Snap-through buckling of initially curved microbeam subject to an electrostatic force

X. Chen and S. A. Meguid

Mechanics and Aerospace Design Laboratory, Department of Mechanical and Industrial Engineering, University of Toronto, 5 King’s College Road, Toronto, Ontario, Canada M5S 3G8

In this paper, the snap-through buckling of an initially curved microbeam subject to an electrostatic force, accounting for fringing field effect, is investigated. The general governing equations of the curved microbeam are developed using Euler–Bernoulli beam theory and used to develop a new criterion for the snap-through buckling of that beam. The size effect of the microbeam is accounted for using the modified couple stress theory, and intermolecular effects, such as van der Waals and Casimir forces, are also included in our snap-through formulations. The snap-through governing equations are solved using Galerkin decomposition of the deflection. The results of our work enable us to carefully characterize the snap-through behaviour of the initially curved microbeam. They further reveal the significant effect of the beam size, and to a much lesser extent, the effect of fringing field and intermolecular forces, upon the snap-through criterion for the curved beam.

1. Introduction

Micro-electro-mechanical systems (MEMS) have aroused great interest for their unique advantages such as small size, high precision and low power consumption. One benchmark of MEMS is the initially straight microbeam system driven by electrostatic force, whose static and dynamic behaviours have been largely investigated in the literature [1–4]. Recently, the bistable MEMS based on initially curved microbeams have drawn more and more attention from the scientific community for their various potential applications such as optical switches, micro-valves and non-volatile memories [5–7].

The initially curved beam (arch) under transverse force may exhibit bistability, and the transition between two stable states is referred to as snap-through...
buckling [8]. The existence of snap-through buckling depends on various factors, e.g. initial arch rise, beam thickness and clamping angle. Pippard [9] conducted experiments to develop a phase diagram of instability in terms of the arch span and the initial angle at the clamped ends. This work was followed by Patricio et al. [10] in which they developed theoretical model simulations to derive a similar phase diagram. As a result of the earlier experiments and model simulations, Krylov et al. [11] revealed that the snap-through buckling occurs at large initial deflections. Pane & Asano [12] conducted energy analysis and further found that the existence of bistable states in an initially curved beam depends on the ratio of its initial deflection to its thickness. Park & Hah [13] conducted theoretical investigations and showed that the existence of bistable states also depends on the residual axial stress in the beam. Das & Batra [14] developed a finite-element model to study the transient snap-through behaviour of the initially curved beam, and found that at high loading rates (i.e. voltage is applied at a high rate), the snap-through buckling is suppressed. Moghimi Zand [15] also developed a finite-element model and found the significant inertia effect on the dynamic snap-through behaviour. Medina et al. [8,16] examined the symmetric buckling and antisymmetrical bifurcation of electrostatically actuated and initially curved microbeams with/without residual stress. They derived the criteria of symmetric and non-symmetric snap-through instability for quasi-static loading conditions.

Careful literature review indicates that many studies consider a uniform mechanical force as the applied load. However, the electrostatic force applied on the curved microbeam is highly non-uniform and strongly depends on the beam deflection. Several studies consider the electrostatic force, but they fail to examine the fringing field effect and/or the influence of the intermolecular forces such as Casimir and van der Waals forces. Furthermore, the size effect at the microscale is neglected in almost all the existing studies.

The size effect on the deformation behaviour of microstructures has already been observed experimentally [17–20], and such size dependency is attributed to the non-local effects, which cannot be described by the classical continuum theories of local character. Various non-classical continuum theories with additional material length-scale parameters have been proposed [19,21–25]. Among them, the modified couple stress theory developed by Yang et al. [25] with a length-scale parameter is one of the most used. Determining the microstructure-dependent length-scale parameters is difficult, so it is desirable to use the theories with only one length-scale parameter [26]. Based on the non-classical continuum theories, the size effect on various behaviours of microbeams has been theoretically studied, including bending, buckling, free vibration, pull-in instability, etc. [27–35].

In this paper, we extend the earlier studies to investigate the size effect on the snap-through behaviour of the initially curved microbeam under electrostatic force. The modified couple stress theory [25] is used. The fringing field effect is taken into account by Meijs–Fokkema formula [36]. The influence of the intermolecular forces is also examined. Based on our model and simulation results, a unified criterion for the existence of snap-through buckling is derived and plotted in a diagram, which can be used for the design of the bistable MEMS.

2. Formulation

(a) General governing equations and boundary conditions

Consider an initially curved rectangular microbeam of span $L$, width $b$ and thickness $h$ undergoing in-plane bending ($x$–$z$ plane in figure 1). The respective displacements $u_x$, $u_y$ and $u_z$ along $x$-, $y$- and $z$-coordinate are assumed to be dependent only on $x$ and $z$. $u_y$ is further assumed to be 0. For a thin beam ($h \ll L$), the Euler–Bernoulli beam theory is applied

$$u_x(x, z) = u(x) - zw'(x), \quad (2.1a)$$

$$u_z(x, z) = w(x) \quad (2.1b)$$

and

$$\theta(x) = -w'(x), \quad (2.1c)$$
where \( u(x) \) and \( w(x) \) are, respectively, the axial (along \( x \)-coordinate) and transverse (along \( z \)-coordinate) displacements of a point on the mid-plane of the beam; \( \theta(x) \) is the rotation of the cross section around \( y \)-coordinate; a superimposed apostrophe denotes a derivative with respect to \( x \). During the snap-through buckling, the mid-plane stretching can be important. To consider this effect, the von Karman nonlinear strain is used. With the aid of equation (2.1), the non-zero strain component (i.e. axial strain \( \varepsilon_{xx}^* \)) can be obtained as [26]

\[
\varepsilon_{xx}^* = u_x' + \frac{1}{2}(u_z')^2 + w'' = u' - zw'' + \frac{1}{2}(w')^2. \tag{2.2}
\]

Considering the initial strain \( \varepsilon_{xx}^0 \) related to the initial deflection \( w_0(x) \) by: \( \varepsilon_{xx}^0 = -zw''_0 + (w'_0)^2/2 \), we calculate the axial strain change \( \varepsilon_{xx} \) from equation (2.2) as

\[
\varepsilon_{xx} = \varepsilon_{xx}^* - \varepsilon_{xx}^0 = u' - z(w'' - w''_0) + \frac{1}{2}((w')^2 - (w'_0)^2). \tag{2.3}
\]

