Design of three-dimensional origami structures based on a vertex approach

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Origami geometric design is fundamental to many engineering applications of origami structures. This paper presents a new method for the design of three-dimensional (3D) origami structures suitable for engineering use. Using input point sets specified, respectively, in the $x-z$ and $y-z$ planes of a Cartesian coordinate system, the proposed method generates the coordinates of the vertices of a folded origami structure, whose fold lines are then defined by straight line segments each connecting two adjacent vertices. It is mathematically guaranteed that the origami structures obtained by this method are developable. Moreover, an algorithm to simulate the unfolding process from designed 3D configurations to planar crease patterns is provided. The validity and versatility of the proposed method are demonstrated through several numerical examples ranging from Miura-Ori to cylinder and curved-crease designs. Furthermore, it is shown that the proposed method can be used to design origami structures to support two given surfaces.

1. Introduction

Origami, an ancient art of paper folding, has aroused considerable research interest in mathematics and engineering in recent years and has received credit for many innovative applications ranging from lightweight sandwich structures [1], to automotive safety devices [2], deployable solar panels [3], medical stents [4], foldable electronics [5,6] and mechanical metamaterials [7–9]. Among the current research on origami mathematics, a great majority is focused on the flat- and/or rigid-foldability problems [10–17], whereas origami geometric
design that studies the folded configurations and/or folding mechanisms of origami structures and is fundamental to many engineering applications is still intractable. Deriving parametric equations that describe the folding kinematics of a certain crease pattern is the most commonly used approach in origami design [18–20]. But this approach suffers from vast complexity and lacking of flexibility, which is particularly the case for one who does not have much knowledge about origami. Lang [21] proposed an origami design method, known as the tree method, for the construction of arbitrary tree-shaped origami figures based on the concept of universal molecule. However, it is virtually impossible to create true three-dimensional (3D) shapes by this method. Tachi [22] developed the first practical method for designing 3D origami structures using edge- and vertex-tucking molecules. However, the origami structures constructed by this method are more for artistic purpose than of engineering use due to the existence of short folds and/or small facets in the edge- and vertex-tucking molecules which are not favoured from manufacturing’s or mechanical property’s point of view. Graphic approaches for designing 3D doubly corrugated surfaces also exist [23,24]. Neither can these approaches be conveniently converted into computer codes nor can they generate the interim folding sequence.

In this paper, an easy-to-implement method for designing 3D origami structures that are suitable for engineering use is reported. The proposed method is based on a vertex generation algorithm, which assumes that the vertices of an origami structure defined in a Cartesian coordinate system can be generated by transposing a set of points defined in the \( y-\) plane into a series of \( x \)-positions followed by an in-plane transposition at each \( x \)-position. The developability of such obtained 3D origami structures is mathematically guaranteed. Moreover, an algorithm to simulate the unfolding process from the folded state to the two-dimensional (2D) crease pattern is provided.

The layout of this paper is as follows. First, the vertex generation and unfolding simulation algorithms of origami structures are described. Then, several numerical examples are given to demonstrate the validity and versatility of the proposed method. The mathematical meaning of the proposed method and its potential application on designing origami structures between two target surfaces are discussed. Finally, a brief summary concludes the paper.

2. The vertex method

(a) Generation of 3D origami structure

Any origami structure can be regarded as consisting of a number of vertices that are connected by fold lines. Given the spatial positions of the vertices, the geometry of an origami structure is entirely determined. Therefore, designing an origami structure is essentially equivalent to determining the coordinates of its vertices in a Cartesian coordinate system.

In our method, the vertices of a 3D origami structure are determined through the following equation:

\[
V_{ij} = \begin{bmatrix} x_{ij} \\ y_{ij} \\ z_{ij} \end{bmatrix} = V^y_j + [A_j] V^x_i, \quad i = 1, \ldots, m; \quad j = 1, \ldots, n,
\]

where \( V^x_i = \begin{bmatrix} x_i^x \\ 0 \\ z_i^x \end{bmatrix} \), \( i = 1, \ldots, m \) are \( m \) input points specified in the \( x-z \) plane; \( V^y_j = \begin{bmatrix} 0 \\ y_j^y \\ z_j^y \end{bmatrix} \), \( j = 0, 1, \ldots, n + 1 \) are \( n + 2 \) input points specified in the \( y-z \) plane; \( [A_j] \) is a \( 3 \times 3 \) matrix given by

\[
[A_j] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & (-1)^j \frac{\cos \theta_{j-1} + \cos \theta_j}{\sin(\theta_{j-1} - \theta_j)} \\ 0 & 0 & (-1)^j \frac{\sin \theta_{j-1} + \sin \theta_j}{\sin(\theta_{j-1} - \theta_j)} \end{bmatrix},
\]
where the angular variable $\theta_j$ can be determined from

$$
\begin{bmatrix}
0 \\ \cos \theta_j \\ \sin \theta_j
\end{bmatrix}^T = \frac{V_{j+1}^y - V_j^y}{\|V_{j+1}^y - V_j^y\|},
$$

(2.3)

where $\|u\|$ denotes the norm of a vector $u$.

