and estimate their number as about 1 in 250 fissions with $^{235}\text{U}$, with ranges significantly greater than the range of the natural $\alpha$-particles from this body. It will be seen that our overall results are readily reconcilable with this estimate, based, like our own, on the assumption of isotropic secondary emission.

REFERENCES

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A connexion between the criterion of yield and the strain ratio relationship in plastic solids

BY SIR GEOFFREY TAYLOR, F.R.S.

(Received 5 March 1947)

The assumption that the work done during a small plastic strain is a maximum as the yield-stress criterion is varied is shown to give rise to a connexion between the yield-stress and the strain-ratio relationship. The strain-ratio relationship is that which exists between the ratios of principal stress differences and the ratios of the corresponding strain differences. It is common to assume that this relationship is one of simple proportionality. Experiments, however, show that this assumption is not true in metals. The observed strain-ratio relationship is used in conjunction with the assumption of maximum work during a given strain to calculate the criterion of yield. It is found that this is very close to, but not identical with, the Mises-Hencky criterion.

INTRODUCTION

The three principal properties which define the plastic behaviour of an isotropic material and must be known or assumed before the solution of any problem in plastic flow can be attempted are:

(a) The criterion of yield (Mises-Hencky, maximum stress difference, etc.).
(b) The relationship between the ratios of small strains parallel to the principal axes and the ratios of principal stress differences. This relationship may be called the strain-ratio relationship.
(c) The amount of strain hardening after yielding.

When problems concerning small strains are considered it is frequently a good enough approximation to assume that there is no strain hardening, so that the criterion of yield is independent of the amount of small strain considered. On the other hand, both (a) and (b) must be known before any problems in which the stress depends on the strain can be solved.
Sir Geoffrey Taylor

It has usually been considered that (a) and (b) are quite unrelated, and from the purely mathematical point of view this is true. Problems are solvable mathematically, for instance, if either the Mises-Hencky or the maximum-stress difference hypothesis is assumed in conjunction either with the strain-ratio relationship observed with viscous fluids or with other strain-ratio assumptions. When the physical nature of a plastic material is considered, however, it seems that a certain relationship between (a) and (b) must exist. A relationship is here deduced using an assumption that the energy dissipation for a given strain is a maximum. The theoretical result is compared with available observations.

**Representation of plasticity conditions**

(a) *Criterion of yield*

If $\sigma_1$, $\sigma_2$, and $\sigma_3$ are the three principal stresses it may be assumed that the yield criterion depends only on the stress differences $\sigma_1 - \sigma_2$, $\sigma_2 - \sigma_3$, $\sigma_3 - \sigma_1$ and is independent of the mean pressure $\bar{p} = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$. It is convenient to express results in terms of the three residual stresses $\sigma'_1$, $\sigma'_2$, $\sigma'_3$, which result from subtracting the mean pressure from the principal stress, so that

$$\sigma'_1 = \frac{2}{3}\sigma_1 - \frac{1}{3}\sigma_2 - \frac{1}{3}\sigma_3, \text{ etc., and } \sigma'_1 + \sigma'_2 + \sigma'_3 = 0. \quad (1)$$

In these co-ordinates the Mises-Hencky criterion of yield is

$$(\sigma'_1)^2 + (\sigma'_2)^2 + (\sigma'_3)^2 = \frac{2}{3}Y^2, \quad (2)$$

and the maximum shear stress criterion is

$$\sigma'_1 - \sigma'_3 = Y, \quad (3)$$

where $Y$ is the yield stress in direct tensile loading and

$$\sigma'_1 \geq \sigma'_2 \geq \sigma'_3. \quad (4)$$

(b) *Strain-ratio relationship*

It will be assumed that the compressibility of the material is unchanged when plastic flow sets in. If $e_1$, $e_2$, $e_3$ are the principal strains (assumed small) it is convenient to use the residual principal strains

$$e'_1 = e_1 - \frac{1}{3}(e_1 + e_2 + e_3), \text{ etc.,} \quad (5)$$

because

$$e'_1 + e'_2 + e'_3 = 0, \quad (6)$$

so that if

$$e'_1 \geq e'_2 \geq e'_3, \quad (7)$$

all possible strain ratios have been considered when $e'_3/e'_1$ varies from $-\frac{1}{2}$ to $-2$.

