An investigation, by the method of characteristics, of the lateral expansion of the gases behind a detonating slab of explosive

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The phenomena occurring when an uncased explosive charge is detonated in a fluid medium are examined by hydrodynamical methods.

Attention is focused chiefly on the pressure and velocity distributions in the gaseous products of the explosion, which expand laterally behind the detonation wave as it travels down the charge, the results being shown in graphical form. To simplify the problem, the charge, and the gas and fluid fields, were treated as two-dimensional.

The hydrodynamical equations were solved numerically using the method of characteristics. This dates back to Monge, but it is only recently that it has been applied to the numerical solution of hyperbolic equations. The methods of numerical integration used in this paper are similar to those developed early in the war by the Research Section of the External Ballistics Department, Ordnance Board, for determining the velocity distributions around projectiles moving at supersonic speeds.

The nature of the boundary conditions made it necessary to find explicit theoretical formulas for the gas field near the charge, and the analysis involved is given at length.

For the problem in which the surrounding medium is air, the shape and position of the shock waves set up by the explosion are calculated. The shock waves are found to be straight to the nominal accuracy of the calculations (1 in 5000) for six charge widths from their intersections with the block of explosive.

1. INTRODUCTION

When a block of explosive is detonated, the gaseous products expand laterally behind the detonation wave. At the wave front their velocity relative to the wave is taken equal to the local velocity of sound, in accordance with the Chapman-Jouguet condition; but elsewhere in the gas field this relative flow is supersonic, and the expansion is so rapid that the pressure along the axis of the charge falls to one-tenth of its value at the face, in a distance comparable with the thickness of the charge.

The velocity of detonation, being far greater than the sonic velocity in the outer medium, causes a shock wave to be formed. For normal conditions in air, the shock wave meets the detonation wave on the surface of the charge, but it is possible, for certain initial conditions of the outer medium, to have the shock wave travelling ahead of the detonation wave, the possibilities being analogous to those for a projectile in flight.

The boundary between the gas and the outer medium is curved, and across it the stream velocities are discontinuous. In general, too, the shock wave is curved, and vorticity, resulting from the variation of entropy along and immediately behind the wave, spreads into the high-pressure region. Along each streamline when the outer medium is a perfect gas, the vorticity is, in fact, proportional to the pressure. Only when the shock wave is straight, the initial conditions in the medium being

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uniform, is the motion irrotational. This has been the case, however, over all the field covered by the present investigation, since the shock wave was found to be straight for a considerable distance behind the detonation wave.

The treatment of these phenomena by hydrodynamical means presents, therefore, a complicated mathematical problem, and it does not seem possible to obtain an analytical solution in finite terms; thus recourse must be had to numerical integration of the equations involved. To make the problem more tractable the block of explosive and the gas and fluid fields are considered to be two-dimensional.

Certain difficulties are inherent in the numerical integration of the equations, on account of the sonic velocity of relative flow immediately behind the detonation wave. It was clear that the problem could only be begun by obtaining analytical solutions which would enable boundary conditions to be stipulated along a line away from the face. The various types of analytical solution which it was necessary to seek are given in §§ 5, 6 and 7, and in a subsequent section the question of accuracy achieved in the fitting of the boundary conditions is discussed.

In § 8 is to be found a general account of the way in which perturbations of the flow are carried through the gas field. It shows how the conditions at a considerable distance from the corner are dependent upon conditions very close to the face on the axis of the charge, and demonstrates that a simple Meyer expansion provides a remarkably accurate solution over a large part of the gas field.

2. Initial conditions

Throughout the present paper c.g.s. units are used. The block of explosive is taken to be T.N.T., of which the unexploded density is 1.5 g./c.c., for which the observed velocity of detonation is 6790 m./sec. The thickness of the block is taken to be 2 cm. The velocity of the gases just behind the detonation wave, relative to the wave itself, is equal to the local velocity of sound $a_0$ (§ 1). $a_0$ is taken to be 4697 m./sec. corresponding to a gas density of 1.895 g./c.c. Tables of pressure-density relations and of density—gas velocity—velocity of sound relations were supplied by Mr C. R. Illingworth, who obtained them by using the results of two papers by Dr H. Jones (unpublished). In calculating the densities, the solid carbon was separated from the detonation products, in preparation for the application of the hydrodynamical equations for the flow of a compressible fluid.