The fold and boundary lines of the origami structure are defined by straight line segments each connecting two adjacent vertices, i.e. $V_{ij}$ and $V_{i,j+1}$ for $1 \leq j < n$ or $V_{ij}$ and $V_{i,j+1}$ for $1 \leq i < m$. Hence, each internal vertex $V_{ij}, 1 < i < m, 1 < j < n$ is a degree-four vertex if $V_{ij}$ does not coincide with $V_{ij-1}$ and $V_{i,j+1}$ and a degree-six vertex if $V_{ij}$ coincides with either $V_{ij-1}$ or $V_{ij+1}$.

It can be proved that the folded origami structures thus obtained satisfy two geometric conditions: (I) vertices $V_{ij}, V_{i,j+1}, V_{i+1,j}, V_{i+1,j+1}$ and $V_{i+2,j}$ for arbitrary $i$ and $j$ are co-planar and (II) the gaus area, i.e. $2\pi$ minus the sum of the sector angles around each internal vertex is equal to zero. Therefore, the origami structures are developable. The details of the proof are provided in appendices A and B.

(b) Generation of 2D crease pattern

Given the positions of the vertices of a folded origami structure, the corresponding 2D crease pattern can be generated by mapping the vertices of the folded origami structure to a set of vertices in a plane under the rules that the distance between any two adjacent vertices and the sector angles around each vertex remain the same before and after mapping. Here, we denote the corresponding vertex in the crease pattern of vertex $V_{ij}$ by $\tilde{V}_{ij} = [\tilde{x}_{ij} \quad \tilde{y}_{ij}]^T$. Equations (B 12) and (B 13) imply that for any given $i$, vertices $\tilde{V}_{i,1}, \tilde{V}_{i,2}, \ldots, \tilde{V}_{i,n}$ locate on a same straight line. Denote the line that passes through $\tilde{V}_{i,1}, \tilde{V}_{i,2}, \ldots, \tilde{V}_{i,n}$ by $\chi_i^y$. According to equations (A 4) and (A 5), lines $\chi_i^x, i = 1, 2, \ldots, m$ are parallel to each other. Therefore, it is convenient to define a 2D rectangular coordinate system with its $y$-axis parallel to lines $\chi_i^y$ and origin located at vertex $\tilde{V}_{1,1},$ as shown in Figure 1. In this coordinate system, the coordinates of $\tilde{V}_{ij}$ can be generated through the following equations:

$$\tilde{V}_{1,1} = [0 \quad 0]^T,$$

(2.4)

$$\tilde{V}_{i,1} = \begin{cases}
\tilde{V}_{i-1,1} + \|V_{i-1,1} - V_{i,1}\| [\sin \varphi_{i-1} \quad \cos \varphi_{i-1}]^T & i = 2, \ldots, m \\
\tilde{V}_{i-1,1} + \|V_{i-1,1} - V_{i,1}\| [\sin \xi_{i-1} \quad -\cos \xi_{i-1}]^T & i = 2, \ldots, n
\end{cases}
$$

(2.5)

and

$$\tilde{V}_{i,j} = \tilde{V}_{i,j-1} + \|V_{i,j-1} - V_{i,j}\|[0 \quad 0]^T, \quad i = 1, \ldots, m; j = 2, \ldots, n,$$

(2.6)

where

$$\cos \varphi_i = \frac{\|V_{i,1} - V_{i,2}\|^2 + \|V_{i,1} - V_{i+1,1}\|^2 - \|V_{i,2} - V_{i+1,1}\|^2}{2\|V_{i,1} - V_{i,2}\|\|V_{i,1} - V_{i+1,1}\|}, \quad \varphi_i \in (0, \pi)
$$

(2.7)

and

$$\cos \xi_i = \frac{\|V_{i+1,1} - V_{i+1,2}\|^2 + \|V_{i+1,1} - V_{i+1,2}\|^2 - \|V_{i+1,2} - V_{i+1,1}\|^2}{2\|V_{i+1,1} - V_{i+1,2}\|\|V_{i+1,1} - V_{i+1,2}\|}, \quad \xi_i \in (0, \pi)
$$

(2.8)

The fold and boundary lines of the crease pattern are defined by straight line segments each connecting two adjacent vertices. Finally, the convexity of a fold line in the crease pattern can be directly known from the folded configuration.