In considering all possible relationships between stress ratios and strain ratios it is clear that any one of them could be represented by a curve in a diagram with $\sigma'_3/\sigma'_1$ as abscissa and $e'_3/e'_1$ as ordinate, the full range extending from $-\frac{1}{2}$ to $-2$ in each case. This diagram is unsymmetrical in the sense that if the yield is the same in compression as in tension the symmetrical diagram would show the compression
Connexion between criterion yield and strain ratio in plastic solids

part of the curve as similar to the part representing extension. The former extends from \( \sigma'_3/\sigma'_1 = -\frac{1}{3} \) to \(-1\), while the latter extends from \(-1\) to \(-2\), so that the diagram is unsymmetrical. To avoid this difficulty Lode (1926) introduced the symmetrical variables

\[
\begin{align*}
\mu &= \frac{2\sigma'_2 - \sigma'_1 - \sigma'_3}{\sigma'_1 - \sigma'_3} \quad \text{or} \quad -3 \left( \frac{\sigma'_1 + \sigma'_3}{\sigma'_1 - \sigma'_3} \right) \\
\nu &= \frac{2\epsilon'_2 - \epsilon'_1 - \epsilon'_3}{\epsilon'_1 - \epsilon'_3} \quad \text{or} \quad -3 \left( \frac{\epsilon'_1 + \epsilon'_3}{\epsilon'_1 - \epsilon'_3} \right)
\end{align*}
\]

The results of experiments on all possible combinations of \(\sigma'_1, \sigma'_2, \sigma'_3, \epsilon'_1, \epsilon'_2, \epsilon'_3\) can be represented on a square diagram (2) showing \(\mu\) as abscissa and \(\nu\) as ordinate and covering the range \(-1 < \mu < +1, -1 < \nu < +1\). It will be seen that when \(\nu\) is positive, \(\epsilon'_2\) is positive. This corresponds with cases where the residual principal strain which has the greatest absolute value is compressive; the point \(\nu = +1\) for instance corresponds with a uniaxial compressive load parallel to \(\epsilon'_3\). When \(\epsilon'_3\) is negative the residual strain which has the greatest absolute value is positive and \(\nu = -1\) corresponds with uniaxial extension parallel to \(\epsilon'_1\). In an isotropic medium the \(\mu, \nu\) curve must pass through the points \(\mu = \nu = -1, \mu = \nu = 0, \mu = \nu = +1\). Otherwise no limitations on the form of the \(\mu, \nu\) curve are imposed by considerations of symmetry alone. The results of experiments in which complex stresses were produced in a tube (a) by simultaneous end-load and internal pressure (Lode 1926), and (b) by simultaneous end-load and torque are given in a paper (Taylor & Quinney 1931) by the present author and the late Mr H. Quinney. It was found that with most of the metals used, namely, mild steel, decarburized mild steel, copper, nickel and aluminium, the \(\mu, \nu\) curve lies below the line \(\mu = \nu\) when \(\mu\) and \(\nu\) are positive (i.e. \(\mu > \nu > 0\)), and above this line when \(\mu\) and \(\nu\) are negative (i.e. \(\mu < \nu < 0\)). With very soft metals like lead and cadmium this was still true, but the experimental \(\mu, \nu\) curves lie nearer to the line \(\mu = \nu\) than with the above-mentioned metals. Experiments made with glass heated till it began to flow gave points lying very closely on the line \(\mu = \nu\). This is to be expected in view of the fact that \(\mu = \nu\) for viscous fluids.