The symmetric curvature tensor \( \chi \) conjugated to the deviatoric couple stress tensor \( \psi \) in the modified couple stress theory is [25]

\[
\chi = \frac{1}{2}(\nabla \omega + (\nabla \omega)^T). \tag{2.4}
\]

where \( \omega = (\nabla \times u)/2 \) is the rotation vector, with \( u = (u_x, u_y, u_z)^T \), being the displacement vector; the superimposed T denotes the transpose of the matrix. With the aid of equation (2.1), the non-zero curvature components in equation (2.4) are

\[
\chi_{xy}^* = \chi_{yx}^* = -\frac{1}{2}w''. \tag{2.5}
\]

Considering the initial non-zero curvature \( \chi_{xy}^0 = \chi_{yx}^0 = -w''_0/2 \) owing to the initial deflection \( w_0 \), we obtain the curvature change \( \chi_{xy} \) and \( \chi_{yx} \) from equation (2.5) as

\[
\chi_{xy} = \chi_{yx} = \chi_{xy}^* - \chi_{xy}^0 = -\frac{1}{2}(w'' - w''_0). \tag{2.6}
\]

To derive the governing equations, the theorem of minimum potential energy is used

\[
\delta U_{\text{elas}} - \delta W_{\text{ext}} = 0, \tag{2.7}
\]

where \( \delta U_{\text{elas}} \) and \( \delta W_{\text{ext}} \) are, respectively, the variations of the elastic strain energy and the work done by the external forces. Considering the non-zero strain component \( \varepsilon_{xx} \) and the non-zero curvature components \( \chi_{xy} \) and \( \chi_{yx} \), we can calculate \( \delta U_{\text{elas}} \) as [25]

\[
\delta U_{\text{elas}} = \int_0^L \int_S (\sigma : \delta \varepsilon + \psi : \delta \chi) \, ds \, dx = \int_0^L \int_S (\sigma_{xx} \delta \varepsilon_{xx} + 2\psi_{xy} \delta \chi_{xy}) \, ds \, dx, \tag{2.8}
\]
where \( \int_S ds \) is the integral over the cross section (y-z plane in figure 1). Introduce equations (2.3) and (2.6) into equation (2.8), integrate the resulting equation by parts with respect to \( x \), and we obtain

\[
\delta U_{\text{elas}} = -\int_0^L N' \delta u \, dx - \int_0^L (M'' + C'' + (Nw')') \delta w \, dx - N(0) \delta u(0) + N(L) \delta u(L)
- (M'(0) + C'(0) + N(0) \delta w(0)) + (M'(L) + C'(L) + N(L) \delta w(L)) \delta w(L)
+ (M(0) + C(0)) \delta w(0) - (M(L) + C(L)) \delta w'(L),
\] (2.9)

where the stress resultants \( N, M \) and \( C \) are defined as

\[
N = \int_S \sigma_{xx} \, ds, \quad \tag{2.10a}
M = \int_S \tau_{zr} \, ds, \quad \tag{2.10b}
\]

and

\[
C = \int_S \psi_{xy} \, ds. \quad \tag{2.10c}
\]

The variation \( \delta W_{\text{ext}} \) of the work done by the external forces is

\[
\delta W_{\text{ext}} = \int_0^L (f_x(x) \delta u + f_z(x) \delta w + m_y(x) \delta \theta) \, dx + N_1 \delta u(0) + N_2 \delta u(L)
+ T_1 \delta w(0) + T_2 \delta w(L) + M_1 \delta \theta(0) + M_2 \delta \theta(L),
\] (2.11)

where \( f_x, f_z \) and \( m_y \) are, respectively, the distributed axial load (along \( x \)-coordinate), transverse load (along \( z \)-coordinate) and body couple (around \( y \)-coordinate) per unit length; \( N_1, N_2, T_1, T_2, M_1 \) and \( M_2 \) are, respectively, the normal forces, transverse forces and bending moments at the two ends of the beam (i.e. ‘1’ at \( x = 0 \), ‘2’ at \( x = L \)). Replace \( \theta \) with equation (2.1c) in equation (2.11), integrate the obtained-equation by parts with respect to \( x \), and we obtain

\[
\delta W_{\text{ext}} = \int_0^L f_x \delta u \, dx + \int_0^L (f_z + m_y) \delta w \, dx + N_1 \delta u(0) + N_2 \delta u(L)
+ (T_1 + m_y(0)) \delta w(0) + (T_2 - m_y(L)) \delta w(L) - M_1 \delta \theta(0) - M_2 \delta \theta(L).
\] (2.12)

Introducing equations (2.9) and (2.12) into equation (2.7), we arrive at

\[
\int_0^L (N' + f_x) \delta u \, dx + \int_0^L (M'' + C'' + (Nw')' + f_z + m_y') \delta w \, dx + (N(0) + N_1) \delta u(0)
- (N(L) - N_2) \delta u(L) + (M'(0) + C'(0) + N(0) \delta w(0) + T_1 + m_y(0)) \delta w(0)
- (M'(L) + C'(L) + N(L) \delta w(L) - T_2 + m_y(L)) \delta w(L)
- (M(0) + C(0) + M_1) \delta w'(0) + (M(L) + C(L) - M_2) \delta w'(L) = 0.
\] (2.13)