(c) Unfolding simulation

The algorithms to generate a 3D origami structure and the corresponding 2D crease pattern discussed above can be summarized as

$$V = f(V^x, V^y)
$$

(2.9)
\[ \tilde{V} = g(V) = g(f(V^x, V^y)), \]

where \( V^x \) and \( V^y \) denote the input point sets in the \( x-z \) and \( y-z \) planes, respectively; \( V \) and \( \tilde{V} \) represent the sets of vertices of the folded origami structure and its crease pattern, respectively; \( f \) represents the projection of \( V^x \) and \( V^y \) to \( \tilde{V} \) given by equation (2.1); and \( g \) represents the mapping of \( V \) to \( \tilde{V} \) given by equations (2.4)–(2.6). Consider a new pair of input point sets in the \( x-z \) and \( y-z \) planes, denoted by \( U^x \) and \( U^y \), respectively. If the vertex set of the resulting crease pattern is the same as \( \tilde{V} \), i.e.

\[ g(f(U^x, U^y)) = g(f(V^x, V^y)), \]

the new 3D origami structure by \( U^x \) and \( U^y \) represents another folded state of the original one. As \( \tilde{V} \) involves \( m \times n \) vertices and \( \tilde{V}_{1,1} \) always locates at the origin, equation (2.11) is in fact an equation set of \( 2(m \times n - 1) \) equations for \( U^x \) and \( U^y \). Note that translating the input set in the \( y-z \) plane only changes the spatial positions of the resulting 3D origami structure along the \( y \)- and/or \( z \)-axis but does not change its shape. When the folded shape of the origami structure is concerned, it is convenient to fix one point (as a rule of thumb \( U^y_{1,1} \)) in \( U^y \) to the origin. Similarly, translating the input set in the \( x-z \) plane along the \( x \)-axis only changes the spatial positions of the resulting 3D origami structure in the \( x \)-direction. It is convenient to fix one point (as a rule of thumb \( U^x_{1,1} \)) in \( U^x \) to the origin. Besides, at least one point in \( U^x \) or \( U^y \) must be in a different position from the corresponding point in \( V^x \) or \( V^y \) so that a different folded state can be found. One coordinate of that point, referred to as the control parameter subsequently, needs to be prescribed. When the folded origami structure unfolds to the plane crease pattern, the \( x \)-directional width increases monotonously from its initial width \( w^\text{ini}_x \) to that of the plane crease pattern \( w^\text{cp}_x \). Therefore, it is a convenient choice to use the \( x \)-directional width as the control parameter in the unfolding simulation algorithm. The nonlinear least-squares optimization method is employed in the sequel to solve equation (2.11). In each increment step of the algorithm, the control parameter is increased by \( (w^\text{cp}_x - w^\text{ini}_x)/N \), where \( N \) is the total number of steps. Equation (2.11) is then solved to gain \( V^{x(t)} \) and \( V^{y(t)} \), i.e. the current input point sets in the \( x-z \) and \( y-z \) planes, where the input point sets in the \( x-z \) and \( y-z \) planes gained in the previous step \( V^{x(t-1)} \) and \( V^{y(t-1)} \) are used as the initial guess for the least-squares optimization solver. Once the input point sets in the \( x-z \) and \( y-z \) planes in all of the steps are obtained, they are used to generate the 3D origami structures step by step to simulate the unfolding process.

![Figure 1](http://rspa.royalsocietypublishing.org/downloads/20150407/471:20150407.png)

**Figure 1.** The 2D rectangular coordinate system in which the crease pattern is generated.
3. Examples

(a) Miura-Ori design

As the first example, consider the input point sets in the $x-z$ and $y-z$ planes shown in figure 2, both of which are characterized by periodic triangular wave patterns (the dashed lines) and can be expressed by

$$V_x^i = \left[ \begin{array}{c} i - \frac{1}{2} T_x \\ 0 \\ 1 + \frac{(-1)^i}{2} h_x \end{array} \right]^T, \quad i = 1, \ldots, m$$

(3.1)

and

$$V_y^j = \left[ \begin{array}{c} 0 \\ \frac{i - 1}{2} T_y \\ 1 + \frac{(-1)^j}{2} h_y \end{array} \right]^T, \quad j = 0, \ldots, n + 1,$$

(3.2)

where $T_x$ and $h_x$ are the period and height of the triangular wave pattern of the input point set in the $x-z$ plane, and $T_y$ and $h_y$ are those of the triangular wave pattern of the input point set in the $y-z$ plane. Substituting equations (3.1) and (3.2) into equation (2.1) yields

$$V_{i,j} = \left[ \begin{array}{c} i - \frac{1}{2} T_x \\ \frac{i - 1}{2} T_y + 1 + \frac{(-1)^i}{2} \frac{h_x}{h_y} \sqrt{\frac{T_y^2}{2} + h_y^2} \\ 1 + \frac{(-1)^j}{2} \frac{h_y}{h_x} \end{array} \right], \quad i = 1, \ldots, m; \quad j = 1, \ldots, n.$$  

