x) < \tan \alpha_i$

$$FS(x) = \frac{\tau_p}{\tau_g + \tau_h} = \frac{1}{\chi f'(x) + k_h} > 1,$$

(5.4)

i.e. the slope outside of the interval $x_i - L_i \leq x \leq x_i$ in figure 1 is stable.
As a result of this analysis, it can be concluded that a slab failure may occur over the length $L_i$ (figure 8). Outside this zone, the slope will remain stable.

(b) Shear band propagation approach

In contrast to the LE, the SBP approach allows for progressive and catastrophic failure in a slope. This implies that if conditions for such failure exist, the peak strength will be mobilized only locally and temporarily, and then softened by external forces. Therefore, for the failure to take place, it is sufficient that the local driving forces exceed the residual (and not the peak) strength.

Similar to the previous section, each location of the nonlinear slope will be considered as a part of an infinite slope with the same inclination. For the slope inclination $f'(x) \geq \tan \alpha_i$, from equations (2.7) and (5.1) it follows that

$$r(x) = \frac{s \chi (f'(x) + k_h) - 1}{s - 1} \geq 1,$$

that is in the interval $x_i - L_i \leq x \leq x_i$ in figure 1 progressive failure can take place as a result of increasing seismic loading, and the slope is unstable with the local safety factor calculated with respect to the residual strength:

$$FS_i(x) = \frac{r_f}{r_g + r_h} = \frac{1}{s_x f'(x) + k_h} \leq 1.$$

For the slope inclination $\tan \alpha_q \leq f'(x) < \tan \alpha_i$,

$$0 \leq r(x) = \frac{s \chi (f'(x) + k_h) - 1}{s - 1} < 1,$$

that is the slope in the intervals $x_q - L_q \leq x < x_i - L_i$ and $x_i < x \leq x_q$ in figure 1 is quasi-stable and the length of the initial failure zone $L_i$ should be compared with the critical length (4.25) using the corresponding average gravitational shear stress ratio:

$$\bar{r} = \frac{s \chi ((H_i/L_i) + k_h) - 1}{s - 1} \geq \frac{s \chi (\tan \alpha_i + k_h) - 1}{s - 1} = 1.$$

If $L_i < L_{cr}$, no catastrophic failure can take place in the intervals $x_q - L_q \leq x < x_i - L_i$ and $x_i < x \leq x_q$, and the slope is stable with the local safety factor calculated with respect to the peak strength

$$FS_q(x) = \frac{r_p}{r_g + r_h} = \frac{1}{s \chi f'(x) + k_h} > 1.$$
If \( L_i \geq L_{cr} \), catastrophic failure in the intervals \( x q - L_q \leq x < x_i - L_i \) and \( x_i < x \leq x_q \) results in the safety factor calculated with respect to the residual strength:

\[
FS_q(x) = \frac{\tau_r}{\tau_g + \tau_h} = \frac{1}{s \chi f'(x) + k_h} \leq 1. \tag{5.10}
\]

For the slope inclination \( f'(x) < \tan \alpha_q \),

\[
r(x) = \frac{s \chi (f'(x) + k_h) - 1}{s - 1} < 0, \tag{5.11}
\]

i.e. the slope in the intervals \( x < x_q - L_q \) and \( x_q < x \) in Figure 1 is stable with the safety factor calculated with respect to the peak strength:

\[
FS_g(x) = \frac{\tau_p}{\tau_g + \tau_h} = \frac{1}{\chi f'(x) + k_h} > 1. \tag{5.12}
\]

(c) Propagation and arrest of failure

In the SBP approach, the safety factors for the initial failure zone \( x_i - L_i \leq x \leq x_i \) of the slope \( f'(x) \geq \tan \alpha_q \) are calculated using the degraded residual rather than the peak strength, because of the progressive nature of the soil failure during the initial shear band formation due to increasing seismic loading. Both LE and SBP approaches identify this zone as unstable, but the SBP approach gives lower safety factors.