To satisfy equation (2.13) with arbitrary variations of displacements \( \delta u \) and \( \delta w \), we obtain the following governing equations

\[
\delta u : N' + f_x = 0 \quad \tag{2.14a}
\]

and

\[
\delta w : M'' + C'' + (Nw')' + f_z + m_y' = 0, \quad \tag{2.14b}
\]

with the following boundary conditions when the corresponding displacements are not specified

\[
\delta u : N(0) = -N_1, \quad N(L) = N_2, \quad \tag{2.15a}
\]

\[
\delta w : M'(0) + C'(0) + N(0)w'(0) = -T_1 - m_y(0), \quad M'(L) + C'(L) + N(L)w'(L) = T_2 - m_y(L) \quad \tag{2.15b}
\]

and

\[
\delta w' : M(0) + C(0) = -M_1, \quad M(L) + C(L) = M_2. \quad \tag{2.15c}
\]
Consider an initially curved double-clamped microbeam under electrostatic force (figure 2). Neglecting the gravity, we reduce the governing equations (equation (2.14)) to

\[ \delta u : N' = 0 \] (2.16a)

and

\[ \delta w : M'' + C'' + (Nw')' + f_z = 0, \] (2.16b)

with the following boundary conditions for the double-clamped beam

\[ \delta u : u(0) = 0, \quad u(L) = 0, \] (2.17a)

\[ \delta w : w(0) = 0, \quad w(L) = 0 \] (2.17b)

and

\[ \delta w' : w'(0) = 0, \quad w'(L) = 0. \] (2.17c)

The distributed transverse load \( f_z \) is composed of

\[ f_z = f_{\text{elec}} + f_{\text{Casi}} + f_{\text{VDW}}, \] (2.18)

where \( f_{\text{elec}}, f_{\text{Casi}} \) and \( f_{\text{VDW}} \) are, respectively, the electrostatic force, Casimir force and van der Waals force per unit length.

The electrostatic force \( f_{\text{elec}} \) per unit length can be calculated using [37,38]

\[ f_{\text{elec}} = \frac{1}{2} V^2 \frac{dC}{dg}, \] (2.19)

where the sign of the force depends on the coordinate system, \( V \) is the applied voltage difference between the beam and the rigid electrode, \( C \) is the capacitance per unit length of the capacitor composed of the beam and the electrode and \( g \) is the gap between the beam and the electrode as being

\[ g(x) = g_0 + w(x), \] (2.20)

with \( g_0 \) being the initial gap (i.e. distance between the clamped beam ends and the rigid electrode (figure 2)). For a small gap \( g (\ll \text{beam length}) \), the beam with the electrode can be regarded

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**Figure 2.** Initially curved double-clamped microbeam under electrostatic force. The directions of electrostatic and intermolecular forces are indicated by arrows.
as a parallel-plate capacitor. To further take into account the fringing fields at the edges of the microbeam, the capacitance $C$ is estimated using the Meijs–Fokkema formula [36]

$$C(g) = \varepsilon_0 \left( \frac{b}{g} + 0.77 + 1.06 \left( \frac{b}{g} \right)^{0.25} + 1.06 \left( \frac{h}{g} \right)^{0.5} \right), \quad (2.21)$$

where $\varepsilon_0 (= 8.8542 \times 10^{-12} \text{ F} \cdot \text{m}^{-1})$ is the vacuum permittivity. It is noted that the error of the estimated capacitance using equation (2.21) is within 6% for the microbeam systems satisfying beam-width-to-gap ratio $(b/g)$ larger than 0.3 and beam-thickness-to-gap ratio $(h/g)$ smaller than 10 [36]. So to ensure the proper application of equation (2.21), this paper studies only the microbeam systems with the width-to-initial-gap ratio $(b/g_0)$ larger than 0.5 (considering $w_{\text{max}} = 0.5g_0$ in equation (2.20)) and the thickness-to-initial-gap ratio $(h/g_0)$ smaller than 5 (considering $w_{\text{min}} = -0.5g_0$). Introduce equations (2.20) and (2.21) into equation (2.19), and after several calculations, we obtain [11]

$$f_{\text{elec}} = -\frac{1}{2} \frac{\varepsilon_0 b V^2}{(g_0 + w)^2} \left( 1 + 0.265 \left( \frac{b}{h} \right)^{-0.75} \left( \frac{g_0 + w}{h} \right)^{0.75} + 0.53 \left( \frac{h}{g} \right)^{-1} \left( \frac{g_0 + w}{h} \right)^{0.5} \right). \quad (2.22)$$

The intermolecular forces can be described by Casimir and van der Waals forces. The former force is attributed to the attraction between two closely spaced conducting surfaces, and the latter one is due to the electrostatic interactions among dipoles at the atomic scale [39,40]. For a small gap ($\ll$ beam length), the parallel-plate approximation is applied [41,42]:

$$f_{\text{casi}} = -\frac{\pi^2 h cb}{240(g_0 + w)^4} \quad (2.23a)$$

and

$$f_{\text{VDM}} = -\frac{Ab}{6\pi(g_0 + w)^3}, \quad (2.23b)$$

where $h (= 1.0546 \times 10^{-34} \text{ J} \times \text{s})$ is the reduced Planck constant; $c (= 3 \times 10^8 \text{ m} \cdot \text{s}^{-1})$ is the speed of light; $A (= \pi^2 k \rho_1 \rho_2)$ is the Hamaker constant, with $k$ being the interaction parameter, $\rho_1$ and $\rho_2$ being the numbers of atoms per unit volume in the microbeam and the rigid electrode.

Suppose the beam material is elastically isotropic with Young’s modulus $E$ and Poisson’s ratio $\nu$. For an Euler–Bernoulli beam undergoing in-plane bending, we consider only the axial stress $\sigma_{xx}$ [26]. Then, the one-dimensional constitutive relation is

$$\sigma_{xx} = E \varepsilon_{xx}. \quad (2.24)$$

The deviatoric couple stress $\psi_{xy}$ is related to the symmetric curvature $\chi_{xy}$ by [25]

$$\psi_{xy} = \frac{E l^2}{(1 + \nu)} \chi_{xy}, \quad (2.25)$$

where $l$ is a length-scale parameter. With equations (2.3), (2.6), (2.24) and (2.25), equation (2.10) is changed to

$$N(x) = ES \left( u' + \frac{1}{2}(w')^2 - \frac{1}{2}(w'_0)^2 \right), \quad (2.26a)$$

$$M(x) = -EI(w'' - w'_0) \quad (2.26b)$$

and

$$C(x) = -\frac{ESl^2}{2(1 + \nu)}(w'' - w'_0), \quad (2.26c)$$

where $S (= bh)$ is the cross-sectional area ($y$–$z$ plane in figure 2); $l (= bh^3/12)$ is the second moment of area. Introduce equation (2.26) into equation (2.16), we have

$$\delta u : ES \left( u' + \frac{1}{2}(w')^2 - \frac{1}{2}(w'_0)^2 \right)' = 0 \quad (2.27a)$$
and
\[
\delta w : EI \left(1 + \frac{6}{(1 + \nu)} \left(\frac{l}{h}\right)^2\right) (w''' - w_0''') - ES \left(u' + \frac{1}{2} (w')^2 - \frac{1}{2} (w_0')^2\right) w' - f_z = 0. \tag{2.27b}
\]