(3.3)

The above equation corresponds to the periodically folded Miura-Ori pattern, an example of which is illustrated in figure 3a, where $m, n, T_x, h_x, T_y$ and $h_y$ are taken as 7, 7, 10, 10, 10 and 10, respectively, and the shaded region represents a repeating cell of the Miura-Ori structure. The corresponding crease pattern, obtained from equations (2.4) to (2.6), is shown in figure 3b, where the valley folds are represented by the dotted lines.

Using the unfolding simulation algorithm given in §2c, the unfolding process of the Miura-Ori structure in figure 3a can be readily simulated in MATLAB (MathWorks Inc., USA), where function *lsqnonlin* is employed to solve equation (2.11). An animation of the unfolding motion is provided by electronic supplementary material, movie S1. It is known in the literature that the dimensions of a Miura-Ori structure are related by

$$w_y = (n - 1)a \cos \beta \sqrt{1 - \frac{w_x^2}{(m - 1)^2 b^2} + b \sqrt{1 - \frac{w_z^2}{(m - 1)^2 b^2}}}$$

(3.4)

and

$$h_z = a \sin^2 \beta - \frac{w_x^2}{(m - 1)^2 b^2} \sqrt{1 - \frac{w_z^2}{(m - 1)^2 b^2}},$$

(3.5)

where $w_x, w_y$ and $h_z$ are the $x$-directional width, $y$-direction width and $z$-directional height of the Miura-Ori, respectively, and $a, b$ and $\beta$ are constants defined in figure 3b. Figure 4 plots the changes of $w_y$ and $h_z$ against $w_x$ obtained from equations (3.4) and (3.5) and from the unfolding simulation. A good agreement between the analytical and simulation results provides an evidence for the validity of the unfolding simulation algorithm.

(b) Cylinder design

Wu [26] shows that finding the closure conditions for a crease pattern that can fold to a 3D state in which two zigzag lines on the opposite edges meet, such as a seamless cylinder, involves much
effort. In our method, the closure conditions are simplified as

\[ V_{n-1}^y = V_0^y, \quad V_n^y = V_1^y \quad \text{and} \quad V_{n+1}^y = V_2^y. \]  

(3.6)
Consider for example the input point sets in the $x-z$ and $y-z$ planes shown in figure 5. They can be expressed by

$$V^x_i = \left[ i - 1 \frac{1}{2} T_x 0 (-1)^{i-1} \frac{1}{2} h_x \right]^T, \quad i = 1, \ldots, m$$

and

$$V^y_j = \left[ 0 d_j \sin(j-1)\beta \quad d_j \cos(j-1)\beta \right]^T, \quad j = 0, \ldots, n + 1,$$

where

$$d_j = \frac{1 - (-1)^j}{2} r_1 + \frac{1 + (-1)^j}{2} r_2.$$  

The closure conditions given by equation (3.6) can be satisfied by imposing $n = 2N + 1$ and $\beta = \pi / N$, where $N$ is an integer larger than 1, and thus cylindrical origami designs are obtained. An example is shown in figure 6, where $m, N, T_x, h_x, r_1$ and $r_2$ equal 3, 6, 10, 10, 15 and 30, respectively.
Figure 7. Changes of $k_1$ and $k_2$ against $w_x$ obtained from equation (3.10) and from the unfolding simulation. (Online version in colour.)

Note that $V_1^x$ in figure 5a does not locate at the origin. For this and all such cases, we find that introducing a temporary point $V_0^x$ at the origin to the input point set in the $x-z$ plane can significantly assist the convergence of the algorithm for the simulation of unfolding. An animation of the unfolding motion of the current example is provided by electronic supplementary material, movie S2. The crease pattern of the current example in figure 6b is referred to as the arc-Miura pattern by Gattas et al. [18]. They found that all V-vertices, i.e. vertices at the intersection of three valley creases and one mountain crease and all M-vertices, i.e. vertices at the intersection of three mountain creases and one valley crease of this pattern lie, respectively, along concentric radii $R_1$ and $R_2$ during folding given by

$$R_i = \sqrt{\frac{a_1^2 + a_2^2 - 2a_1a_2 \cos \eta_i}{2 - 2 \cos (\eta_1 - \eta_2)}}, \quad i = 1, 2,$$

(3.10)

where

$$\eta_i = \cos^{-1} \left( 1 - \frac{2 \cos^2 \phi_i}{1 - \frac{w_x^2}{(m-1)^2 b_i^2}} \right), \quad i = 1, 2.$$

