A model for cells structured by size undergoing growth and division leads to an initial boundary value problem that involves a first-order linear partial differential equation with a functional term. Here, size can be interpreted as DNA content or mass. It has been observed experimentally and shown analytically that solutions for arbitrary initial cell distributions are asymptotic as time goes to infinity to a certain solution called the steady size distribution. The full solution to the problem for arbitrary initial distributions, however, is elusive owing to the presence of the functional term and the paucity of solution techniques for such problems. In this paper, we derive a solution to the problem for arbitrary initial cell distributions. The method employed exploits the hyperbolic character of the underlying differential operator, and the advanced nature of the functional argument to reduce the problem to a sequence of simple Cauchy problems. The existence of solutions for arbitrary initial distributions is established along with uniqueness. The asymptotic relationship with the steady size distribution is established, and because the solution is known explicitly, higher-order terms in the asymptotics can be readily obtained.

1. Introduction

This paper concerns the study of a first-order functional partial differential equation (PDE) that arises in a model of cell growth, where the cell population is structured by size. Here ‘size’ can be mass or DNA content. Let \( n(x, t) \) denote the density distribution of cells structured by size \( x \) at time \( t \). The differential equation models cells that are...
each growing at a constant rate \( g > 0 \), and dividing at a constant rate \( b > 0 \) into \( \alpha > 1 \) daughter cells of size \( x/\alpha \) (usually \( \alpha = 2 \)). The cells are also assumed to have a constant per capita death rate \( \mu > 0 \). A mass balance across the cell cohort yields

\[
\frac{\partial}{\partial t} n(x,t) + g \frac{\partial}{\partial x} n(x,t) = \frac{\alpha^2 b n(a x, t)}{\alpha^2 x} - \frac{b n(x, t)}{a^2} - \mu n(x, t). \tag{1.1}
\]

The above equation is supplemented by a given initial distribution

\[ n(x,0) = n_0(x), \tag{1.2} \]

where \( n_0 \) is a probability distribution function, and the boundary condition,

\[ n(0,t) = 0. \tag{1.3} \]

This problem is thus of the initial boundary value type involving a functional differential equation. Equation (1.1) was derived in detail by Sinko & Streifer [1,2].

The cell division problem was studied by Hall & Wake [3]. The motivation for the study came from experimental results for certain plant cells [4] that suggested solutions of the type

\[ n(x,t) = w(t) y(x), \tag{1.4} \]

at least as a long-term approximation. Here, \( y \) is a probability density function. Hall and Wake called this solution ‘the steady size distribution’ (SSD) and showed that it was unique. The SSD solution brings to the fore a connection with the well-known pantograph equation. Briefly, substituting this solution form (1.4) into equation (1.1) yields

\[ w(t) = k e^{(\lambda - \mu) t}, \]

where \( k \) is a constant and \( \lambda \) is a constant arising from the separation of variables. The function \( y \) satisfies

\[ g y' + (b + \lambda) y = b a^2 y(a x), \tag{1.5} \]

along with the conditions \( y(0) = 0, y(x) \to 0 \) as \( x \to \infty \), and

\[ \int_0^\infty y(x) \, dx = 1. \]

This leads to

\[ \lambda = b(a - 1). \]

Equation (1.5) is an example of the pantograph equation, which arises in a number of applications including the collection of current in an electric train [5], light absorption in the Milky Way [6] and a ruin problem [7]. A detailed analysis of the equation is given in [8,9], and the equation has been studied in the complex plane [10–12]. We note also that the equation has been studied in the context of probability [13] and the cell growth problem has been interpreted in this framework [14]. Similar functional PDEs have been studied in [15,16], with an advanced non-local argument. Both papers are devoted to the classes of uniqueness for Cauchy problem to PDE with linearly transformed arguments.

The cell division problem has been generalized to include dispersion [17,18] and this led to the study of second-order pantograph equations [19]. The problem has also been studied for certain non-constant coefficients [20,21], and a multi-compartment model has been developed for an application to the treatment of cancer [22].

All the studies focused exclusively on SSD solutions for cases where the eigenvalue can be determined explicitly. In general, the separation constant \( \lambda \) cannot be determined explicitly for given functions \( b(x) \) and \( g(x) \), and this prompts questions concerning the existence of an eigenvalue and corresponding positive eigenfunction. These results have been established for more general aggregation-fragmentation models by Doumic & Gabriel [23]. Some further results
for more general choices of $b$ and $g$ were obtained by da Costa et al. [24], Perthame & Ryzhik [25] and Michel et al. [26]. In particular, Perthame and Ryzhik proved the existence of a positive eigenfunction for a class of division rate functions that are positive, bounded and bounded away from zero. Under suitable decay conditions, they also showed that any solution to the cell division problem is asymptotic to this eigenfunction as $t \to \infty$.

Although much is known about SSD solutions to the cell division problem, little is known about the solution to the problem for a given initial distribution, except that it is asymptotic to the SSD solution. In §2, we obtain a solution valid for $x \geq t$. We then use this solution to derive a solution that is valid for $0 \leq x < t$ in §3. This solution is given in terms of an arbitrary function $G_0$ and thereby glean the time asymptotics for the solution including higher order terms. We show that as $t \to \infty$,

$$n(x, t) \sim e^{bt} \theta_0(x) + e^{bt} \theta_1(x) + O(e^{(b/a)t}),$$

where the $\theta_k(x)$ are given by Dirichlet series that decay rapidly as $x \to \infty$. These $\theta_k$, in fact, correspond to a known class of eigenfunctions associated with the pantograph equation. Uniqueness of the solution is established in §5.