An even more dramatic difference to the conventional LE approach can be expected in the stability analysis of slopes with inclinations in the range of \( \tan \alpha_q \leq f'(x) < \tan \alpha_i \), outside the initial shear zone. In contrast to the LE approach, which considers all the slopes outside the initial shear zone as stable, in the SBP approach the zone \( x_q - L_q \leq x < x_i - L_i \) and \( x_i < x \leq x_q \) become quasi-stable (Figure 7), because if the condition \( L_q \geq L_{cr} \) is satisfied, the initial shear band will start propagating into it. The important question to answer is: how far will the shear band propagate into the quasi-stable zone?

Indeed, with the growth of the shear band length \( L \), the average gravitational shear stress ratio \( \bar{r} \) in equation (4.26) decreases, so that the critical length \( L_{cr} \) in equation (4.25) increases and, in principle, exceed the current length of the shear band. In reality, however, if the shear band starts propagating, it will always reach the boundaries of the quasi-stable zone, as long as the average gravitational shear stress ratio \( \bar{r} \) remains positive (always true within the quasi-stable zone, where any average slope is larger than \( \tan \alpha_q \)). This can be easily shown by applying propagation criteria (4.25) and (4.26) to the shear band of the length \( L \), propagating within the quasi-stable zone with slope \( \tan \alpha_q \leq f'(x) < \tan \alpha_i \) (orange line in Figure 9):

\[
L \bar{r} \geq L_{icr} = \left( 1 + \frac{E_1}{E_u} \right) \sqrt{\frac{s \chi}{s - 1} \frac{2 E_u \delta}{\gamma'}}. \tag{5.13}
\]

It follows, that for SBP to begin, the following sufficient condition has to be satisfied for the initial failure zone:

\[
L_i \bar{r}_i \geq L_{icr}. \tag{5.14}
\]

The condition for the shear band to reach the boundaries of the quasi-stable zone is

\[
L_q \bar{r}_q \geq L_{icr}, \tag{5.15}
\]

which would be satisfied automatically if

\[
L_q \bar{r}_q \geq L_i \bar{r}_i. \tag{5.16}
\]

Combining inequality (5.16) with equations (4.26) and (5.2), we obtain a sufficient condition for SBP through the entire quasi-stable zone

\[
\frac{H_q - H_i}{L_q - L_i} \geq \frac{1}{s \chi} - k_h = \tan \alpha_q. \tag{5.17}
\]
Figure 9. Propagating shear band (orange) and extent of failure (yellow) predicted by the SBP approach.

Figure 10. Geometric interpretation of the sufficient condition for the SBP in the quasi-stable zone.

Geometric interpretation of condition (5.17) is presented in figure 10, where the initial failure zone section $L_i$ in figure 1 has been removed from the slope surface and remaining slope sections connected together. It can be observed that inequality (5.17) is always satisfied for any monotonically increasing function $f(x)$, and the SBP will not arrest until the end of the quasi-stable zone is reached. This is, of course, assuming that pseudo-static acceleration, excess pore water pressures and degradation index do not change during the SBP. Because the seismic part of the loading is short-lived, a premature arrest of the SBP could take place during or immediately after the earthquake.

The final question to answer is whether the shear band will propagate further, into the stable zone (blue line in figure 9). Indeed, it is likely to be that the condition (5.13) will be still satisfied within a certain area outside of the quasi-stable zone. The boundary of this area can be determined by solving equation (5.13) with respect to $L$.

This analysis does not, however, account for the dynamic effects of the SBP, which suggest that inertia and wave propagation could extend the final shear band length beyond its static estimates. Analytical solution of the problem of dynamic SBP [45] suggests the shear band arrest criterion of $\bar{r} = 0$. This is consistent with the upper bound static estimate (5.13), according to which, for $\bar{r} = 0$, no finite length of the shear band could be sufficient for its further propagation.