(3.11)

In the above equations, $a_1$, $a_2$, $b_1$, $b_2$, $\phi_1$ and $\phi_2$ are constants defined in figure 6b. Through equation (3.10), the relationships between $R_i$, $i = 1, 2$ and $w_x$ are established. During the unfolding simulation, $R_1$ and $R_2$ can be obtained by means of fitting a cylindrical surface to all V-vertices and all M-vertices, respectively. Figure 7 plots the changes of the curvatures $k_1$ and $k_2$, i.e. the reciprocals of $R_1$ and $R_2$ against $w_x$ obtained from equation (3.10) and the unfolding simulation. Again, the validity of the unfolding simulation algorithm is demonstrated by the good match between the theoretical and simulation results.

(c) Curved-crease design

Origami structures with curved creases can be designed and virtually unfolded using the vertex method with relative ease. To illustrate this, consider for example the input point set in the $x-z$ plane shown in figure 8, where the input points lie on a cosine wave (the dashed line) whose height and period are $h_x$ and $T_x$, and there are $M+1$ input points in each period. Given that the cosine wave has $N_T$ periods, the input point set in the $x-z$ plane can be expressed by

$$V_i^x = \begin{bmatrix} i - \frac{1}{M} & h_x \left( 1 - \cos \frac{2\pi(i-1)}{M} \right) \end{bmatrix}^T, \quad i = 1, \ldots, N_T M + 1.$$  

(3.12)
Figure 8. The input point set in the $x-z$ plane for example (c).

Figure 9. (a) An origami structure with curved creases and (b) the corresponding crease pattern. (Online version in colour.)

Various origami designs can be obtained by combining equation (3.12) with different input point sets in the $y-z$ plane. Figure 9a shows an example obtained from equations (3.12) and (3.2), where $M$, $N_T$, $n$, $T_x$, $h_x$, $T_y$ and $h_y$ are taken as 50, 3, 7, 10, 10, 10 and 10, respectively. It is evident that curved creases extending in the $x$-direction can be approached asymptotically as $M$ increases. When $M$ is sufficiently large, the band bonded by two adjacent curved creases is subject to bending instead of folding at discrete fold lines. Therefore, there is no need to define creases extending in the $y$-direction in the crease pattern (figure 9b).

An animation of the unfolding motion of the origami structure in figure 9a is provided by electronic supplementary material, movie S3. To validate the simulation result, it is compared to the unfolding motions simulated using the finite-element (FE) method and the planar quadrilateral (PQ) mesh method [27], respectively. In the FE model, half of a repeating cell is modelled due to symmetry and meshed with four-node shell elements with a small mesh size of 0.2 (figure 10). The nodes on the thick-dashed and dashed-dotted lines are constrained in the $x$- and $z$-directions, respectively. The curved crease (the dotted line) is modelled as an ideal one whose rotational stiffness is set to zero. The parameters in the PQ-mesh method are defined in figure 9a. For the current case, longitudinal side length $a$ equals 11.1803 and the values of lateral sector angle $\varphi_k$ and side length $b_k$, $k = 1, \ldots, 25$ are listed in table 1. The changes of $w_y$ and $h_z$ against $w_x$ given by the FE method, the PQ-mesh method and our algorithm are plotted in figure 10. Again, a good agreement between the FE, PQ-mesh and simulation results is observed.
Figure 10. Changes of $w_y$ and $h_z$ against $w_x$ given by the FE method, the PQ-mesh method and the unfolding simulation algorithm. The FE model is shown in the blank. (Online version in colour.)

Table 1. The values of lateral sector angle $\phi_k$ and side length $b_k$, $k = 1, \ldots, 25$.

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4. Discussion

The mathematical meaning of equation (2.1) is that the set of input points in the $y$–$z$ plane are first transposed into each of $m$ parallel planes that pass through the $m$ $x$–$z$ plane input points, respectively, and in the $i$th plane, the $j$th point is displaced by an offset distance of $z_j^\eta_j$, where $\eta_j = \sec \angle V^y_j V^y_{j-1} V^y_{j+1}/2$, along the angle bisector of the angle formed by two vectors $V^y_{j-1} V^y_j$ and $V^y_{j+1} V^y_{j-2}$ with the latter rotating counter-clockwise to the former. For any two consecutive points, the signs of the offset distances are alternating where the negative offset distance means that the point is displaced along the negative direction of the above-defined angle bisector. Figure 11 shows an example for three consecutive points $V_{ij}$ to $V_{ij+2}$ in the $i$th plane, where the dashed and dotted arrows indicate, respectively, the positive and negative directions of the angle bisectors of the angles formed by rotating $V^y_{j+k-1} V^y_{j+k}$ counter-clockwise to $V^y_{j+k-2} V^y_{j+k-1}$, $k = 1, 2, 3$. In the light of the mathematical meaning of equation (2.1), various potential applications of the vertex method can be devised. One such application is to design origami structures with changing thickness to support two non-parallel surfaces. To illustrate this, consider first six consecutive points $V^y_{j-1}$ to $V^y_{j+4}$ in the input point set in the $y$–$z$ plane that features a square wave (dashed line), as shown in figure 12a or b, where the lengths of segments $V^y_j V^y_{j+1}$ and $V^y_{j+2} V^y_{j+3}$ equal...
\[ z_i^\gamma > 0 \quad (z_i^\gamma < 0) \]