The initial conditions in the outer medium were taken for half-saturated air at 60° F, with $\gamma$, the ratio of specific heats, taken equal to 1.405, the velocity of sound 341.359 m./sec., and the pressure 1015.9 mb. The Mach number for the incident air, moving with a velocity equal to the velocity of detonation, is 19.8911.

3. Mathematical formulation of the problem

The problem is considered to be two-dimensional, a section of the block normal to its surface and to the detonation wave (considered plane) being a rectangle. For the purposes of mathematical treatment, the rectangle is taken to be semi-infinite in length, although finite in breadth.
Except where otherwise stated, Cartesian axes moving with the detonation wave are used. The ordinates are measured parallel to the wave, and the abscissae parallel to the axis of symmetry. The origin is at the centre of the block face.

The detonation wave is supposed to travel steadily down the block, thus enabling the equations for steady motion in conjunction with the moving axes, to be applied to the field behind the wave.

The general features of the gas and fluid fields are shown in figure 1. OD marks the line of separation between the gas and the outer medium and is a streamline for both media. The gas and fluid velocities are, in general, not equal along OD, but the pressure is continuous across OD.

![Diagram](image)

**Figure 1.** The subdivisions of the fields of product gases and air. OS is a shock wave in air, and OD is the boundary between air and product gases.

The gas region is divided into the primary field and the secondary field, OC marking the boundary between them. The distinction between these two fields is chiefly of mathematical significance. In the primary gas field, defined by the singularity at the corner and the boundary conditions along the detonation wave and axis of symmetry, all streamlines originate from the charge face and there is no streamline from the corner. But the line of separation of the gas and the outer medium must be a streamline from the corner in both fluids. Mathematically, the reconciliation is effected by the possibility of adjoining a secondary gas field to the primary field. The line of separation OC must be a characteristic (Goursat 1922) from the corner in the primary field, and the velocity and pressure must be continuous
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across it. The actual characteristic and the velocity and pressure distribution throughout the secondary field are then uniquely determined by the initial conditions in the outer medium (in contradistinction to the distribution in the primary field which is independent of the outer medium). It is important to notice that there is no singularity at the corner in the secondary field, so allowing the existence of a streamline from the corner. In the case of the semi-infinite block, where characteristics from the corner are radii and conditions in the primary field are constant along radii, the secondary region is one of uniformity of velocity and pressure. For the finite block, \( OC \) is the characteristic whose initial direction is that of the line of separation in the semi-infinite block solution (for the same conditions in the outer medium). The hydrodynamical equations of motion (§ 4) for the secondary region are exactly those for the primary region.

Referring again to figure 1, \( OS \) is the shock wave in the surrounding fluid and the region bounded by \( OS \) and \( OD \) is filled with fluid which has passed through the shock wave and which will, in general, be in vortex motion. Irrotational motion implies a straight shock wave (with uniform initial conditions in the outer medium).

4. General theory of gas field

Referred to the axes described in § 3, the equation satisfied by the velocity potential \( \phi \) of the relative motion is

\[
(u^2 - a^2) \frac{\partial^2 \phi}{\partial x^2} + 2uv \frac{\partial^2 \phi}{\partial x \partial y} + (v^2 - a^2) \frac{\partial^2 \phi}{\partial y^2} = 0.
\]  
\[(4.1)\]

\((u, v)\) are the components of relative stream velocity in the \( x \) and \( y \) directions. \( a \) is the local velocity of sound, given in terms of the total velocity by Bernoulli's equation

\[
u^2 + v^2 = a^2_0 - 2 \int_{\rho_0}^{\rho} \frac{a^2 d\rho}{\rho},
\]  
\[(4.2)\]

where the suffix zero denotes values on the face. Equation (4.1) is integrated using the method of characteristics (Goursat 1922). The equations of the two sets of characteristics are

\[
dy = \mu_1 dx \quad \text{and} \quad dy = \mu_2 dx,
\]  
\[(4.3)\]

where \( \mu_1, \mu_2 \) are the roots of

\[
(u^2 - a^2) \mu^2 - 2uv \mu + (v^2 - a^2) = 0,
\]

and are real since the flow is everywhere supersonic.