2. The existence of a solution for $x \geq t$

We begin the construction of a general solution to the cell division problem by first constructing a solution that is valid for $x \geq t \geq 0$. The hyperbolic character of the differential equation and the nature of the initial data prompt the study of solutions in this region, which after a simple transformation is the domain of definition for the data. In the absence of the functional term $n(\alpha x, t)$, equation (1.1) is a classical Cauchy problem that can be solved readily by the method of characteristics. The functional term complicates matters, but since $\alpha > 1$, the domain of definition remains the same. The strategy is to define the solution as a series of functions each of which satisfies a simple Cauchy problem that can be readily solved.

We establish the existence of a non-negative solution $n$ to equation (1.1) that satisfies equation (1.2) under moderate conditions for the initial data. Specifically, we assume that $n_0$ is a bounded probability density function.

Before we embark on constructing the solution, we make some simplifications to the differential equation (1.1). Let

$$n(x, t) = e^{-(b+\mu)t} \tilde{n}(x, t).$$

Then,

$$\frac{\partial}{\partial t} \tilde{n} + g \frac{\partial}{\partial x} \tilde{n} = b\alpha^2 \tilde{n}(\alpha x, t),$$

and this can be further simplified using the transformation $x = \hat{g}\hat{x}$ to obtain

$$\frac{\partial}{\partial \hat{t}} \hat{n}(\hat{x}, \hat{t}) + \frac{\partial}{\partial \hat{x}} \hat{n}(\hat{x}, \hat{t}) = \alpha^2 b \hat{n}(\alpha \hat{x}, \hat{t}),$$

where $\hat{n}(\hat{x}, \hat{t}) = \tilde{n}(g\hat{x}, t)$. Dropping circumflexes and tildes, it is clear that we can reduce the functional differential equation problem to

$$\frac{\partial}{\partial \hat{t}} n(\hat{x}, \hat{t}) + \frac{\partial}{\partial \hat{x}} n(\hat{x}, \hat{t}) = \alpha^2 b n(\alpha \hat{x}, \hat{t}),$$

and retain conditions (1.2) and (1.3).
If we restrict our attention to solutions of (2.1) that are integrable with respect to $x$ on $[0, \infty)$ for any fixed $t > 0$, then the transformation

$$m(x, t) = \int_x^\infty n(\xi, t) \, d\xi,$$  
(2.2)

yields

$$\frac{\partial}{\partial t} m(x, t) + \frac{\partial}{\partial x} m(x, t) = b\alpha m(\alpha x, t).$$  
(2.3)

Integrating equation (2.1) from 0 to $\infty$ w.r.t $x$ and applying condition (1.3) gives

$$\frac{\partial}{\partial t} m(0, t) = b\alpha m(0, t).$$

The initial distribution $n_0$ can be regarded as a probability density function so that $m(0, 0) = 1$; therefore,

$$m(0, t) = e^{b\alpha t}.$$  
(2.4)

The initial condition for equation (2.3) is

$$m(x, 0) = m_0(x) = \int_x^\infty n_0(\xi) \, d\xi.$$  
(2.5)

It turns out that it is easier to work with the ‘cumulative density function’ $m$ for the extension to $0 \leq x \leq t$. We construct a solution $n$ and it is clear that the same construction will work for equation (2.3).

**Theorem 2.1 (Existence of solution for $x \geq t$).** Let $W_0 \subseteq \mathbb{R}^2$ denote the set $\{(x, t) : x \geq t \geq 0\}$. There exists a non-negative solution $Q$ to equation (2.1) that satisfies condition (1.2) and is valid for $(x, t) \in W_0$.

**Proof.** We construct a sequence of functions $\{N_k(x, t)\}$, defined by a sequence of PDEs such that

$$Q(x, t) = \sum_{k=0}^\infty N_k(x, t)$$  
(2.6)

is a solution to equation (2.1) that satisfies the initial condition (1.2) and is valid for $x \geq t$. The functional differential equation problem can be converted to a sequence of Cauchy problems by defining the following sequence:

$$N_0(x, t) = n_0(x - t),$$

and for $k \geq 1$,

$$\frac{\partial}{\partial t} N_k(x, t) + \frac{\partial}{\partial x} N_k(x, t) = b\alpha^2 N_{k-1}(\alpha x, t),$$  
(2.7)

with

$$N_k(x, 0) = 0.$$  
(2.8)

Note that $N_0$ satisfies the Cauchy problem

$$\frac{\partial}{\partial x} n(x, t) + \frac{\partial}{\partial t} n(x, t) = 0, \quad n(x, 0) = n_0(x),$$
and each problem given by equation (2.7) and condition (2.8) is a Cauchy problem that can be solved by the method of characteristics. The characteristic projections $\xi$ and $\eta$ are given by $\xi = t$ and $\eta = x - t$. In terms of $\xi$ and $\eta$, let

$$N_k(x, t) = N_k(\xi + \eta, \xi) = \tilde{N}_k(\xi, \eta)$$

and

$$N_k(\alpha x, t) = N_k(\alpha \xi + \alpha \eta, \xi) = \tilde{N}_k(\xi, \eta).$$