To summarize, in the SBP approach, the failure is not limited to the slab failure over the length $L_i$ with the slope inclination $f'(x) \geq \tan \alpha_i$, predicted by the LE. If the conditions are right, the shear band will propagate, extending the failure into the quasi-stable zone over the length $L_q$ with the slope inclination $f'(x) \geq \tan \alpha_q$, in which case the safety factor for the quasi-stable zone should be calculated using the degraded residual strength and not the peak strength. The length $L_q$ can be considered as a lower bound for the final shear band length $L_f$, with the upper bound determined from equation $\bar{r}(L_f) = 0$. 
6. Effects of the slope shape

In order to illustrate the potential effects of the slope shape on the results of the proposed stability analysis, consider two particular types of analytical functions (hyperbolic and exponential) to approximate skew-symmetric S-shaped slope surfaces (figure 11). Both functions are described by two parameters, inclination at zero \( \tan \alpha_0 \) and the final value at infinity \( H \), but are rather different in terms of the rate of arriving to this final value, so that a broad range of possible S-shaped slope geometries are bounded by these two functions.

(a) Exponential slope surface

Consider an exponential slope surface:

\[
z = f(x) = \begin{cases} 
-H \left(1 - \exp \left(\frac{x}{H} \tan \alpha_0\right)\right) & \text{for } x < 0 \\
H \left(1 - \exp \left(-\frac{x}{H} \tan \alpha_0\right)\right) & \text{for } x \geq 0.
\end{cases}
\]  

(6.1)

The boundaries of the initial and quasi-stable zones can be obtained using equations (5.1) and (5.2.), respectively,

\[f'(x_i) = \tan \alpha_0 \cdot \exp \left(\frac{-x_i}{H} \tan \alpha_0\right) = \tan \alpha_i = \frac{1}{\chi} - k_h \]  

(6.2)

and

\[f'(x_q) = \tan \alpha_0 \cdot \exp \left(\frac{-x_q}{H} \tan \alpha_0\right) = \tan \alpha_q = \frac{1}{s\chi} - k_h. \]  

(6.3)

Then the lengths of the initial and quasi-stable zones are given by

\[L_i = 2x_i = \frac{2H}{\tan \alpha_0} \ln \frac{\chi \tan \alpha_0}{1 - \chi k_h} \]  

(6.4)

and

\[L_q = 2x_q = \frac{2H}{\tan \alpha_0} \ln \frac{\chi \tan \alpha_0}{1/s - \chi k_h}. \]  

(6.5)
The average slope and gravitational shear ratio for the initial failure zone are given by
\[
\tan \bar{\alpha} = \frac{2f(x_i)}{L_i} = \frac{2H (1 - \exp \left(\frac{x_i}{H} \tan \alpha_0\right))}{L_i} = \frac{2H}{L_i} \left(1 - \frac{1 - \chi k_h}{\chi \tan \alpha_0}\right) \tag{6.6}
\]
and
\[
\bar{r} = \frac{s}{s-1} \left(\frac{\chi \tan \alpha_0 - 1 + \chi k_h}{\ln((\chi \tan \alpha_0)/(1 - \chi k_h))} + \chi k_h\right) - \frac{1}{s-1}. \tag{6.7}
\]

Using equation (6.13), propagation criterion \(L_i \bar{r} \geq L_{icr}\) in this case can be expressed as
\[
\frac{L_i}{L_{cr}} = \frac{2Hs \chi}{L_{cr} (s-1)} \left(1 - \frac{1 - \chi k_h}{\chi \tan \alpha_0} + \frac{1 - s \chi k_h}{s \chi \tan \alpha_0} \ln \left(\frac{1 - \chi k_h}{\chi \tan \alpha_0}\right)\right) \geq 1. \tag{6.8}
\]

The LE approach produces the following safety factors along the slope:
\[
FS(x) = \frac{1}{\chi f'(x) + k_h} = \frac{1}{\chi \tan \alpha_0 \cdot \exp(-|x|/H \cdot \tan \alpha_0) + \chi k_h}. \tag{6.9}
\]

By contrast, the SBP approach gives for the slope in the initial failure zone in the interval \(x_i - L_i \leq x \leq x_i\) in figure 1:
\[
FS_i(x) = \frac{1}{s \chi f'(x) + k_h} = \frac{1}{s \chi \tan \alpha_0 \cdot \exp(-|x|/H \cdot \tan \alpha_0) + s \chi k_h}. \tag{6.10}
\]