Values of \(\mu\) and \(\nu\) found in experiments with annealed tubes of pure copper are given in columns 1 and 2 of table 1, and the corresponding values of \(\sigma'_3/\sigma'_1 = (\mu + 3)/(\mu - 3)\) are given in column 3.

**Table 1**

<table>
<thead>
<tr>
<th>(\mu)</th>
<th>(\nu)</th>
<th>(\sigma'_3/\sigma'_1)</th>
<th>(\chi)</th>
<th>(1/\chi - \sigma'_3/\sigma'_1)</th>
<th>(m)</th>
<th>(1/m)</th>
<th>(1 + \sigma'_3/\sigma'_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.021</td>
<td>-0.011</td>
<td>0.9860</td>
<td>1.022</td>
<td>0.498</td>
<td>0.025</td>
<td>0.907</td>
<td>1.038</td>
</tr>
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<td>-0.242</td>
<td>-0.171</td>
<td>0.8507</td>
<td>1.412</td>
<td>0.442</td>
<td>0.280</td>
<td>0.885</td>
<td>1.325</td>
</tr>
<tr>
<td>-0.471</td>
<td>-0.393</td>
<td>0.7286</td>
<td>2.295</td>
<td>0.331</td>
<td>0.515</td>
<td>0.946</td>
<td>1.756</td>
</tr>
<tr>
<td>-0.597</td>
<td>-0.502</td>
<td>0.6680</td>
<td>3.016</td>
<td>0.271</td>
<td>0.68</td>
<td>0.978</td>
<td>1.658</td>
</tr>
<tr>
<td>-0.652</td>
<td>-0.554</td>
<td>0.6429</td>
<td>3.484</td>
<td>0.240</td>
<td>0.70</td>
<td>0.980</td>
<td>1.779</td>
</tr>
<tr>
<td>-0.773</td>
<td>-0.659</td>
<td>0.5902</td>
<td>4.865</td>
<td>0.183</td>
<td>0.80</td>
<td>0.978</td>
<td>1.983</td>
</tr>
<tr>
<td>-0.874</td>
<td>-0.775</td>
<td>0.5487</td>
<td>7.889</td>
<td>0.118</td>
<td>0.90</td>
<td>0.997</td>
<td>1.993</td>
</tr>
<tr>
<td>-0.954</td>
<td>-0.876</td>
<td>0.5174</td>
<td>15.13</td>
<td>0.064</td>
<td>0.95</td>
<td>0.983</td>
<td>1.987</td>
</tr>
</tbody>
</table>
POSSIBLE MAXIMUM ENERGY DISSIPATION ASSUMPTION

In a paper published recently (Hill, Lee & Tupper 1947) it is shown that if the strain-ratio relationship expressed by the equation

$$
\mu = \nu
$$

(9)

holds, the work done during a given small strain is a maximum if the Mises-Hencky criterion of flow applies. Since experiments with most metals reveal a marked divergence from the ideal relationship (9), it is of interest to find out what criterion of flow would correspond with the maximum of work during a small strain when the experimentally observed relationship between $\mu$ and $\nu$ is used instead of (9). The work done during a small strain is

$$
W = \sigma_1' e_1' + \sigma_2' e_2' + \sigma_3' e_3'
$$

or

$$
W = \sigma_1'(2e_1' + e_3') + \cdots + \sigma_3'(e_1' + 2e_3').
$$

(10)

A stationary value of $W$ for a given strain occurs when

$$
\frac{d\sigma_3'}{d\sigma_1'} = -\frac{2e_1' + e_3'}{e_1' + 2e_3'}.
$$

(11)