For simplicity, we drop the circumflex when there is no danger of confusion, but retain the bar to denote an advanced argument. Now,

$$\frac{\partial}{\partial \xi} N_k(\xi, \eta) = b\alpha^2 \tilde{N}_{k-1}(\xi, \eta),$$

so that the solution to (2.7) that satisfies (2.8) is

$$N_k(\xi, \eta) = b\alpha^2 \int_0^\xi \tilde{N}_{k-1}(\sigma, \eta) \, d\sigma. \quad (2.9)$$

Now, $N_0(x, t) = n_0(\eta)$, and therefore

$$N_1(\xi, \eta) = b\alpha^2 \int_0^\xi \tilde{N}_0(\sigma, \eta) \, d\sigma$$

$$= \frac{b\alpha^2}{\alpha - 1} \int_{\alpha \eta}^{\alpha(\xi - t)} n_0(w) \, dw. \quad (2.10)$$

In terms of $x$ and $t$,

$$N_1(x, t) = \frac{b\alpha^2}{\alpha - 1} \int_{\alpha(x - t)}^{\alpha x - t} n_0(w) \, dw$$

$$= \frac{b\alpha^2}{\alpha - 1} (T_1(\alpha x - t) - T_1(\alpha(x - t))),$$

where $T_1$ is an antiderivative of $n_0$. It is straightforward to show that

$$N_k(x, t) = \sum_{j=0}^k d_{k,j} T_k(w_{k,j}(x, t)), \quad (2.11)$$

where $T_0(w) = n_0(w), T_{k+1}'(w) = T_k(w)$; and $d_{0,0} = 1$,

$$d_{k,j} = \frac{b\alpha^2 d_{k-1,j-1}}{\alpha^{k-j}(\alpha^j - 1)},$$

and

$$d_{k,0} = -\sum_{j=1}^k d_{k,j}.$$
converges uniformly in any set of $W_0$ of the form \( \{(x, t) \in W_0 : t \leq D \} \), where $D$ is any fixed positive number. Let $M$ be an upper bound for $n_0$. Then $N_0(x, t) \leq M$; hence, equation (2.10) implies

$$N_1(x, t) \leq \frac{b \alpha^2 M}{\alpha - 1} \int_{a(x-t)}^\alpha dw = b \alpha^2 Mt,$$

i.e.

$$N_1(\xi, \eta) \leq b \alpha^2 M \xi.$$

We can continue in this manner to show that

$$N_k(x, t) \leq M \left( b \alpha^2 t \right)^k \frac{1}{k!}$$

for all $x \geq t$, and this leads to

$$\sum_{k=0}^\infty N_k(x, t) \leq M \sum_{k=0}^\infty \left( b \alpha^2 t \right)^k \frac{1}{k!} = Me^{b \alpha^2 t}.$$

The series thus converges uniformly in any set $\{(x, t) \in W_0 : 0 \leq t \leq D \}$. Evidently, $Q(x, t) \geq 0$ for all $(x, t) \in W_0$ and is a solution to equation (2.1) that satisfies condition (1.2).  

Although a solution to equation (2.3) can be gleaned from $Q$, this equation could also be solved directly using the same approach. In particular, the solution found by integrating $Q$ is also given by a solution to equation (2.3) for $x \geq t$ that satisfies condition (2.5) and is of the form

$$P_0(x, t) = \sum_{k=0}^\infty M_k(x, t),$$

where $M_k(x, t) = \sum_{j=0}^k c_{k,j} F_k(w_{k,j}(x, t))$.

$$F_0(w) = m_0(w)$$

and

$$F_{k+1}'(w) = F_k(w).$$

Here, the $c_{k,j}$ are given by $c_{0,0} = 1$, and for $k \geq 1$ and $j = 1, \ldots, k$,

$$c_{k,j} = \frac{b \alpha c_{k-1,j-1}}{\alpha^{k-j}(\alpha^j - 1)}$$

and

$$c_{k,0} = - \sum_{j=1}^k c_{k,j}.$$  

### 3. Extension of the solution for $0 \leq x < t$

We use the solution constructed to equation (2.3) in §2 to construct a solution that also satisfies condition (2.4) and is valid for all $x \geq 0$, $t \geq 0$. The functional character of equation (2.3) can be exploited to continue the solution (2.12) via a sequence of ‘wedges’. For $n \geq 1$, let

$$W_n = \left\{ (x, t) : \frac{t}{\alpha^n} \leq x \leq \frac{t}{\alpha^{n-1}} \right\}$$

(figure 1). The key here is that equation (2.3) is not functional in $W_n$ if the solution is known in $W_{n-1}$. In this case, the problem reduces to a non-homogeneous first-order linear PDE that can be readily solved. It is required that the solution be continuous across the wedge boundaries, and this provides the initial data. The extension to $W_1$ differs from the other extensions in that the initial data are on the characteristic projection $x = t$. The first extension thus introduces an arbitrary function. Further extensions induce non-characteristic data so that there is only one
Figure 1. Constructing wedges to extend the solution to $0 \leq x < t$. (Online version in colour.)

arbitrary function in the construction. In the next section, we use condition (2.4) to determine this function.