If \(L_i \leq L_{cr}\), the slope in the quasi-stable zone in the intervals \(x_q - L_q \leq x < x_i - L_i\) and \(x_i < x \leq x_q\) in figure 1 is stable with the SBP approach predicting the safety factor of
\[
FS_q(x) = \frac{1}{\chi \tan \alpha_0 \cdot \exp(-|x|/H \cdot \tan \alpha_0) + \chi k_h}. \tag{6.11}
\]

If \(L_i \geq L_{cr}\), the slope in the intervals \(x_q - L_q \leq x < x_i - L_i\) and \(x_i < x \leq x_q\) in the quasi-stable zone in figure 1 is unstable with the safety factor
\[
FS_q(x) = \frac{1}{s \chi \tan \alpha_0 \cdot \exp(-|x|/H \cdot \tan \alpha_0) + s \chi k_h}. \tag{6.12}
\]

Finally, the slope in the intervals \(x < x_q - L_q\) and \(x_q < x\) in figure 1 is stable with the SBP approach predicting the safety factor of
\[
FS_q(x) = \frac{1}{\chi \tan \alpha_0 \cdot \exp(-|x|/H \cdot \tan \alpha_0) + \chi k_h}. \tag{6.13}
\]

(b) Hyperbolic slope surface

Consider a hyperbolic slope surface
\[
z = f(x) = \begin{cases} 
\frac{Hx \tan \alpha_0}{H - x \tan \alpha_0}, & \text{for } x < 0 \\
\frac{Hx \tan \alpha_0}{H + x \tan \alpha_0}, & \text{for } x \geq 0.
\end{cases} \tag{6.14}
\]

The boundaries of the initial and quasi-stable zones can be obtained using equations (5.1) and (5.2), respectively:
\[
f'(x_i) = \frac{\tan \alpha_0}{(1 + x_i/H \cdot \tan \alpha_0)^2} = \tan \alpha_i = \frac{1}{\chi} - k_h \tag{6.15}
\]
and
\[
f'(x_q) = \frac{\tan \alpha_0}{(1 + x_q/H \cdot \tan \alpha_0)^2} = \tan \alpha_q = \frac{1}{s \chi} - k_h. \tag{6.16}
\]
Then the lengths of the initial and quasi-stable zones are given by

\[ L_i = 2x_i = \frac{2H}{\tan\alpha_0} \left( \sqrt{\frac{\chi \tan\alpha_0}{1 - \chi k_h} - 1} \right) \]  

(6.17)

and

\[ L_q = 2x_q = \frac{2H}{\tan\alpha_0} \left( \sqrt{\frac{\chi \tan\alpha_0}{1/s - \chi k_h} - 1} \right). \]  

(6.18)

The average slope and gravitational shear ratio for the initial failure zone are given by

\[ \tan \tilde{\alpha} = \frac{2f(x_i)}{L_i} = \frac{1}{L_i/H} \frac{2Hx_i \tan\alpha_0}{H + x_1 \tan\alpha_0} = \sqrt{\tan\alpha_0 \cdot (1/\chi - k_h)} \]  

(6.19)

and

\[ \tilde{r} = \frac{s\chi}{s - 1} \left( \frac{\sqrt{\tan\alpha_0 \cdot (1/\chi - k_h) + k_h} - 1}{\tan\alpha_0} \right). \]  

(6.20)

So that, using equation (6.18), propagation criterion \( L_i \tilde{r} > L_{i,cr} \) can be expressed as

\[ \frac{L_i}{L_{i,cr}} = \frac{2H}{L_{i,cr}} \left( \sqrt{\frac{\tan\alpha_0}{1 - \chi k_h} - 1} \right) \frac{\sqrt{\tan\alpha_0 \cdot (1/\chi - k_h) + k_h} - 1}{(s - 1) \tan\alpha_0} \geq 1. \]  

(6.21)

The LE approach produces the following safety factors along the slope:

\[ \text{FS}(x) = \frac{1}{\chi \ f'(x) + k_h} = \frac{1}{(\chi \tan\alpha_0)/(1 + |x|/H \cdot \tan\alpha_0)^2 + \chi k_h}. \]  

(6.22)