Along \( dy = \mu_1 dx \), \( u \) and \( v \) satisfy the relation

\[
du + \mu_2 dv = 0,
\]  
\[(4.4)\]

and along \( dy = \mu_2 dx \),

\[
du + \mu_1 dv = 0.
\]

These equations are converted into difference relations, using mean values, and then integrated numerically. The details of this process are given in § 9.
A general picture of the characteristics is given in figure 2. The two sets radiate from the corners of the block, one set from each corner. This may be compared with the case of the semi-infinite block where one set of characteristics are simply radius vectors from the corner. The values of \( (u, v) \) at any point of the field are obtained by integrating along the two characteristics through the point using the known boundary values. The boundary conditions in this problem are the given velocity on the face, the vanishing of \( v \) along the axis of symmetry and the singularities expressing the existence of the corner. No difficulty arises from the axis condition, nor from the singularities, since the velocities immediately at the corners are provided by the Meyer solution. But a serious difficulty presents itself in the conditions near to and on the face, for the gradients of the characteristics tend to infinity as the face is approached, and in the limit the characteristics initially making a zero angle with the face coincide entirely with it. It is, therefore, impossible to use

\[\text{Figure 2. Typical characteristics in the primary gas field. The broken lines represent the limit to which the analytical solutions were taken.}\]
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Equations (4·3) and (4·4), and in order to start on the problem at all, recourse must be had to some other means. The method adopted here is to obtain analytical solutions in series form, valid near to the face and close to the corners. Effectively, this substitutes for the boundary conditions on \( x = 0 \) boundary conditions some distance out into the field where the gradients of the characteristics are sufficiently small to permit using equations (4·3) and (4·4) to the degree of accuracy required.

Three different types of series solutions are used:

(a) A solution in powers of \( x \), covering the field near the face (§ 5).

(b) A solution in powers of \( \theta \) (the angular distance between the face and a radius from the corner), the necessary boundary conditions being given by (a) (§ 6).

(c) A solution in powers of \( r \), using boundary conditions given by (b) (§ 7).

Finally, in § 8, a more general method of approach is used to indicate the extent of the region of validity of the solutions, and the way in which the existence of an axis of symmetry modifies the Meyer expansion near the block.

5. Solution near the face

\((u, v)\) denote Cartesian components of velocity referred to the axes described in § 4.

The equation of continuity is

\[
\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0.
\]  

(5·1)

Bernoulli’s equation is

\[
u^2 + v^2 = a_0^2 - 2 \int_{\rho_s}^\rho \frac{a^2 d\rho}{\rho},
\]  

(5·2)

where suffix zero refers to values on the face.

It is required to find a solution in the form of a power series in \( x \), valid in some region near the face. Assume the velocity potential \( \phi \) is expressible as

\[
\phi(x, y) = a_0 x + P_2(y) x^2 + P_3(y) x^3 + \ldots,
\]  

(5·3)

where

\[
u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}.
\]

The axis condition gives

\[
P_2'(0) = 0, \quad P_3'(0) = 0, \quad \text{etc.}
\]  

(5·4)

If

\[
\rho(x, y) = \rho_0 + F_1(y) x + F_2(y) x^2 + \ldots,
\]

then

\[
\int_{\rho_s}^{\rho} \frac{a^2 d\rho}{\rho} = \frac{a_0^2}{\rho_0} (F_1 x + F_2 x^2 + \ldots + x_2 (F_1 x + F_2 x^2 + \ldots)^2 + \ldots,
\]

where

\[
x_2 = \frac{1}{2! d\rho_0} \left( \frac{a_0^2}{\rho_0} \right), \quad \text{etc.}
\]

Substituting the series into (5·1) and (5·2) and equating coefficients of powers of \( x \), we obtain successively:
From the coefficients of \( x^0 \) in (5·1) and \( x \) in (5·2), the same relation, viz.
\[
a_0 F_1 + 2 \rho_0 P_2 = 0.
\]