The solution to equation (2.3) valid in the wedge $W_n$ will be denoted by $h_n$ for $n \geq 1$. In addition, we introduce the notation

$$P_n = \sum_{k=n}^{\infty} \sum_{j=n}^{k} c_{k,j} F_k(w_{k,j})$$

(3.1) for $n \geq 0$. If $(x, t) \in W_1$ then $(\alpha x, t) \in W_0$. The function $h_1$ thus satisfies

$$\frac{\partial}{\partial t} h_1 + \frac{\partial}{\partial x} h_1 = b\alpha P_0(\alpha x, t).$$

In characteristic coordinates, the above PDE is

$$\frac{\partial}{\partial \xi} h_1 = b\alpha \sum_{k=0}^{\infty} \sum_{j=0}^{k} c_{k,j} F_k(\tilde{w}_{k,j}),$$

and using the relation $\tilde{w}_{k,j} = w_{k+1,j+1}$ along with equation (2.14) we have

$$h_1(\xi, \eta) = \sum_{k=0}^{k} \sum_{j=0}^{k} \frac{b\alpha c_{k,j}}{\alpha^{k-j}(\alpha^j - 1)} F_{k+1}(w_{k+1,j+1}) + G_0(w_{0,0}),$$

where $G_0$ is an arbitrary function of $w_{0,0} = \eta$. Equation (2.15) thus implies

$$h_1(\xi, \eta) = P_1(\xi, 0) + G_0(0).$$

(3.2)

We require the solution to be continuous across the characteristic projection $\eta = 0$, and this condition will not determine $G_0$ uniquely. Now,

$$\lim_{\eta \to 0^-} h_1(\xi, \eta) = P_1(\xi, 0) + G_0(0)$$

and

$$\lim_{\eta \to 0^+} P_0(\xi, \eta) = P_1(\xi, 0) + \lim_{\eta \to 0^+} \sum_{k=0}^{\infty} c_{k,0} F_k(w_{k,0}).$$

The continuity condition and the relation $w_{k,0} = \alpha^k \eta$ thus give

$$G_0(0) = \lim_{\eta \to 0^+} \sum_{k=0}^{\infty} c_{k,0} F_k(0).$$

The function $F_{k+1}$ can be any antiderivative of $F_k$, since condition (2.16) ensures that condition $M_k(x, 0) = 0$ is satisfied. We can thus choose the $F_k$ such that $F_k(0) = 0$ for $k \geq 1$. Equation (2.13), however, requires that $F_0(0) = 1$. With this choice of $F_k$, we thus have $G_0(0) = 1.
We now consider the extension of the solution to the wedge $W_2$ and from this analysis extract a general form for $h_n$. Proceeding as before

$$\frac{\partial}{\partial \xi} h_2 = b\alpha h_1(\xi, \eta),$$

which by using equation (3.2) gives

$$\frac{\partial}{\partial \xi} h_2 = \tilde{P}_1(\xi, \eta) + \tilde{G}_0(\eta). \tag{3.3}$$

Integrating equation (3.3) with respect to $\xi$, using the definition of $F_n$ and the recursion relation (2.15) leads to

$$h_2(\xi, \eta) = P_2(\xi, \eta) + \frac{\alpha G_1(w_{1,1})}{\alpha - 1} + H(\eta),$$

where $H$ is an arbitrary function and $G'_1(u) = G_0(u)$. (The $1/(\alpha - 1)$ factor comes from $w_{1,1} = (\alpha - 1)\xi + \alpha\eta$.) The function $G_1$ can be any antiderivative of $G_0$, so we choose $G_1$ such that $G_1(0) = 0$. To get $H$, we impose the continuity condition on the line $x = t/\alpha$, i.e. $w_{1,1} = 0$. Thus,

$$\lim_{w_{1,1} \to 0^-} h_2 = \lim_{w_{1,1} \to 0^-} P_2 + \lim_{w_{1,1} \to 0^-} \frac{\alpha G_1(0)}{\alpha - 1} + \lim_{w_{1,1} \to 0^-} H(\eta)$$

and

$$\lim_{w_{1,1} \to 0^+} h_1 = \lim_{w_{1,1} \to 0^+} P_2 + \lim_{w_{1,1} \to 0^+} \sum_{k=1}^{\infty} c_k F_k(w_{k,1}) + \lim_{w_{1,1} \to 0^+} G_0(\eta).$$

Since

$$w_{k,1} = \alpha^{k-1}(\alpha - 1)\xi + \alpha^k\eta = \alpha^{k-1}w_{1,1},$$

we have

$$\lim_{w_{1,1} \to 0^+} \sum_{k=1}^{\infty} c_k F_k(w_{k,1}) = \sum_{k=1}^{\infty} c_k F_k(0) = 0.$$

The continuity of $P_2$ in $W_2 \cup W_1$ means that

$$\lim_{w_{1,1} \to 0^-} P_2 = \lim_{w_{1,1} \to 0^+} P_2,$$

and the continuity condition

$$\lim_{w_{1,1} \to 0^-} h_2 = \lim_{w_{1,1} \to 0^+} h_1$$

yields

$$\lim_{w_{1,1} \to 0^-} H(\eta) = \lim_{w_{1,1} \to 0^+} G_0(\eta), \tag{3.4}$$

since $G_1(0) = 0$. Now, $w_{1,1} = 0$ implies

$$\eta = -\frac{(\alpha - 1)}{\alpha} \xi,$$

and this means condition (3.4) must be satisfied for all $\xi$ on the line $w_{1,1} = 0$, i.e. for all $\xi$ on this line

$$H\left(-\frac{(\alpha - 1)}{\alpha} \xi\right) = G_0\left(-\frac{(\alpha - 1)}{\alpha} \xi\right).$$