By contrast, the SBP approach gives for the slope in the initial failure zone in the interval \( x_i - L_i \leq x \leq x_i \) in figure 1

\[ \text{FS}_i(x) = \frac{1}{s\chi f'(x) + k_h} = \frac{1}{(s\chi \tan\alpha_0)/(1 + |x|/H \cdot \tan\alpha_0)^2 + s\chi k_h}. \]  

(6.23)

If \( L_i < L_{i,cr} \), the slope in the quasi-stable zone in the intervals \( x_q - L_q \leq x < x_i - L_i \) and \( x_i < x \leq x_q \) in figure 1 is stable, with the SBP approach predicting the safety factor of

\[ \text{FS}_q(x) = \frac{1}{(\chi \tan\alpha_0)/(1 + |x|/H \cdot \tan\alpha_0)^2 + \chi k_h}. \]  

(6.24)

If \( L_i \geq L_{i,cr} \), the slope in the quasi-stable zone in the intervals \( x_q - L_q \leq x < x_i - L_i \) and \( x_i < x \leq x_q \) in figure 1 is unstable with the safety factor

\[ \text{FS}_q(x) = \frac{1}{s\chi \tan\alpha_0/(1 + |x|/H \cdot \tan\alpha_0)^2 + s\chi k_h}. \]  

(6.25)

Finally, the slope in the intervals \( x < x_q - L_q \) and \( x_q < x \) in figure 1 is stable with the SBP approach predicting the safety factor of

\[ \text{FS}_m(x) = \frac{1}{\chi \tan\alpha_0/(1 + |x|/H \cdot \tan\alpha_0)^2 + \chi k_h}. \]  

(6.26)

7. Example

The following example demonstrates application of the proposed procedure to a nonlinear slope. The calculations are performed for two different slope geometries described in the previous section.

(a) Geometry and soil parameters

Let us consider a slope (figure 1) with the maximum inclination \( \alpha_0 = 40^\circ \) and the total slope height \( 2H = 40 \text{ m} \). The corresponding slope surfaces described by hyperbolic and exponential
Table 1. SBP parameters for different slope geometries.

<table>
<thead>
<tr>
<th>slope surface</th>
<th>$L_i$ (m)</th>
<th>$L_q$ (m)</th>
<th>$L_i(\bar{r} = 0)$ (m)</th>
<th>$\bar{\alpha}$ (°)</th>
<th>$\bar{r}$</th>
<th>$L_{cr}$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>hyperbolic</td>
<td>105</td>
<td>498</td>
<td>1428</td>
<td>3.384</td>
<td>1.304</td>
<td>146</td>
</tr>
<tr>
<td>exponential</td>
<td>192</td>
<td>716</td>
<td>1932</td>
<td>3.400</td>
<td>1.314</td>
<td>145</td>
</tr>
</tbody>
</table>

functions are presented in figure 11. The pseudo-static coefficient $k_h$ and the degradation index $\delta$ are

$$k_h = 0.1 \quad \text{and} \quad \delta_d = 0.6.$$  

The soil is a normally consolidated low plasticity clay with the total unit weight of

$$\gamma = 20 \text{ kN m}^{-3},$$

the strength parameters

$$k = 0.25, \quad s = 1.25, \quad \delta = 0.5 \text{ m}, \quad r_u = 0.0, \quad \delta_d \frac{k(1 - \delta)}{\chi} = 1 = 0.15$$

and stiffness parameters

$$E_u/\tau_p = 500 \quad \text{and} \quad E_l/E_u = 0.5.$$

(b) Limiting equilibrium approach

For the LE approach, the safety factors are obtained for exponential and hyperbolic slope surfaces using equations (6.9) and (6.22), respectively. Distributions of the safety factors along the slope are given in figure 12a. As a result of this analysis, it can be concluded that a slab failure will occur over the length $L_i = 105$ m for the hyperbolic surface and $L_i = 192$ m for the exponential slope surface, obtained from equations (6.17) and (6.4), respectively. Outside these zones, the slope will remain stable.