\[
\begin{align*}
V_{j-1}^y & \quad V_j^y \quad V_{j+1}^y \\
V_{j-2}^y & \quad V_{j-1}^y \\
V_{j+2}^y & \quad V_{j+1}^y \\
V_{j+3}^y & \quad (V_{j+2}^y)
\end{align*}
\]

**Figure 11.** An example for three consecutive points \( V_{ij} \) to \( V_{ij+2} \) showing the mathematical meaning of equation (2.1).

\[
\begin{align*}
V_{ij} (V_{ij+1}) & \quad 2z_i^x \\
V_{ij-1} & \quad 2z_i^x \\
\end{align*}
\]

**Figure 12.** (a–b) Vertices \( V_{ij} \) to \( V_{ij+3} \) resulted from six consecutive points \( V_{i-1}^y \) to \( V_{i+4}^y \) in the input point set in the \( y-z \) plane that features a square wave (dashed line) when (a) \( z_i^\gamma > 0 \) and (b) \( z_i^\gamma < 0 \); (c) Generation of vertices that support two surfaces \( z = f_u(y) \) and \( z = f_l(y) \), where the blue and green dots correspond to the positive and negative \( z_i^\gamma \), respectively. (Online version in colour.)
Figure 13. An example showing a designed origami structure that supports two surfaces: (a) a view along the x-axis, (b) 2D crease pattern, (c) 3D view and (d) a physical model. (Online version in colour.)

2|z^x_i|}. The resulting vertices $V_{ij}$ to $V_{i,j+3}$ for positive $z^x_i$ are shown in figure 12a, where $V_{ij}$ and $V_{i,j+1}$ are at the same position $|z^x_i|$ above segment $V^y_i V^y_{i+1}$ and on the bisector of the segment, and the resulting vertices $V_{ij}$ to $V_{i,j+3}$ for negative $z^x_i$ are shown in figure 12b, where $V_{i,j+2}$ and $V_{i,j+3}$ are at the same position $|z^x_i|$ below segment $V^y_{i+2} V^y_{i+3}$ and on the bisector of the segment. Denote by $z = f_u(y)$ and $z = f_l(y)$, $y \in [y_1, y_2]$ the upper and lower surfaces to be supported, as shown in figure 12c. The following steps can be taken to design an origami structure to support these surfaces:

(a) Given that the square wave of the input point set in the $y-z$ plane consists of $N_y$ periods, the wavelength $\lambda_y$ is then equal to $(y_2 - y_1)/N_y$.
(b) In the input point set in the $x-z$ plane, the z-coordinates of all points are given the same absolute value equal to $\lambda_y/4$ but may have different signs, i.e. $|z^x_i| = \lambda_y/4, i = 1, \ldots, m$.
(c) Move the upper surface in the negative z-direction by $|z^x_i|$ and denote the new surface by $z = f'_u(y)$. Similarly, move the lower surface in the positive z-direction by $|z^x_i|$ and denote the new surface by $z = f'_l(y)$.
(d) To the input point set in the $y-z$ plane, adjust individually the $z$-positions of the high and low levels of the square wave so that the middle points (hollow dots) of the high and low levels lie on $z = f'_u(y)$ and $z = f'_l(y)$, respectively.

Then, in planes that pass through $x-z$ plane input points with positive $z$-coordinates, the resulting vertices lie either on the upper surface or between the upper and lower surfaces (see the blue dots in figure 12c). Similarly, in planes that pass through $x-z$ plane input points with negative $z$-coordinates, the resulting vertices lie either on the lower surface or between the upper and lower surfaces (see the green dots in figure 12c). Collectively, the upper and lower surfaces are supported by an origami structure defined by these vertices. A design example as well as its physical model is shown in figure 13, where the upper and lower surfaces are given by

$$z = f_u(y) = 7.5 - 2.5 \cos \frac{\pi y}{25}, \quad y \in [0, 50], \quad (4.1)$$

and

$$z = f_l(y) = -6.25 + 1.25 \cos \frac{\pi y}{25}, \quad y \in [0, 50], \quad (4.2)$$

respectively. $N_y$ is taken as 10, and the input point set in the $x-z$ plane is given by

$$V^i_x = \begin{bmatrix} 5i - 2.5 & 0 & (-1)^i \frac{50}{4N_y} \end{bmatrix}^T, \quad i = 1, \ldots, 11. \quad (4.3)$$