We thus conclude that $H(u) = G_0(u)$, and the solution is thus

$$h_2 = P_2 + \frac{\alpha G_1(w_{1,1})}{\alpha - 1} + G_0(w_{0,0}).$$

We can determine $h_3$ in a similar manner to get

$$h_3 = P_3 + \frac{(b\alpha)^2}{(\alpha - 1)(\alpha^2 - 1)} G_2(w_{2,2}) + \frac{b\alpha}{\alpha - 1} G_1(w_{1,1}) + G_0(w_{0,0}).$$
where \( G_1'(u) = G_1(u) \) and \( G_2(0) = 0 \). For the general wedge \( W_n, n \geq 2 \), we find

\[
h_n = P_n + G_0(w_{0,0}) + \sum_{k=1}^{n-1} \frac{(b\alpha)^k}{\prod_{m=1}^{k}(\alpha^m - 1)} G_k(w_{k,k}),
\]

(3.5)

where, for \( k \geq 1 \), \( G_{k+1}'(u) = G_k(u) \) and \( G_k(0) = 0 \). A solution \( m \) to equation (2.3) that satisfies the initial condition (2.5) can thus be defined piecewise by the sequence \( \{h_n\} \), viz.

\[
m(x, t) = \begin{cases} 
  h_0 \equiv P_0, & \text{if } (x, t) \in W_0 \\
  h_1 = P_1 + G_0, & \text{if } (x, t) \in W_1 \\
  \vdots \\
  h_n = P_n + G_0(w_{0,0}) + \sum_{k=1}^{n-1} \frac{(b\alpha)^k}{\prod_{m=1}^{k}(\alpha^m - 1)} G_k(w_{k,k}), & \text{if } (x, t) \in W_n. \\
  \vdots 
\end{cases}
\]

(3.6)

By construction, the solution is continuous on the wedge boundaries. If the initial data \( m_0 \) is smooth, then the construction also shows that \( m_\xi \) is smooth for \( 0 \leq t/\alpha^n \leq x \). The function \( G_0 \) in solution (3.6) is arbitrary. If it is required that \( m \) have a continuous derivative with respect to \( \eta \), then \( G_0' \) would have to be continuous but this does not ensure continuity on the line \( \eta = 0 \). Now,

\[
\frac{\partial}{\partial \eta} h_0 = \sum_{k=0}^{\infty} c_{k,0} F_k'(w_{k,0}) \alpha^k + \frac{\partial}{\partial \eta} P_1
\]

and

\[
\lim_{\eta \to 0^+} \frac{\partial}{\partial \eta} h_0 = c_{1,0} F_1'(0) + \lim_{\eta \to 0^+} \frac{\partial}{\partial \eta} P_1 = \frac{-b\alpha^2}{\alpha - 1} + \lim_{\eta \to 0^+} \frac{\partial}{\partial \eta} P_1.
\]

Here we have used \( F_{k+1}'(0) = F_k(0) = 0 \) for \( k \geq 1 \), \( F_1'(0) = F_0(0) = m_0(0) = 1 \) and \( F_0'(0) = m_0'(0) = -n_0(0) = 0 \). The continuity condition

\[
\lim_{\eta \to 0^+} \frac{\partial}{\partial \eta} h_0 = \lim_{\eta \to 0^-} \frac{\partial}{\partial \eta} h_1
\]

thus gives

\[
G_0'(0) = \frac{-b\alpha^2}{(\alpha - 1)}. \tag{3.7}
\]

Similar calculations on the other wedge boundaries show that (3.7) is in fact the only requirement on \( G_0 \) apart from the continuity of \( G_0' \). In the next section, we determine \( G_0 \) and show that it satisfies this continuity condition and condition (3.7).

### 4. The limiting solution and asymptotics as \( t \to \infty \)

In this section, we determine \( G_0 \) from the boundary condition (2.4) at \( x = 0 \). To apply this boundary condition, it is necessary to look at the limiting function \( h_n \) as \( n \to \infty \). For a fixed value of \( t \), the limit \( x \to 0^+ \) corresponds to \( (x, t) \) in \( W_n \) as \( n \to \infty \). We thus consider \( \lim_{n \to \infty} h_n(x, t) \).
The series defining $P_0$ is convergent; hence, $P_n \to 0$ as $n \to \infty$; and

$$h(0, t) = \lim_{n \to \infty} h_n = G_0(-t) + \sum_{k=1}^{\infty} \frac{(b\alpha)^k}{\prod_{m=1}^{k}(a^m - 1)} G_k(-t).$$

Now $h(0, t) = m(0, t) = e^{b\alpha t}$ by condition (2.4), and therefore

$$e^{-b\alpha u} = G_0(u) + \sum_{k=1}^{\infty} \frac{(b\alpha)^k}{\prod_{m=1}^{k}(a^m - 1)} G_k(u), \quad (4.1)$$

where $u = -t$. Taking the Laplace transform of both sides of equation (4.1) gives

$$\frac{1}{s + b\alpha} = f(s) \left(1 + \sum_{k=1}^{\infty} \frac{(b\alpha/s)^k}{\prod_{m=1}^{k}(a^m - 1)}\right), \quad (4.2)$$

where $f(s)$ is the Laplace transform of $G_0$. The infinite series in (4.2) can be converted into an infinite product by use of Euler’s identity [27]

$$\prod_{k=0}^{\infty} (1 + zq^k) = 1 + \sum_{k=1}^{\infty} \frac{zq^k}{k} \prod_{m=1}^{k}(1 - q^m),$$

for $|q| < 1$. Now,

$$1 + \sum_{k=1}^{\infty} \frac{(b\alpha/s)^k}{\prod_{m=1}^{k}(a^m - 1)} = 1 + \sum_{k=1}^{\infty} \frac{(b/s)^k q^k}{\prod_{m=1}^{k}(1 - q^m)},$$

where $q = 1/\alpha$, and we apply Euler’s identity with $z = b/s$ to get

$$\frac{1}{s + b\alpha} = f(s) \prod_{k=0}^{\infty} \left(1 + \frac{b}{a^k s}\right),$$

so that

$$f(s) = \frac{1}{s(1 + b\alpha/s)(1 + b/s)(1 + b/\alpha s)(1 + b/\alpha^2 s) \ldots}.$$