(c) Shear band propagation approach

For the SBP approach, the safety factors are obtained for exponential and hyperbolic slope surfaces using equations (6.10)–(6.13) and (6.23)–(6.26), respectively. Distributions of the safety factors along the slope are given in figure 12b. In this approach, it is crucial to check the SBP criterion to determine if the failure propagates into the quasi-stable zone. This is achieved by calculating average critical length and comparing it with $L_i$ using equations (6.6)–(6.8) and (6.19)–(6.21) for exponential and hyperbolic slope surfaces, respectively, together with the value of $L_{icr} = 191$ m calculated from equation (5.13).

Results of the evaluation of the SBP criterion are given in table 1. Note, that in spite of different algebraic equations, numerical values of the average critical length obtained in this example for both types of surfaces are remarkably close (about 145 m). The length of the initial failure zone is, however, rather different, dramatically affecting the length of the potential landslide. In fact, for the exponential case, the initial zone is 192 m, i.e. larger than critical, and the shear band propagates to reach the total length of 716 m (with the dynamic upper bound $\bar{r} = 0$ reaching 1932 m). By contrast, in the hyperbolic case, the initial shear band of 105 m (smaller than critical) does not propagate and stays within the initial failure zone. Such a strong dependency of the failure length on slope geometry is intriguing and offers significant insights for the understanding of naturally occurring submarine landslides. For example, the common occurrence of long and relatively shallow submarine landslides in mild slopes may
simply be explained by the initial failure in locally steeper portions of the slope followed by the catastrophic SBP.

(d) Sensitivity of the propagation criterion

Sensitivity of the SBP criterion $L_i/L_{cr} \geq 1$ was explored by varying, one at a time, each of the parameters of equations (6.8) and (6.21) for exponential and hyperbolic slope surfaces, respectively, around the values used in the example. Results are presented in figure 13.

As is seen, the ratio $L_i/L_{cr}$ is rather sensitive to the seismic parameters (figure 13a,b) and the strength parameter $k$ (figure 13c), less sensitive to the slope inclination (figure 13d), and the sensitivity of clay (figure 13e). For the exponential slope surface, the ratio $L_i/L_{cr}$ is more sensitive to the changes of the parameters than for the hyperbolic one, mainly due to much faster changes
in the slope inclination of the exponential surface with growing distance from the point of the steepest slope.

8. Summary and conclusion

The paper extends the SBP approach, originally developed for a case of an infinite slope, to the true nonlinear slope geometry. It assumes that the initial weak zone would appear in the steepest part of the slope where the combined action of the gravity and seismic loads overcome the degraded peak shear resistance of the soil. If the length of this steepest part is larger than the critical length, the shear band will propagate into the quasi-stable parts of the slope, where the gravitational and seismically induced shear stresses are smaller than the peak but larger than the residual shear strength of the soil. This extension eliminates the main challenge of application of the SBP approach—determination of the length of the initial weak zone—because the slope geometry can readily be obtained from submarine site investigations. Accounting for the true nonlinear slope geometry in the SBP approach also helps to identify conditions for the early arrest of the shear band, before the occurrence of failure in the sliding layer or change in loading and excess pore water pressures.

A simple approach to slope stability analysis has been developed through definition of the shear stress ratio and assessment of the energy balance conditions for SBP and arrest in a nonlinear submarine slope. Growth of a shear band is strongly dependent on the shape of the slope, seismic parameters and the strength of soil and less dependent on the slope inclination and
the sensitivity of clay. For the slope surface with faster changing inclination, the criterion is more sensitive to the changes of the parameters.

For a nonlinear slope, the SBP approach demonstrates that failure is not always limited to a slab failure over the initial length predicted by the LE. If the initial failure zone is longer than the critical length, the shear band will propagate into the entire quasi-stable zone, in which case the safety factors for both zones should be calculated using the degraded residual strength and not the peak strength. The boundary between the quasi-stable and stable zones can be considered as a lower bound for the arrest of the SBP. In fact, due to dynamic effects, SBP can continue into the stable zone and arrest only when the average shear stress ratio along the shear band drops to zero. Depending on the slope geometry, loading and soil parameters, the difference in the size of the landslide predicted by LE and SBP approaches can reach orders of magnitude, providing an explanation for the immense dimensions and common occurrence of submarine landslides.

Data accessibility. There are no data for this article.

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