With equation (2.27a), equation (2.27b) can be reduced to
\[
EI \left(1 + \frac{6}{(1 + \nu)} \left(\frac{l}{h}\right)^2\right) (w''' - w_0''') - ES \frac{L}{2L} \left(\int_0^L (w')^2 - (w_0')^2 \, dx\right) w'' - f_z = 0. \tag{2.28}
\]

Equation (2.27a) shows that the axial force \(N\) is constant, so \(N\) can be estimated as the average value calculated from equation (2.26a) being \(ES/L(\int_0^L (u' + 1/2(w')^2 - (1/2)(w_0')^2) \, dx)\). In equation (2.28), we replace \(N\) with the average value, and considering the boundary condition (equation (2.17a)) we obtain
\[
EI \left(1 + \frac{6}{(1 + \nu)} \left(\frac{l}{h}\right)^2\right) (w''' - w_0''') - ES \frac{L}{2L} \left(\int_0^L (w')^2 - (w_0')^2 \, dx\right) w'' = 0. \tag{2.29}
\]

Introducing equations (2.18), (2.22) and (2.23) into equation (2.29), and we have
\[
EI \left(1 + \frac{6}{(1 + \nu)} \left(\frac{l}{h}\right)^2\right) (w''' - w_0''') - ES \frac{L}{2L} \left(\int_0^L (w')^2 - (w_0')^2 \, dx\right) w'' + \frac{1}{2} \frac{\nu_0 b V^2}{(g_0 + w)^2} \left(1 + 0.265 \left(\frac{b}{h}\right)^{-0.75} \left(\frac{g_0 + w}{h}\right)^{0.75} + 0.53 \left(\frac{b}{h}\right)^{-1} \left(\frac{g_0 + w}{h}\right)^{0.5}\right) \nonumber
\]
\[\quad \quad + \frac{\pi^2 h c b}{240(g_0 + w)^4} + \frac{Ab}{6\pi (g_0 + w)^3} = 0. \tag{2.30}
\]

It is seen from the governing equation (equation (2.30)) that the length-scale parameter \(l\) has the effect of increasing the effective bending stiffness \((EI)_{\text{eff}}\), being
\[
(EI)_{\text{eff}} = EI \left(1 + \frac{6}{(1 + \nu)} \left(\frac{l}{h}\right)^2\right). \tag{2.31}
\]

For thin beams (beam thickness \(h\) close to \(l\)), the effective bending stiffness can be as large as \((1 + (6)/(1 + \nu)) \approx 5.7\) times the conventional bending stiffness \((EI)\), whereas for thick beams \((h \gg l)\), the effective bending stiffness is nearly equal to the conventional one, indicating that the size effect is negligible.

Rewrite equation (2.30) in the following non-dimensional form \([8,43]\)
\[
(\tilde{\tilde{w}}''' - \tilde{\tilde{w}}_0''' - \alpha \left(\int_0^1 (\tilde{\tilde{w}}')^2 - (\tilde{\tilde{w}}_0')^2 \, dx\right) (\tilde{\tilde{w}}'') + \lambda_{\text{VDW}} \frac{1}{(1 + \tilde{\tilde{w}})^3} + \lambda_{\text{casi}} \frac{1}{(1 + \tilde{\tilde{w}})^4}) \nonumber
\]
\[\quad \quad = - \frac{\beta_v}{(1 + \tilde{\tilde{w}})^2} \left(1 + 0.265 \left(\frac{b}{h}\right)^{-0.75} \left(1 + \tilde{\tilde{w}}\right)^{0.75} + 0.53 \left(\frac{b}{h}\right)^{-1} \left(1 + \tilde{\tilde{w}}\right)^{0.5}\right), \tag{2.32}
\]

where the non-dimensional quantities are defined in Table 1, a superimposed apostrophe in the non-dimensional equations denotes a derivative with respect to the normalized coordinate \(\tilde{x}\). The non-dimensional boundary conditions from equation (2.17) can be expressed as
\[
\delta \tilde{\tilde{w}} : \tilde{\tilde{w}}(0) = 0, \quad \tilde{\tilde{w}}(1) = 0 \tag{2.33a}
\]
and
\[
\delta \tilde{\tilde{w}}' : \tilde{\tilde{w}}'(0) = 0, \quad \tilde{\tilde{w}}'(1) = 0. \tag{2.33b}
\]
with the values of the constants in table 2, we calculate equation (2.35) as

\[
\frac{\lambda_{VDW}}{\beta_v} = \frac{A}{3\pi \varepsilon_0 \varepsilon_0 V^2}
\]  

(2.35a)

and

\[
\frac{\lambda_{casi}}{\beta_v} = \frac{\pi^2 h c l^4}{120 \varepsilon_0 \varepsilon_0^2 V^2}.
\]  

(2.35b)

Consider \( g_0 \approx 10^{-6} \) m for the microscale systems and \( V \approx 10^1 \) V for the order of applied voltage, and with the values of the constants in table 2, we calculate equation (2.35) as

\[
\frac{\lambda_{VDW}}{\beta_v} \approx 10^{-5}
\]  

(2.36a)

and

\[
\frac{\lambda_{casi}}{\beta_v} \approx 3 \times 10^{-6}
\]  

(2.36b)

With equations (2.34) and (2.36), the force ratios can be estimated as

\[
\left| \frac{f_{VDW}}{f_{elec}} \right| < 10^{-5} \frac{1}{1 + \bar{w}}
\]  

(2.37a)

and

\[
\left| \frac{f_{casi}}{f_{elec}} \right| < 3 \times 10^{-6} \frac{1}{(1 + \bar{w})^2}.
\]  

(2.37b)
\[
\cosh(\lambda) \approx \frac{1}{1 - \frac{\lambda^2}{2}} \quad \text{for} \quad \lambda \ll 1
\]

The maximum force ratios are determined by the minimum stable deflection, i.e. deflection at the pull-in instability, which is roughly half gap \((\bar{w} = -0.5)\) \([37,44,45]\). Then, equation (2.37) leads to

\[
\max\left| f_{\text{VDW}} / f_{\text{elec}} \right| \approx 2 \times 10^{-5} \quad \text{(2.38a)}
\]

and

\[
\max\left| f_{\text{casi}} / f_{\text{elec}} \right| \approx 1 \times 10^{-5} \quad \text{(2.38b)}
\]

Equation (2.38) shows that the intermolecular forces (van der Waals and Casimir forces) are negligible with respect to the electrostatic force when studying the snap-through buckling.