It is clear that $f$ has simple poles at $s = -b\alpha - k$, for $k = -1, 0, 1, 2, \ldots$, and Mittag-Leffler’s theorem [28] implies that $f(s)$ can be represented in the form

$$f(s) = \frac{a_{-1}}{s + b\alpha} + \frac{a_0}{s + b} + \frac{a_1}{s + b/\alpha} + \ldots + r(s),$$

where $r$ is an entire function. The inverse transform of $f$ is therefore

$$G_0(u) = \sum_{n=-1}^{\infty} a_n e^{-b\alpha - n u}$$

where $a_n = \text{Res}_{s=-b/\alpha^n} f(s)$.

Now,

$$a_{-1} = \lim_{s \to -b\alpha} \frac{(s + b\alpha)}{(s + b\alpha)(1 + b/s)(1 + b/\alpha s)(1 + b/\alpha^2 s) \ldots} = \frac{1}{\prod_{k=1}^{\infty} (1 - 1/\alpha^k)} = R(\alpha),$$

and it can be shown that for $k \geq 0$,

$$a_k = \frac{(-1)^{k+1}}{\prod_{m=1}^{k+1}(\alpha^m - 1)} R(\alpha);$$
and it can be confirmed directly from Euler’s identity that $G_0 = 0$. Integrating equation (4.3) yields

$$G_1(u) = -\frac{R(\alpha)}{b\alpha} \left( e^{-b\alpha u} + \sum_{n=1}^{\infty} \frac{(-1)^n n e^{-b\alpha u}}{\prod_{m=1}^{n}(\alpha^m - 1)} \right),$$

and it can be confirmed directly from Euler’s identity that $G_1(0) = 0$. In general, for $n \geq 0$, it can be shown that

$$G_n(u) = \frac{(-1)^n R(\alpha)}{(b\alpha)^n} \left( e^{-b\alpha u} + \sum_{k=1}^{\infty} \frac{(-1)^k \alpha^k}{\prod_{m=1}^{k}(\alpha^m - 1)} \right),$$

and that $G_n(0) = 0$ for $n \geq 1$. Substituting $u = w_n = \alpha^n x - t$ into equation (4.4) yields

$$G_n(\alpha^n x - t) = \frac{(-1)^n R(\alpha)}{(b\alpha)^n} \left( e^{-b\alpha^n x + b\alpha t} + \sum_{k=1}^{\infty} \frac{(-1)^k \alpha^{kn} e^{-b\alpha^{kn}x + b\alpha^{kn}t}}{\prod_{m=1}^{k}(\alpha^m - 1)} \right),$$

and the limiting function is therefore

$$h(x, t) = R(\alpha) \left( \frac{\alpha}{\alpha - 1} \right)^n \left( e^{-b\alpha x} + \sum_{n=1}^{\infty} \frac{(-1)^n \alpha^n e^{-b\alpha x}}{\prod_{m=1}^{n}(\alpha^m - 1)} \right),$$

The above series can be rearranged to collect the factors of $e^{b\alpha x}$, $e^{b\alpha t}$, $e^{b\alpha t/\alpha}$, etc. In particular, the $e^{b\alpha x}$ term is

$$e^{b\alpha x} \left( e^{-b\alpha x} + \sum_{n=1}^{\infty} \frac{(-1)^n \alpha^n e^{-b\alpha x}}{\prod_{m=1}^{n}(\alpha^m - 1)} \right) = e^{b\alpha x} V_0(x),$$

and the $e^{b\alpha t}$ term is

$$-\frac{e^{b\alpha t}}{(\alpha - 1)} \left( e^{-b\alpha x/\alpha} + \sum_{n=1}^{\infty} \frac{(-1)^n \alpha^n e^{-b\alpha x/\alpha}}{\prod_{m=1}^{n}(\alpha^m - 1)} \right) = e^{b\alpha t} V_1(x).$$

In general,

$$-\frac{(-1)^k e^{b\alpha t/\alpha^k}}{\prod_{m=1}^{k}(\alpha^m - 1)} \left( e^{-b\alpha x/\alpha^k} + \sum_{n=1}^{\infty} \frac{(-1)^n \alpha^n e^{-b\alpha x/\alpha^k}}{\prod_{m=1}^{n}(\alpha^m - 1)} \right) = e^{b\alpha t/\alpha^k} V_k(x),$$

and the limit function can thus be expressed as

$$h(x, t) = R(\alpha) \sum_{k=0}^{\infty} e^{b\alpha t/\alpha^k} V_k(x).$$

We note that the constant $R(\alpha)$ can be evaluated or represented a number of ways. Hall & Wake [3] show that Euler’s pentagonal number theorem can be invoked to convert this product to an infinite series. They also note a representation of $R(\alpha)$ in terms of a Jacobi elliptic function. More generally, Morgan [29] considers this constant as a special case and shows that it can be represented in terms of a Dedekind eta function or implicitly in terms of a theta function.
Finally, we note that $G_0$ is a smooth function and

$$G_0'(0) = -b\alpha R(\alpha) \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha^n \prod_{m=1}^{n}(\alpha^m - 1)} \right\}.$$

Euler’s identity implies

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha^n \prod_{m=1}^{n}(\alpha^m - 1)} = \prod_{k=0}^{\infty} \left( 1 - \frac{1}{\alpha^{k+2}} \right) = \frac{1}{(1 - 1/\alpha)R(\alpha)},$$

so that equation (3.7) is satisfied. In summary, we have the following result.