It has been seen that experiments on the strain-ratio relationship give $\sigma_3'/\sigma_1'$ in terms of $e_3'/e_1'$, so that (11) will enable the criterion of yield to be determined by numerical integration. Alternatively, if the relationship between $\sigma_3'$ and $\sigma_1'$ is known by experiment, (11) gives directly the strain-ratio relationship. To determine the criterion of yield from the observed relationship between $\sigma_3'/\sigma_1'$ and $e_3'/e_1'$ (11) may be written

$$
\frac{d\sigma_3'}{d\sigma_1'} = \sigma_1' \frac{d}{d\sigma_1'} \left( \frac{\sigma_3'}{\sigma_1'} \right) + \frac{\sigma_3'}{\sigma_1'} = -\frac{(2 + e_3'/e_1')}{(1 + 2e_3'/e_1')}.
$$

(12)

Taking the experimental value of $e_3'/e_1'$ corresponding with each value of $\sigma_3'/\sigma_1'$, the corresponding experimental value of $-\frac{(2 + e_3'/e_1')}{(1 + 2e_3'/e_1')}$ can be found. Calling this $\chi$, (11) can be integrated, giving

$$
\text{constant} + \log_{e} \sigma_1' = \int \frac{d(\sigma_3'/\sigma_1')}{\chi - (\sigma_3'/\sigma_1')} = I.
$$

(13)

Expressed in terms of $\nu$, $\chi = \frac{1 - \nu}{1 + \nu}$. Values of this for copper are given in column 4, table 1, and values of the integrand of (12) in column 5. To perform the integration the values given in column 5, table 1, were plotted against $\sigma_3'/\sigma_1'$. The values of the integral $I$ were then determined from this diagram for values of $\sigma_3'/\sigma_1'$ ranging from $-1$ to $-0.5$. These are given in column 2, table 2. The value of the constant in (13) is determined by the fact that when $\sigma_3'/\sigma_1' = -0.5$, $\sigma_1' = \frac{2}{3} Y$. In this way the values given in column 3 of table 2 were found for $3\sigma_1'/2Y$. 


Connexion between criterion yield and strain ratio in plastic solids

Table 2

<table>
<thead>
<tr>
<th>$\sigma_3/\sigma_1$</th>
<th>I</th>
<th>$3\sigma_3/2Y$</th>
<th>$\sqrt{3}\left(1 + \frac{\sigma_3}{\sigma_1} + \left(\frac{\sigma_3}{\sigma_1}\right)^2\right)^{-1}$</th>
<th>$3\left(1 - \frac{\sigma_3}{\sigma_1}\right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0</td>
<td>0.849</td>
<td>0.866</td>
<td>0.750</td>
</tr>
<tr>
<td>-0.95</td>
<td>0.0247</td>
<td>0.870</td>
<td>0.888</td>
<td>0.770</td>
</tr>
<tr>
<td>-0.90</td>
<td>0.0487</td>
<td>0.891</td>
<td>0.908</td>
<td>0.789</td>
</tr>
<tr>
<td>-0.85</td>
<td>0.0716</td>
<td>0.912</td>
<td>0.927</td>
<td>0.810</td>
</tr>
<tr>
<td>-0.80</td>
<td>0.0926</td>
<td>0.931</td>
<td>0.945</td>
<td>0.834</td>
</tr>
<tr>
<td>-0.75</td>
<td>0.1114</td>
<td>0.949</td>
<td>0.960</td>
<td>0.856</td>
</tr>
<tr>
<td>-0.70</td>
<td>0.1278</td>
<td>0.964</td>
<td>0.974</td>
<td>0.882</td>
</tr>
<tr>
<td>-0.65</td>
<td>0.1416</td>
<td>0.978</td>
<td>0.985</td>
<td>0.909</td>
</tr>
<tr>
<td>-0.60</td>
<td>0.1526</td>
<td>0.989</td>
<td>0.992</td>
<td>0.938</td>
</tr>
<tr>
<td>-0.55</td>
<td>0.1605</td>
<td>0.996</td>
<td>0.997</td>
<td>0.969</td>
</tr>
<tr>
<td>-0.525</td>
<td>0.1629</td>
<td>0.999</td>
<td>0.999</td>
<td>0.985</td>
</tr>
<tr>
<td>-0.50</td>
<td>0.1640</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Comparison with Mises-Hencky and maximum stress difference criteria