(d) Reduced-order model

In §2c, we have proved that the intermolecular forces can be neglected in the study of snap-through buckling. So the governing equation (equation (2.32)) can be further reduced to

\[
(\bar{w}^{(m)} - \bar{w}_0^{(m)}) - \alpha \left( \int_0^1 (\bar{w}')^2 - (\bar{w}_0')^2 \, d\bar{x} \right) \bar{w}'' = - \frac{\beta_v}{(1 + \bar{w})^2} \left( 1 + 0.265 \left( \frac{b}{h} \right)^{-0.75} \left( 1 + \frac{\bar{w}}{h} \right)^{0.75} + 0.53 \left( \frac{b}{h} \right)^{-1} \left( 1 + \frac{\bar{w}}{h} \right)^{0.5} \right) \quad \text{(2.39)}
\]

Equation (2.39) with the boundary conditions expressed in equation (2.33) can be solved by using the Galerkin decomposition of the dimensionless deflection \(\bar{w}(\bar{x})\) as \([8,11,43]\)

\[
\bar{w}(\bar{x}) \approx \sum_{k=1}^n q_k \phi_k(\bar{x}), \quad \text{(2.40)}
\]

where \(\phi_k (k = 1, 2, \ldots, n)\) is the \(k\)th linear undamped eigenmode of the straight beam, and \(q_k\) is its generalized coordinate. For a double-clamped straight beam, we have

\[
\phi_k(\bar{x}) = A_k \left( \cosh(\lambda_k \bar{x}) - \cos(\lambda_k \bar{x}) - \frac{\sinh(\lambda_k) + \sin(\lambda_k)}{\cosh(\lambda_k) - \cos(\lambda_k)} (\sinh(\lambda_k \bar{x}) - \sin(\lambda_k \bar{x})) \right), \quad \text{(2.41)}
\]

where \(A_k\) is a constant satisfying \(\max_{\bar{x} \in [0,1]} |\phi_k(\bar{x})| = 1\), and \(\lambda_k\) is a frequency parameter satisfying \(\cosh(\lambda_k) \cos(\lambda_k) = 1\).

It is shown in \([43]\) that the numerical simulations of snap-through buckling using \(n \geq 6\) in equation (2.40) are indistinguishable from each other. It is further stated that a reasonably accurate response of the beam can be given by considering only the first mode \((n = 1)\) \([43]\), which indicates that the first mode approximation of the deflection can capture the characteristics of the snap-through behaviour. So, in order to simplify our study for an analytical snap-through criterion, we decided to make a first mode approximation here. Suppose the dimensionless initial deflection
\( \bar{w}_0(\bar{x}) \) is also in the first mode, then we have
\[
\bar{w}(\bar{x}) = q_1 \phi_1(\bar{x}) \tag{2.42a}
\]
and
\[
\bar{w}_0(\bar{x}) = q_0 \phi_1(\bar{x}) \tag{2.42b}
\]
where \( q_1 \) is the dimensionless midpoint deflection; \( q_0 = r/g_0 \) is the dimensionless initial arch rise, with \( r \) being the initial arch rise (i.e. initial deflection at the midpoint). Introduce equation (2.42) into equation (2.39), multiply the result by \( \phi_1 \), and then integrate over the domain \([0, 1]\). Further integrate by parts with respect to \( \bar{x} \) and consider the boundary conditions (equation (2.33)), we obtain
\[
\beta_v = -\frac{\alpha s_{11}^2}{I_1(q_1)} q_1^3 - \frac{b_{11} - \alpha s_{11} q_0^2}{I_1(q_1)} q_1 + \frac{b_{11} q_0}{I_1(q_1)} \tag{2.43}
\]
where \( b_{11}, s_{11} \) and \( I_1 \) are
\[
b_{11} = \int_0^1 (\phi_1')^2 \, d\bar{x}, \tag{2.44a}
\]
\[
s_{11} = \int_0^1 (\phi_1')^2 \, d\bar{x} \tag{2.44b}
\]
and
\[
I_1(q_1) = \int_0^1 \frac{\phi_1}{(1 + q_1 \phi_1)^2} \left( 1 + 0.265 \left( \frac{b}{h} \right)^{-0.75} \left( \frac{1 + q_1 \phi_1}{h} \right)^{0.75} \right.
+ 0.53 \left( \frac{b}{h} \right)^{-1} \left( \frac{1 + q_1 \phi_1}{h} \right)^{0.5} \left) \, d\bar{x}. \tag{2.44c}
\]

The values of \( b_{11} \) and \( s_{11} \) are given in table 3. Equation (2.43) describes the evolution of the voltage parameter \( \beta_v \) with the dimensionless midpoint deflection \( q_1 \), which will be used to study the snap-through behaviour in the following section.