**Theorem 4.1 (Existence).** A solution $m$ to equation (2.3) that satisfies conditions (2.4) and (2.5) is given by equation (3.6), where the $P_n$ are defined by (3.1) and $G_0$ is defined by (4.3). The smoothness of this solution is limited only by the smoothness of the initial function $m_0$.

It is known (cf. [25]) that any solution to equation (1.1) that satisfies conditions (1.2) and (1.3) also satisfies

$$n(x,t) \sim e^{bat}y(x)$$

as $t \to \infty$. Here, $y$ is the steady size distribution derived by Hall & Wake [3]. Thus, for any initial probability density function $n_0$, the ‘long-term’ solution approaches asymptotically the same function. We can deduce this asymptotic relationship directly from our general solution.

Fix any $x > 0$. The solution $m$ is given by a function in the sequence $\{hn\}$, and it is clear that

$$n(x,t) \to \infty$$

as $t \to \infty$. Here, $y$ is the steady size distribution derived by Hall & Wake [3]. Thus, for any initial probability density function $n_0$, the ‘long-term’ solution approaches asymptotically the same function. We can deduce this asymptotic relationship directly from our general solution.

Finally, we note that the Dirichlet series defined by the $V_k$ correspond to the eigenfunctions derived by van-Brunt and Vlieg-Hulstman [30,31] for the pantograph equation.

5. **Uniqueness**

We show the solution $m$ in theorem 4.1 is unique. Suppose that $m_1$ and $m_2$ are distinct solutions to equation (2.3) that satisfy equations (2.4) and (2.5). Let $u(x,t) = m_1(x,t) - m_2(x,t)$. Then $u$ satisfies

$$u_t + u_x = b\alpha u(\alpha x, t),$$

along with $u(x,0) = 0$, and $u(0,t) = 0$. Let

$$u(x,t) = e^{bat}p(x,t).$$

Then $p$ satisfies

$$p_t + p_x = b\alpha (p(\alpha x, t) - p(x,t)), \quad (5.1)$$

$$p(x,0) = 0 \quad (5.2)$$

and

$$p(0,t) = 0. \quad (5.3)$$

The next lemma shows that the only solution to the above problem is $p = 0$. 

Lemma 5.1. Let \( W = ((x, t) : x \geq 0 \text{ and } t \geq 0), \) and suppose that \( p \) is a solution to equation (5.1) valid in \( W \) that satisfies conditions (5.2) and (5.3). Suppose further that \( p_t \) and \( p_x \) are continuous in \( W \) and that for any \( T \geq 0 \) and any \( \epsilon > 0 \) there are positive numbers \( \delta_\epsilon \) and \( X_\epsilon \) such that

\[
|p(x, t)| < \epsilon
\]

whenever \( t \in [T, T + \delta_\epsilon] \) and \( x > X_\epsilon \). Then \( p(x, t) = 0 \) for all \((x, t) \in W \). 

Proof. Suppose there is a point \((x_0, t_0) \in W \) at which \( p(x_0, t_0) \neq 0 \). Without loss of generality, we can assume \( p(x_0, t_0) > 0 \). Conditions (5.3) and (5.4) imply that \( p_0(x) = p(x, t_0) \) must have a global maximum \( \gamma_0 > 0 \) at some \( x \in (0, \infty) \). Condition (5.4) also indicates that there must be a largest value of \( x \), say \( m_0 \), at which \( p_0(m_0) = \gamma_0 \). Let \( l_0 > m_0 \) and define the set

\[
R_0 = \{(x, t) \in W : x \leq l_0, t \leq t_0\}.
\]

Now \( p \) is continuous on \( R_0 \), so there must be a point \((x_1, t_1) \in R_0 \) at which \( p \) attains its maximum value \( \lambda_0 \geq \gamma_0 \). Since \( m_0 \) is the position of the ‘last’ global maximum for \( p(x, t_0) \), we have \( p_0(m_0, t_0) = 0 \) and

\[
p_t(m_0, t_0) = \alpha(p(xm_0, t_0) - p(m_0, t_0)) < 0.
\]

The above inequality shows that there must be a \( t < t_0 \) at which \( p(m_0, t) > \gamma_0 \); hence, \( \lambda_0 \) cannot be attained on the line \( t = t_0 \). Clearly, \( \lambda_0 \) is not attained on the lines \( x = 0 \) or \( t \); thus, it must be attained at either an interior point of \( R_0 \) or on the line segment \( L_0 = \{(x, t) : x = l_0, 0 < t < t_0\} \). If it occurs on \( L_0 \), then \( p_t(x_1, t_1) = 0 \) and \( p_x(x_1, t_1) \geq 0 \); hence, \( p(\alpha x_1, t_1) \geq p(x_1, t_1) = \lambda_0 \). If \( \lambda_0 \) is not attained on \( L_0 \), then \((x_1, t_1)\) is an interior point; hence, \( p_x(x_1, t_2) = p_t(x_1, t_1) = 0 \) and consequently

\[
p(\alpha x_1, t_1) = p(x_1, t_1) = \lambda_0.
\]

If \((\alpha x_1, t_1)\) is an interior point of \( R_0 \), then the above argument can be applied to \((\alpha x_1, t_1)\). Eventually, \( \alpha^n x_1 > l_0 \) for \( n \) large, so that in this manner we can assert the existence of a point \((x^*, t_1)\) with \( x^* > l_0 \) at which \( p(x^*, t_1) \geq \lambda_0 > \gamma_0 \).