The Mises-Hencky criterion (2) may be written

$$\sigma_1^2 + \sigma_1^2 \sigma_2 + \sigma_3^2 = \frac{1}{3} Y^2,$$

so that

$$\frac{3\sigma_1^2}{2Y} = \frac{\sqrt{3}}{2} \left(1 + \frac{\sigma_3}{\sigma_1} + \left(\frac{\sigma_3}{\sigma_1}\right)^2\right)^{-\frac{1}{2}}.$$

Values of this function are given in column 4 of table 2.

The maximum stress difference criterion is

$$\sigma_1 - \sigma_3 = Y \quad \text{or} \quad \frac{\sigma_1}{Y} = \frac{1}{1 - \frac{\sigma_3}{\sigma_1}}.$$

For comparison the values of $\frac{3}{2(1 - \sigma_3/\sigma_1)}$ are shown in column 5 of table 2. The values given in columns 3, 4, and 5 are shown graphically in figure 1.

Comparison with observed criterion

The observations made with copper tubes subjected to combined end-load and torsion (Taylor & Quinney 1931) were carried out by first loading the tube till it stretched, say, 0.5%.

If the longitudinal stress for this pure extension was $Y$ the load was reduced to $P = mY$ ($0 < m < 1$) and torque applied till plastic flow occurred. It was found that though the beginning of inelastic displacement was not easy to measure the torque at which steady plastic flow occurred was definite. The shear stress corresponding with this condition being $S$, the angle $\theta$ between the axis of the tube and the axis of the greatest principal stress is given by

$$\tan 2\theta = \frac{2S}{P}.$$
The principal stresses are

\[ \sigma_1 = \frac{1}{2} P(1 + \sec 2\theta), \quad \sigma_2 = 0, \quad \sigma_3 = \frac{1}{2} P(1 - \sec 2\theta), \tag{18} \]

so that

\[ \frac{\sigma'_1}{Y} = \frac{1}{3} m(1 + \sec 2\theta) = \frac{1}{3} m, \tag{19} \]

and

\[ \frac{\sigma'_1}{\sigma'_3} = \frac{1}{3} - \frac{\sec 2\theta}{\sec 2\theta} \quad \text{or} \quad \cos 2\theta = 3 \frac{\sigma'_3 + \sigma'_1}{\sigma'_3 - \sigma'_1}. \tag{20} \]

Hence

\[ \frac{3\sigma'_1}{2Y} = \frac{1}{2} \frac{m}{1 + (\sigma'_3/\sigma'_1)}. \tag{21} \]

Figure 1. (a) Mises-Hencky. (b) Calculated by stationary energy dissipation hypothesis. (c) Maximum stress difference (Mohr). × × Observed points (Taylor & Quinney 1931).

The experimental values of \( m \) and \( \sigma'_3/\sigma'_1 \) are given in columns 6 and 3 of table 1. The values of \( 3\sigma'_1/2Y \) found by inserting those in (21) for values of \( m > 0.28 \) are given in column 7 of table 1. When \( m \) is very small (21) tends to the form 0/0, so that another expression for \( \sigma'_1 \) is more accurate. The form used when \( m = 0.025 \) and \( m = 0.28 \) was

\[ \frac{3\sigma'_1}{2Y} = \frac{P}{Y} \left( \frac{1}{4} + \frac{3}{4} \sqrt{\left(1 + \frac{4S^2}{P^2}\right)} \right). \tag{22} \]

The experimental points are plotted in figure 1, where it will be seen that the material behaves very nearly in accordance with the prediction based on the assumed stationary energy principle and the measured strain ratio relationship. Both the observed points and those calculated by the minimum energy relationship lie close to the Mises-Hencky line.

References