Note that the fold lines connected to the boundary vertices (i.e. vertices that lie on the supported surfaces) form an angle of $45^\circ$ or $135^\circ$ with the $y$-axis when viewed along the $x$-axis. Therefore, at the boundary vertices, the slope of the supported faces in the $y-z$ plane should be within the range of $[-1,1]$ to avoid conflicts between the faces and the fold lines. Besides, the upper and lower surfaces should be at least $2|z^*_i|$ so that the moved surfaces do not intersect. Since $|z^*_i|$ is inversely proportional to $N_y$, the lower limit for $N_y$ increases as the minimum distance between the upper and lower surfaces decreases.

5. Summary and final remark

In this paper, we have proposed a synthetic method, known as the vertex method, of designing a developable 3D origami structure, generating its 2D crease pattern and simulating its unfolding process out of input point sets specified respectively in the $x-z$ and $y-z$ planes of a Cartesian coordinate system. Since the vertex method is based on a pure mathematical procedure, it can be easily implemented computationally by anyone with or without experience in origami design. The validity and versatility of the method have been illustrated through three numerical examples including Miura-Ori, cylindrical and curved-crease designs. Cylindrical origami structures have found applications in energy absorption devices [2,26], whereas curved-creased foldcores have been found to outperform straight-creased foldcores under impact loads [28]. Using the vertex method can greatly facilitate the research on these topics. Furthermore, according to the mathematical meaning of the vertex method, we have successfully shown through an example the potential of the method to design origami structures between two given surfaces, which can lead to broad applications such as morphing wing design [29].

The main limitation of the vertex method is that it is difficult, if not impossible, to design doubly curved origami structures with the method due to the fact that the planes into which the input point set in the $y-z$ plane are transposed are parallel. Introducing proper rotational transformation to these planes is a potential solution to overcome this limitation, which is beyond the scope of the discussion in this paper and will be considered in the future work.

Data accessibility. The data used except the movies are all in the manuscript. The movies supporting this article have been uploaded as the electronic supplementary material.

Authors’ contributions. X.Z. designed the study, derived the equations, carried out the analysis and drafted the manuscript; H.W. coordinated the study and commented on the manuscript; Z.Y. conceived of the study and commented on the manuscript. All authors gave final approval for publication.
Appendix A. Proof of geometric condition (I)

Denote the vector pointing from \( V_{ij} \) to \( V_{ij+1} \) by \( V_{ij}V_{ij+1} \) and the vector from \( V_{i+1,j} \) to \( V_{i+1,j+1} \) by \( V_{i+1,j}V_{i+1,j+1} \). According to equation (2.1), there are

\[
V_{ij}V_{ij+1} = V_{ij+1} - V_{ij} = (V'^{y}_{j+1} - V'^{y}_{j}) - k_i \begin{bmatrix} 0 \\ \cos \theta_j \\ \sin \theta_j \end{bmatrix},
\]

and

\[
V_{i+1,j}V_{i+1,j+1} = V_{i+1,j+1} - V_{i+1,j} = (V'^{y}_{j+1} - V'^{y}_{j}) - k_{i+1} \begin{bmatrix} 0 \\ \cos \theta_j \\ \sin \theta_j \end{bmatrix},
\]

where coefficient \( k_i \) is given by

\[
k = \frac{\sin(\theta_{j-1} - \theta_j) + \sin(\theta_{j-1} - \theta_{j+1}) + \sin(\theta_j - \theta_{j+1})}{\sin(\theta_{j-1} - \theta_j) \sin(\theta_j - \theta_{j+1})} \frac{(-1)^j}{\|V'^{y}_{j+1} - V'^{y}_{j}\|}.
\]

Substituting equation (2.3) into equations (A 1) and (A 2) yields

\[
V_{ij}V_{ij+1} = \left(1 - \frac{k_i}{\|V'^{y}_{j+1} - V'^{y}_{j}\|}\right) (V'^{y}_{j+1} - V'^{y}_{j}),
\]

and

\[
V_{i+1,j}V_{i+1,j+1} = \left(1 - \frac{k_{i+1}}{\|V'^{y}_{j+1} - V'^{y}_{j}\|}\right) (V'^{y}_{j+1} - V'^{y}_{j}).
\]

The above equations indicate that both vectors \( V_{ij}V_{ij+1} \) and \( V_{i+1,j}V_{i+1,j+1} \) are parallel to \( V'^{y}_{j+1} - V'^{y}_{j} \). Therefore, vertices \( V_{ij}, V_{ij+1}, V_{i+1,j} \) and \( V_{i+1,j+1} \) are co-planar.