The function \( p_1(x) = p(x, t_1) \) must have a largest value of \( x \), say \( m_1 \), at which \( p_1 \) achieves its global maximum \( \gamma_1 \geq \lambda_0 \). Choose any number \( l_1 \) such that \( l_1 > \max\{m_1, \alpha l_0\} \) and let

\[
R_1 = \{(x, t) \in W : x \leq l_1, t \leq t_1\}.
\]

We can repeat the arguments used on \( R_0 \) to assert the existence of a point \((\tilde{x}, t_2)\), where \( l_1 < \tilde{x} \) and \( t_2 < t_1 \), at which \( p(\tilde{x}, t_2) \geq \lambda_1 \geq \gamma_1 > \gamma_0 \). Here, \( \lambda_1 \) denotes the maximum of \( p \) in \( R_1 \). Evidently, we can continue this process ad infinitum, and thus construct sequences \( \{t_k\}, \{m_k\} \) and \( \{\gamma_k\} \), where \( p_k(x) = p(x, t_k) \) has its last global maximum \( \gamma_k \) at \( x = m_k \). All of these sequences are monotonic: in particular, \( \{t_k\} \) is monotonic strictly decreasing and bounded below by \( 0; \{m_k\} \) is monotonic strictly increasing and satisfies \( m_k > \alpha^{k-1} l_0 \); and \( \{\gamma_k\} \) is monotonic strictly increasing so that specifically \( \gamma_k > \gamma_0 > 0 \) for all \( k \). Clearly, there must be a \( \tau \geq 0 \) such that \( t_k \to \tau \) as \( k \to \infty \); and \( m_k \to \infty \) as \( k \to \infty \). For each \( k \geq 1 \), \( p(m_k, t_k) > \gamma_0 > 0 \), so that if we choose \( T = \tau \) and \( \epsilon = \gamma_0 \), it is clear that there is no \( \delta_\epsilon > 0 \) that satisfies (5.4).

Theorem 5.2 (Uniqueness). Let \( m \) be defined by equation (3.6). Then for any \( \epsilon > 0 \) and any \( T \geq 0 \) there is a \( \delta_\epsilon > 0 \) and an \( X_\epsilon \) such that

\[
|m(x, t)| < \epsilon
\]

whenever \( t \in [T, T + \delta_\epsilon] \) and \( x > X_\epsilon \). The function defined by equation (3.6) is unique among functions with continuous partial derivatives that satisfy equations (2.3)–(2.5) and (5.6).

Proof. Lemma 5.1 shows that the solution \( m \) of Theorem 4.1 is unique provided \( m \) satisfies the appropriate decay condition. Let

\[
m(x, t) = e^{\alpha t}p(x, t).
\]
We show that $p$ satisfies condition (5.4). Choose $T \geq 0$ and $\epsilon > 0$. For $x > t$, the solution $m$ is given by equation (2.12), and the arguments used to establish the uniform convergence of the series (2.6) can be readily adapted to show that

$$
0 \leq m(x, t) \leq e^{\beta t} \Lambda(x, t),
$$

where $\Lambda(x, t) = \sup_{z \geq \alpha(x-t)} m_0(z)$.

Choose any $\delta_{\epsilon} > 0$. Since $m_0(x) \to 0$ as $x \to \infty$, there is an $X_{\epsilon}$ such that $X_{\epsilon} > T + \delta_{\epsilon}$ and $m_0(z) < \epsilon$, for all $z \geq \alpha(X_{\epsilon} - (T + \delta_{\epsilon}))$, i.e.

$$
p(x, t) \leq \Lambda(x, t) < \epsilon
$$

for all $x > X_{\epsilon}$ and $t \in [T, T + \delta_{\epsilon}]$.

## 6. Conclusion

In this paper, we have developed a new method whereby an initial boundary value problem involving a first-order linear functional PDE can be solved. The method is not restricted to the functional equation studied in this paper: the same strategy can be employed to deal with more general functional PDEs with advanced arguments. For example, if the division rate $b$ is not constant with respect to $x$, the same approach in principle can be used. The crux, however, is finding the limiting function. Certainly, future work would include such generalizations.

In terms of the cell division model, the general solution developed in this paper provides more detailed information about how the cell size distribution depends on the initial distribution. It is well known that solutions are asymptotic to the SSD solution as $t \to \infty$, but the analysis underlying this relation does not fully explain or illustrate why the initial data have such a weak influence on the long-term solution and how the SSD solution arises. The weak dependence is a result of the hyperbolic character of the differential operator and the advanced argument. We have shown that the SSD solution arises as the leading-order term in an expansion for the limiting function, which represents the solution as $t \to \infty$. By contrast, this limiting solution depends strongly on the boundary data. The expansion also provides the higher order terms in the asymptotic expansion, and these terms correspond to eigenfunctions for the pantograph equation.

### Authors’ contributions

This is a collaborative project with all three contributing to this work. It was included in the PhD thesis of A.A.Z. which is to be awarded this month (April 2015).

### Competing interests

There is no conflict of interest.

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