From the coefficients of \( x \) in (5·1) and \( x^2 \) in (5·2), on elimination of \( P_3 \)
\[
\left( 3 + \frac{2 \alpha_2 \rho_0^2}{\alpha_0^2} \right) F_1^2 = 0.
\]

The dimensionless quantity
\[
\frac{2 \alpha_2 \rho_0^2}{\alpha_0^2}
\]
will, in future, be denoted by \( \beta \). Its value, obtained from the data described in § 3, is 4·8496. Since \( 3 + \beta = 0, \ F_1 = 0, \) and so
\[
P_2 = 0.
\]
Hence
\[
a_0 F_2 + 3 \rho_0 P_3 = 0.
\]

From the coefficients of \( x^2 \) in (5·1) and \( x^3 \) in (5·3), two equations are obtained from which \( P_3 \) and \( F_4 \) can be simultaneously eliminated to give a differential equation for \( P_3 \), viz.
\[
P_3'' = \frac{P_3^2}{\alpha_0^2} \frac{18}{a_0} (3 + \beta).
\]

Integrating and using (5·4)
\[
(P_3^2)' = \frac{12 (3 + \beta)}{a_0} [P_3^2(y) - P_3^2(0)],
\]
therefore
\[
y \left[ \frac{12 (3 + \beta)}{a_0} P_3(0) \right]^\frac{1}{4} = \int_1^t \frac{dt}{(t^3 - 1)^{\frac{1}{4}}},
\]
where \( t = P_3(y)/P_3(0) \). \( P_3(0) \) is determined by the singularity at \( y = L \) (taking \( L \) to be half the width of the face). This is expressed mathematically by making the pole of the elliptic function involved in the solution of (5·6) correspond to \( y = L \). The justification of the assumption that \( P_3(y) \) has a singularity at \( y = L \) is given in § 6.

The equation for \( P_3(0) \) then becomes
\[
\left[ \frac{12 (3 + \beta)}{a_0} L^2 P_3(0) \right]^\frac{1}{4} = \int_1^\infty \frac{dt}{(t^3 - 1)^{\frac{1}{4}}} = \frac{3}{4} \pi \Gamma \left( \frac{3}{4} \right) \Gamma \left( \frac{3}{2} \right)
\]
(Whittaker & Watson 1927). Taking \( L = 1 \), it is found that \( P_3(0) = 0·294 \).

In the usual notation, if \( \varphi(z) \) denotes the Weierstrassian elliptic function with invariants
\[
g_2 = 0 \quad \text{and} \quad g_3 = 4 P_3^3(0),
\]
it is not difficult to show that
\[
P_3(1 - r) = \varphi \left[ \left( \frac{3 (3 + \beta)}{a_0} \right)^{\frac{1}{4}} r \right].
\]

The asymptotic expansion of \( P_3(1 - r) \) for small \( r \) is
\[
P_3(1 - r) \sim \frac{1}{A r^2} + \frac{P_3^3(0)}{7} (Ar^2)^2 + \frac{1}{13} \frac{P_3^3(0)^2}{7^2} (Ar^2)^5 + \ldots,
\]

(5·9)
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where \( A = 3(3+\beta)/a_0 \). Putting in numerical values, with \( L = 1 \),

\[
P_3(1-r) \approx \frac{0.19946}{r^2} + 0.0913r^4 + 0.0032r^{10} + \ldots.
\]

This expression serves to calculate \( P_3 \) as far as \( r = 1 \) to the accuracy required here.

It was not found necessary to continue the series for \( \phi \) beyond the \( x^3 \) term, and

so the actual approximations used take the form

\[
u = a_0 + 3P_3(y)x^2, \quad v = P_3(y)x^3.
\] (5.10)

The region over which these formulae were taken to hold will be described in \$9.\$

6. Solution for small \( \theta \) with origin at a corner

Taking a corner as origin, and measuring \( \theta \) from the face of the block, it is required

to find a solution of the equations, valid for small \( \theta \).