### 3. Results and discussion

#### (a) Influence of initial arch rise on snap-through behaviour

Let us consider an electrostatically actuated microbeam system described by the dimensional quantities in table 4, which is obtained from the experiments in [11]. The corresponding non-dimensional quantities can be calculated with the aid of table 1: stretching parameter \( \alpha = 17–96 \) (with \( l/h = 0–1, \nu = 0.27 \) [46]), beam width-to-thickness ratio \( b/h = 12 \), dimensionless thickness \( \bar{h} = 0.25 \), dimensionless initial arch rise \( q_0 = 0–0.5 \), voltage parameter \( \beta_v = 0–306 \) (with \( l/h = 0–1, E = 160 \text{ GPa} \) and \( \nu = 0.27 \) [46]). Taking \( \alpha = 95.2798 \) \( l/h = 0.04 \) with the material length-scale parameter \( l \) having the order of magnitude of \( 10^{-1} \mu m \) for both silicon and polysilicon [47], \( b/h = 12, \bar{h} = 0.25, b_{11} = 198.4626 \) and \( s_{11} = 4.8777 \) (table 3), we plot equation (2.43) in figure 3 at different levels of \( q_0 \) (0–0.5). The experimental results from [11] are also shown. It is seen that the model (equation (2.43)) can approximately describe the snap-through behaviour observed from the experiments.
Figure 3. Evolution of voltage parameter $\beta_v$ with dimensionless midpoint deflection $q_1$ at different levels of dimensionless initial arch rise $q_0$. The extreme points $q_s$, $q_r$ and $q_p$ correspond, respectively, to the critical points of the snap-through buckling, the release (snap-back) and the pull-in instability. (Online version in colour.)

The difference in the critical voltages (i.e. voltage parameters at the extreme points) is possibly owing to the non-ideal clamping conditions, residual stresses, initial imperfections in the beam shape and variations of beam geometry owing to the low fabrication tolerances [11].

Figure 3 shows that the existence of the snap-through buckling depends on the level of the dimensionless initial arch rise $q_0$. For very small $q_0$ (e.g. $q_0 = 0$ in figure 3a), there is only one extreme point $q_p$ on the $\beta_v - q_1$ curve corresponding to the pull-in instability. With the increase of the voltage ($\beta_v$ increases), the microbeam bends towards the rigid electrode owing to the electrostatic force. The equilibrium position of the beam can be determined by the balance of the elastic and electrostatic forces. Therefore, the beam deflection decreases gradually (see the loading path $A \rightarrow q_p$ in figure 3a). When the critical point $q_p$ is reached, the microbeam becomes unstable (i.e. the elastic force can no longer resist the electrostatic force), so it collapses onto the rigid electrode ($q_p \rightarrow B$). This behaviour is called pull-in instability.

For a larger value of $q_0$ (e.g. 0.35 in figure 3b), two more extreme points $q_s$ and $q_r$ appear, which correspond, respectively, to the snap-through buckling and the release (snap-back). With the increase of $\beta_v$, the beam deflection decreases gradually ($C \rightarrow q_s$ in figure 3b) until reaching the critical point $q_s$ where two stable states ($q_s$ and $D$) coexist. A slight increase in $\beta_v$ causes a sudden transition from the initial stable state $q_s$ to the second stable state D, so the beam deflection suddenly decreases ($q_s \rightarrow D$). Such transition is called snap-through buckling.

Table 4. Values of dimensional quantities from experiments of microbeams made of single crystal silicon in [11].

<table>
<thead>
<tr>
<th>quantity</th>
<th>meaning</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>beam width</td>
<td>30 $\mu$m</td>
</tr>
<tr>
<td>$g_0$</td>
<td>gap between beam ends and rigid electrode</td>
<td>10 $\mu$m</td>
</tr>
<tr>
<td>$h$</td>
<td>beam thickness</td>
<td>2.5 $\mu$m</td>
</tr>
<tr>
<td>$L$</td>
<td>span of arch</td>
<td>1000 $\mu$m</td>
</tr>
<tr>
<td>$r$</td>
<td>initial arch rise</td>
<td>0–5 $\mu$m</td>
</tr>
<tr>
<td>$V$</td>
<td>applied voltage</td>
<td>0–120 V</td>
</tr>
</tbody>
</table>
the snap-through buckling, the beam deflection continues to decrease gradually with $\beta_v$ ($D \rightarrow q_s$) until reaching the pull-in instability where the beam collapses onto the rigid electrode ($q_p \rightarrow E$).

If $q_0$ is large (e.g. 0.48 in figure 3c), the voltage parameter $\beta_v$ at the snap-through buckling point $q_s$ is larger than that at the pull-in instability point $q_p$. Then, the snap-through and the pull-in take place simultaneously, as shown by the loading path $F \rightarrow q_s \rightarrow G$ in figure 3c. In this case, the observed behaviour of the microbeam is similar to that of the ordinary pull-in instability. In summary, to exhibit the snap-through behaviour (separately from the pull-in instability), the microbeam should have a dimensionless initial arch rise $q_0$ in a certain range.

(b) Size and fringing field effects on snap-through behaviour

The fringing field effect owing to the finite beam width on the snap-through behaviour is shown in figure 4a by plotting equation (2.43) at different levels of width-to-thickness ratio $b/h$ ($2 \sim +\infty$) with $q_0 = 0.35$, $\alpha = 95.2798$, $\bar{h} = 0.25$, $b_{11} = 198.4626$ and $s_{11} = 4.8777$. It is seen that with the decrease of $b/h$, the voltage parameter at the snap-through buckling (point $q_s$) decreases. This is due to the fact that when reducing $b/h$ (beam width decreases), the fringing field effect becomes more significant, which increases the total voltage between the beam and the rigid electrode. Therefore, less applied voltage (normalized as $\beta_v$) is needed to induce the snap-through buckling.

The size effect (considering the length-scale parameter $l$) on the snap-through behaviour is shown in figure 4b by taking $q_0 = 0.35$, $b/h = 12$, $\bar{h} = 0.25$, $b_{11} = 198.4626$ and $s_{11} = 4.8777$ in equation (2.43). At different levels of the length-scale parameter $l/h = 0–1$, the stretching parameter $\alpha$ is calculated with the expression and the microbeam system dimensions given in tables 1 and 4. Figure 4b shows that the critical points ($q_s$, $q_r$) of snap-through buckling
disappear when $l/h$ is larger than 0.7, which indicates that $q_0 = 0.35$ is not in the domain of snap-through buckling for thin microbeams with $l/h \geq 0.7$. Size effect influences the domain of the dimensionless initial arch rise $q_0$ for the snap-through buckling. In §3c, we derive an analytical expression of the domain of $q_0$ (i.e. analytical criterion) for the existence of snap-through buckling.