Appendix B. Proof of geometric condition (II)

According to equation (2.1), the \( x \)-coordinates of vertices \( V_{i,j-1}, V_{ij} \) and \( V_{i+1,j} \) are all equal to \( x_j \). For convenience, we define an auxiliary coordinate system \( x'y'z' \) whose origin is located at \( \begin{bmatrix} x_j & 0 & 0 \end{bmatrix}^T \), as shown in figure 14. According to equation (2.3), \( \theta_j \) denotes the angle formed by rotating the positive direction of \( y \)-axis counter-clockwise to vector \( V'^{y}_{j+1} \). Since vector \( V_{ij}V_{ij+1} \) and the \( y' \)-axis are parallel to vector \( V'^{y}_{j}V'^{y}_{j+1} \) and the \( y \)-axis, respectively, the angle formed by rotating the positive direction of \( y' \)-axis counter-clockwise to vector \( V_{ij}V_{ij+1} \) also equals \( \theta_j \). In the \( y'-z' \) plane, denote the angle bisector of the exterior angle of \( \angle V_{ij-1}V_{ij}V_{ij+1} \) by \( \chi_1 \) (the dotted line in figure 14). The angle between line \( \chi_1 \) and \( y' \)-axes is found by

\[
y_s = \frac{\theta_{j-1} + \theta_j}{2} - \pi.
\]

Hence, the slope of \( \chi_1 \) is obtained by

\[
k_s = \tan y_s = \frac{\sin \theta_{j-1} + \sin \theta_j}{\cos \theta_{j-1} + \cos \theta_j}.
\]
According to the cosine law, there hold

\[ \cos \angle V_{i+1,j} V_{ij} V_{ij-1} = \frac{\| V_{ij} - V_{i+1,j} \|^2 + \| V_{ij} - V_{ij-1} \|^2 - \| V_{ij-1} - V_{i+1,j} \|^2}{2 \| V_{ij} - V_{i+1,j} \| \| V_{ij} - V_{ij-1} \|} \]

\[ \cos \angle V_{i+1,j} V_{ij} V_{ij+1} = \frac{\| V_{ij} - V_{i+1,j} \|^2 + \| V_{ij} - V_{ij+1} \|^2 - \| V_{ij+1} - V_{i+1,j} \|^2}{2 \| V_{ij} - V_{i+1,j} \| \| V_{ij} - V_{ij+1} \|}. \]

Furthermore, because line \( V_{ij} V_{i+1,j} \) bisects \( \angle V_{ij-1} V_{ij} V_{ij+1} \), there holds

\[ \angle V_{i+1,j} V_{ij} V_{ij+1} + \angle V_{i+1,j} V_{ij} V_{ij-1} = \pi. \]
Combining equations (B4)–(B9) yields
\[
\frac{\|V_{i+1,j} - V_{i,j}\|^2 + \|V_{i,j} - V_{i,j-1}\|^2 - \|V_{i+1,j} - V_{i,j-1}\|^2}{\|V_{i,j} - V_{i,j-1}\|} + \frac{\|V_{i+1,j} - V_{i,j}\|^2 + \|V_{i,j} - V_{i,j+1}\|^2 - \|V_{i+1,j} - V_{i,j+1}\|^2}{\|V_{i,j} - V_{i,j+1}\|} = 0. \tag{B 10}
\]

Applying the cosine law to equation (B10) yields
\[
\cos \angle V_{i+1,j}V_{i,j}V_{i,j-1} + \cos \angle V_{i+1,j}V_{i,j}V_{i,j+1} = 0,
\]
which gives
\[
\angle V_{i+1,j}V_{i,j}V_{i,j-1} + \angle V_{i+1,j}V_{i,j}V_{i,j+1} = \pi. \tag{B 12}
\]
Similarly, one can get
\[
\angle V_{i-1,j}V_{i,j}V_{i,j-1} + \angle V_{i-1,j}V_{i,j}V_{i,j+1} = \pi. \tag{B 13}
\]
Combining equations (B12) and (B13) yields
\[
\angle V_{i+1,j}V_{i,j}V_{i,j-1} + \angle V_{i+1,j}V_{i,j}V_{i,j+1} + \angle V_{i-1,j}V_{i,j}V_{i,j-1} + \angle V_{i-1,j}V_{i,j}V_{i,j+1} = 2\pi. \tag{B 14}
\]

References

5. Song Z et al. 2014 Origami lithium-ion batteries. Nat. Commun. 5, 3140. (doi:10.1038/ncomms4140)