Using polar co-ordinates, let \( u \) be the component velocity along the radius vector,

and \( v \) the transverse component. Assume the expansion for the velocity potential \( \phi \) to be

\[
\phi = \sum_{1}^{\infty} Q_n(r) \theta^n, \quad \text{ (6.1)}
\]

giving

\[
u = \frac{\partial \phi}{\partial r} = \sum_{1}^{\infty} Q_n(r) \theta^n \quad \text{ (6.2)}
\]

and

\[
v = \frac{\partial \phi}{r \partial \theta} = \sum_{0}^{\infty} (n+1) \frac{Q_{n+1}(r)}{r} \theta^n. \quad \text{ (6.3)}
\]

Further, assume

\[
\rho = \sum_{0}^{\infty} G_n(r) \theta^n. \quad \text{ (6.4)}
\]

Then if \( a_0 = \) velocity of sound at the block face, and \( \rho_0 \) is the density of the gases

there,

\[
G_0(r) = \rho_0, \quad Q_1(r) = a_0 r. \quad \text{ (6.5)}
\]

Bernoulli’s equation, giving relations between \( Q \)’s and \( G \)’s, is, as before,

\[
w^2 + v^2 = a_0^2 - 2a_0^3 \frac{\rho \rho_0}{\rho_0} (\rho - \rho_0) - 2a_2 (\rho - \rho_0)^2 - 2a_3 (\rho - \rho_0)^3. \quad \text{ (6.6)}
\]

The equation of continuity is

\[
\frac{\partial}{\partial r} (\rho u) + \frac{\partial}{\partial \theta} (\rho v) = 0. \quad \text{ (6.7)}
\]

On substitution of equations (6.2), (6.3) and (6.5) in (6.7) and equating coefficients

of \( \theta^0, \theta \),

\[
G_1 a_0 r + 2 \rho_0 Q_2 = 0, \quad \text{ (6.8)}
\]

\[
\rho_0 a_0 + 4G_1 \frac{Q_3}{r} + 2G_2 a_0 + 6 \rho_0 \frac{Q_3}{r} = 0, \quad \text{ (6.9)}
\]
and from equation (6.6), by equating coefficients of $\theta$, $\theta^2$,

$$4a_0 \frac{Q_2}{r} = -\frac{2a_0^3}{\rho_0} G_1,$$  \hspace{1cm} (6.10)

$$a_0^2 + 6a_0 \frac{Q_2}{r} \cdot \frac{4Q_2^2}{\rho_0} = -\frac{2a_0^3}{\rho_0} G_2 - 2\alpha_2 G_2^2.$$  \hspace{1cm} (6.11)

Equations (6.8) and (6.10) are the same, but combining with equations (6.9) and (6.11) it follows that

$$Q_2^2 \left(3 + 2\alpha_2 \frac{\rho_0^2}{a_0^3}\right) = 0.$$  

The expression in brackets is non-zero (§ 5). It follows, therefore, that

$$Q_2(r) = 0$$

and hence

$$G_1(r) = 0.$$  \hspace{1cm} (6.12)

Equating coefficients of $\theta^3$ in equation (6.7) and of $\theta^3$ in equation (6.6) supplies only one new equation, namely,

$$4 \frac{Q_3}{r} + \frac{a_0}{\rho_0} G_3 = 0.$$  \hspace{1cm} (6.13)

Equating coefficients of $\theta^5$ in the continuity equation and of $\theta^4$ in Bernoulli’s equation, and eliminating the $G$ functions, it follows eventually that

$$rQ_3^2(r) - 6Q_4(r) = \frac{a_0}{2} (1 + \beta) + 6(1 + \beta) \frac{Q_3(r)}{r} + \frac{18(3 + \beta)}{a_0} \left\{ \frac{Q_3(r)}{r^2} \right\},$$  \hspace{1cm} (6.14)

where $\beta$ is equal to $2\alpha_2(\rho_0^2/a_0^3)$ as in § 5. Two particular integrals of this equation can be seen at once to be $Q_3 = \alpha r$, where

$$\alpha = -\frac{1}{6} a_0 \text{ or } -\frac{1}{6} a_0 \lambda \text{ with } \lambda = (1 + \beta)/(3 + \beta) = 0.7452.$$  

It can be shown that the former value of $\alpha$ contributes a term towards a solution representing steady flow with velocity $a_0$, and that the latter contributes a term towards the expansion of Meyer’s solution in which the velocities are constant along a radius vector.