(c) Size and fringing field effects on snap-through criterion

The extreme points $q_s$, $q_t$ and $q_p$ on $\beta_v - q_0$ curve (refer to figure 3) can be obtained by solving the following equation with the aid of equation (2.43)

$$\frac{d\beta_v}{dq_0} = 0 \Rightarrow \alpha s_{11}^2 I_2 q_0^3 - 3\alpha s_{11}^2 I_1 q_0^2 + I_2 (b_{11} - \alpha s_{11}^2 q_0^2)q_1 - (b_{11} - \alpha s_{11}^2 q_0^2)I_1 - b_{11} I_2 q_0 = 0,$$

where $I_2$ is calculated from equation (2.44a) as

$$I_2 = \frac{dI_1}{dq_0} = \int_0^1 \frac{-2\phi_1^2}{(1 + q_1\phi_1)^3} \left(1 + 0.165625 \left(\frac{b}{h}\right)^{-0.75} \left(\frac{1 + q_1\phi_1}{h}\right)^{0.75} + 0.3975 \left(\frac{b}{h}\right)^{-1} \left(\frac{1 + q_1\phi_1}{h}\right)^{0.5}\right) d\bar{x}. \tag{3.2}$$

Equation (3.1) containing integrals ($I_1$, $I_2$) cannot be solved analytically. So we solve the equation numerically, and show the typical results in figure 5. It is seen that $q_0$ must be larger than a critical value $q_0^{\text{min}}$ for the existence of the critical points $q_s$ and $q_t$ related to the snap-through buckling. At $q_0 = q_0^{\text{min}}$, both $q_s$ and $q_t$ are near 0. So for a first approximation, we take $q_1 = 0$ in equation (3.1) and find

$$q_0^{\text{min}} = \left(\frac{b_{11}}{\alpha s_{11}^2} + \left(\frac{b_{11} m_1}{\alpha s_{11}^2 f_1^*}\right)^2 - \frac{b_{11} m_1^*}{\alpha s_{11}^2 f_1^*}\right) \tag{3.3}$$

where the expressions and values of $b_{11}$ and $s_{11}$ are given in table 3; $f_1^*$ and $m_1^*$ are

$$f_1^* = \left(1 + 0.265 \left(\frac{b}{h}\right)^{-0.75} \left(h\right)^{-0.75} + 0.53 \left(\frac{b}{h}\right)^{-1} \left(h\right)^{-0.5}\right) \int_0^1 \phi_1 d\bar{x}, \tag{3.4a}$$

and

$$m_1^* = \left(1 + 0.165625 \left(\frac{b}{h}\right)^{-0.75} \left(h\right)^{-0.75} + 0.3975 \left(\frac{b}{h}\right)^{-1} \left(h\right)^{-0.5}\right) \int_0^1 \phi_2^2 d\bar{x}. \tag{3.4b}$$

When the dimensionless initial arch rise $q_0$ exceeds a critical value $q_0^{\text{max}}$, the voltage parameter $\beta_v^{\text{snap-through}}$ at the snap-through buckling (point $q_s$) becomes larger than that $\beta_v^{\text{pull-in}}$ at the pull-in instability (point $q_p$), so the snap-through and the pull-in take place simultaneously (figure 3c in §3a). To determine the critical value $q_0^{\text{max}}$, we adopt the following procedures: (i) numerically solve equation (3.1) to obtain the evolutions of the extreme points $q_s(q_0)$ and $q_t(q_0)$ with $q_0$. (ii) Take $q_1 = q_s(q_0)$ and $q_0(q_0)$, respectively, in equation (2.43) to obtain the evolutions of the voltage parameters $\beta_v^{\text{snap-through}}(q_0)$ and $\beta_v^{\text{pull-in}}(q_0)$ at the extreme points. (iii) Compare $\beta_v^{\text{snap-through}}(q_0)$ with $\beta_v^{\text{pull-in}}(q_0)$, and determine the critical value $q_0^{\text{max}}$ at which $\beta_v^{\text{snap-through}} = \beta_v^{\text{pull-in}}$. The obtained $q_0^{\text{max}}$ at different levels of the stretching parameter $\alpha$ and width-to-thickness ratio $b/h$ is shown in figure 6, from which it is found that $q_0^{\text{max}}$ varies slightly (0.39–0.44) over the wide ranges of $\alpha$ (54–2400 with $\bar{h} = 1/3–1/20$ and $l/h = 0–1$) and $b/h$ (1.5–100). Because the relative variation of $q_0^{\text{max}}$ is within 10%, we neglect the dependence of $q_0^{\text{max}}$ on $\alpha$ and $b/h$, and take: $q_0^{\text{max}} = 0.4$. With
Equation (3.3), we obtain the following domain of the dimensionless initial arch rise $q_0$ for the existence/observation of snap-through buckling:

$$\sqrt{\frac{b_{11}}{\alpha^2_{11}}} + \left(\frac{b_{11}m_{11}^*}{\alpha^2_{11}f_{11}}\right)^2 - \frac{b_{11}m_{11}^*}{\alpha^2_{11}f_{11}} \leq q_0 \leq 0.4.$$  (3.5)

Equation (3.5) shows that the lower bound of $q_0$ (terms on the left of $q_0$) depends on the stretching parameter $\alpha$, which is a function of the length-scale parameter $l$ (table 1). So, the size effect is implicitly included in equation (3.5). In the rest of this section, we derive a criterion of snap-through buckling from equation (3.5), which is explicitly expressed in term of $l$.

By introducing equation (3.4) into equation (3.5) and replacing the non-dimensional quantities ($\alpha$, $q_0$) with the expressions in table 1, we obtain the following criterion for the existence/observation of snap-through buckling:

$$\left(\frac{r}{h}\right)_{\text{min}} \leq \frac{r}{h} \leq \left(\frac{r}{h}\right)_{\text{max}}.$$  (3.6)

The minimum and maximum allowable ratios ($\left(\frac{r}{h}\right)_{\text{min}}$, $\left(\frac{r}{h}\right)_{\text{max}}$) between the initial arch rise $r$ and beam thickness $h$ are given by

$$\left(\frac{r}{h}\right)_{\text{min}} = \frac{b_{11}}{\alpha^2_{11}} \left[1 + \frac{6}{1 + v} \left(\frac{l}{h}\right)^2\right] + \left(\frac{b_{11}m_{11}}{6\alpha^2_{11}f_{11}}\right)^2 \left[1 + \frac{6}{1 + v} \left(\frac{l}{h}\right)^2\right]^2 \left[1 + 0.165625(b/h)^{-0.75} (\bar{h}^{-0.75} + 0.3975(b/h)^{-1} (\bar{h}^{-0.5})^{-0.5})\right]$$

and

$$\left(\frac{r}{h}\right)_{\text{max}} = 0.4(\bar{h}^{-1}).$$  (3.7b)
where the values of the constants $b_{11}$, $s_{11}$, $f_1$ and $m_{11}$ are given in Table 3; $(\tilde{h}) = h/g_0$ is the dimensionless beam thickness.

It is noted that the ratio $(r/h)$ should also be below a critical value to suppress the asymmetric second mode [11,46]. By comparing with the symmetry breaking criterion [8], we find that for the small gap-to-thickness ratio $g_0/h$ (less than or equal to 4.5), the maximum allowable ratio $(r/h)_{\text{max}}$ from our criterion is smaller than that from the symmetry breaking criterion. So, $(r/h)_{\text{max}}$ of our criterion can be used at the small ratio $g_0/h$ (≤4.5), whereas at large ratio, $g_0/h$ (>4.5), $(r/h)_{\text{max}}$ from the symmetry breaking criterion [8] should be used.

The size effect (by introducing the length-scale parameter $l$, normalized as $l/h$) and the fringing field effect (considering the finite beam width $b$, normalized as $b/h$) on the minimum allowable ratio $(r/h)_{\text{min}}$ are shown, respectively, in Figure 7a,b from equation (3.7a). Both effects increase $(r/h)_{\text{min}}$ and the size effect is much more significant. Equation (2.31) shows that the size effect $(l/h)$ increases the effective bending stiffness, so the microbeam becomes stiffer and more difficult to exhibit snap-through buckling. As a result, the minimum allowable ratio $(r/h)_{\text{min}}$ increases.

With $l/h = 0.04$ and $b/h = 12$, the snap-through criterion (i.e. equation (3.6): domain of the ratio $(r/h)$ between the initial arch rise and beam thickness) is plotted in Figure 7c. It is seen that for the existence of the domain $(r/h)$ of snap-through buckling, the gap-to-beam-thickness ratio $g_0/h$ should be large enough (e.g. $g_0/h > 2$ in Figure 7c). The experimental results in the literature [11,46] are also shown in the figure, and it is found that most of the observed snap-through buckling takes place within the predicted area. The snap-through criterion can be used as a design guideline for the bistable MEMS based on the initially curved microbeam: consider a microbeam system of prescribed dimensions (beam thickness $h$ and width $b$) and made of prescribed material (silicon, polysilicon, epoxy, etc.), determine the length-scale parameter $l$ of the material, and then with the calculated $(b/h)$ and $(l/h)$, plot equation (3.6) in a figure similar to Figure 7c. The initial arch rise $r$ of the microbeam and the gap $g_0$ between the beam ends and the rigid electrode can be chosen in the snap-through area on the figure.

It is noted at last that we can compare the experiments of the two research groups in the same figure (Figure 7c) by taking the same values of $l/h$ and $b/h$ in equation (3.7a), because (i) the microbeams tested in these two groups are made of the same material—silicon, and the order
of the material length-scale parameter $l$ for silicon is $10^{-1}\mu m$ [47]. Moreover, the beam thicknesses $h$ are very close: $h = 2.5 [11]$ and $2.6 \mu m$ [46]. So, we choose one value of $l/h = 10^{-1}\mu m/2.5\mu m = 0.04$. (ii) The width-to-thickness ratios $b/h$ are also close: $b/h = 12 [11]$ and $19 [46]$, and it is seen from figure 7b that the variation of $(r/h)_{\text{min}}$ over a wide range of $b/h (2.5 \sim +\infty)$ is very small. So, we need to consider only one value of $b/h = 12$.

4. Conclusions

This paper is concerned with a unified study of the snap-through behaviour of an initially curved microbeam subjected to an electrostatic force and accounting for fringing field and intermolecular effects. The governing equations were developed with the aid of Euler–Bernoulli beam theory and used to develop a new snap-through criterion in terms of the microbeam system dimensions; accounting for the beam size and the fringing field effect in the development of that criterion. The governing equations were solved using the Galerkin decomposition method and used to develop a limit design chart for the characteristic snap-through behaviour of the beam. Our results, which are based on the first mode approximation, reveal that the size of the microbeam plays a major role in dictating the existence of the snap-through behaviour of the beam, whereas the fringing field and intermolecular forces play an insignificant role.
It is noted that to derive an analytical snap-through criterion, the first mode approximation of the beam deflection was taken. So the model can only approximately describe the evolution of the beam deflection with the applied voltage. To accurately describe the deflection evolution, more modes (greater than or equal to 5) should be considered, and some cases such as non-ideal clamping conditions and imperfections in the beam shape should also be taken into account. The derived snap-through criterion is only valid for the microbeam systems with gap-to-beam-thickness ratio smaller than 4.5, whereas for larger ratios, the symmetry breaking criterion [8] with two modes considered should be used. Another limitation is that the work is only valid for long thin beams (beam thickness \( \ll \) beam length) where the Euler–Bernoulli beam theory can be applied. Moreover, the Meijs–Fokkema formula for narrow beams was used to take into account the fringing field effect. To ensure the application of the formula, this work only studied the microbeam systems with the dimensions satisfying beam-width-to-gap ratio larger than 0.5 and beam-thickness-to-gap ratio smaller than 5 (i.e. gap-to-beam-thickness ratio larger than 0.2).

Data accessibility. There is no data to report in this manuscript.

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References


36. van der Meijs NP, Fokkema JT. 1984 VLSI circuit reconstruction from mask topology. Integr. VLSI J. 2, 85–119. (doi:10.1016/0167-9260(84)90016-6)