If $\alpha$ is allowed to be a function of $r$ in order to get a more general solution the following alternative forms of solution are obtained:

$$Q_3(r) = -\frac{a_0 \lambda}{6} r + A_1 r^7 + A_2 r^{13} + \ldots$$  \hspace{1cm} (6.15)

ascending in powers of $r^d$ with $A_1$ arbitrary and $A_n \ (n > 1)$ a function of $A_1$,

$$Q_3(r) = -\frac{a_0 r}{6} + B_1 r^3 + B_2 r^5 + \ldots$$

ascending in powers of $r^3$ with $B_1$ arbitrary and $B_n \ (n > 1)$ a function of $B_1$,

$$Q_3(r) = -\frac{a_0 r}{6} + C_1 r^4 + C_2 r^7 + \ldots$$
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ascending in powers of $r^3$ with $C_1$ arbitrary and $C_n \ (n > 1)$ a function of $C_1$. Of these solutions the last two are discarded, since they do not tend to Meyer's solution near the corner.

Equation (6·15) is now used to justify the assumption in § 5 that $P_3(y)$ has a singularity at $y = 1$. $P_3(y)$ is expanded in a series of positive and negative powers of $(1 - y)$, and then the substitution $y = 1 - r(1 - \frac{1}{3} \theta^2 + \ldots)$, $x = r(\theta - \frac{1}{3} \theta^3 + \ldots)$ is made in the expression for the velocity potential in powers of $x$. The coefficients of $\theta$, $\theta^3$ are compared with the corresponding coefficients in the expression for the velocity potential in powers of $\theta$, viz.

$$\phi = a_0 r \theta - \left[ \frac{a_0 \lambda}{6} r - A_1 r^7 - A_2 r^{13} - \ldots \right] \theta^3 + \ldots \quad (6·16)$$

It is easily deduced that $P_3(y)$ has a pole of the second order at $y = 1$.

The series for $P_3(1 - r)$ in the neighbourhood of $r = 0$, found in equation (5·9), may now be used to obtain $A_1$, $A_2$, etc., by further equating of coefficients above.

It is found that $A_1 = 0·0913$, $A_2 = 0·0032$, since $A_1 = \frac{A^2}{7} P_3^2(0)$, $A_2 = \frac{1}{13 \cdot 7^2} P_3^6(0)$, using the notation of (5·9).

The components of velocity in the $\theta$ expansion are

$$u_r = a_0 \theta - \left[ \frac{a_0 \lambda}{6} r - A_1 P_3^2(0) r^6 - A_2 P_3^6(0) r^{13} - \ldots \right] \theta^3 + \ldots,$$

$$v_\theta = a_0 - 3 \left[ \frac{a_0 \lambda}{6} r - \frac{A^2}{7} P_3^2(0) r^6 - \frac{A^5}{13 \cdot 7^2} P_3^6(0) r^{12} - \ldots \right] \theta^2 + \ldots$$

7. Solution near the corner

$u(r, \theta)$ and $v(r, \theta)$ denote $(r, \theta)$ components of velocity, where $r$ is the distance from one corner and $\theta$ the angular distance from the face. $u(\theta)$, $v(\theta)$, $\rho(\theta)$, $a(\theta)$, denote quantities in the semi-infinite block solution, constant along radius vectors. A solution is sought for the field near to and around a corner. The expansion in powers of $r$ assumed for the velocity potential is

$$\phi(r, \theta) = ru(\theta) + r^n R_n(\theta) + \text{higher powers of } r, \quad (7·1)$$

where $n$ is some integer $\geq 2$ to be determined. The velocities are given by

$$u(r, \theta) = u(\theta) + nr^{n-1} R_n(\theta) + \ldots, \quad (7·2)$$

$$v(r, \theta) = v(\theta) + nr^{n-1} R_n(\theta) + \ldots, \quad (7·3)$$

remembering that $v(\theta) = du/d\theta$.

The corner condition is automatically satisfied since in the limit as $r \to 0$, $u(r, \theta)$ and $v(r, \theta)$ tend to the semi-infinite block solution. For the density it is assumed that

$$\rho(r, \theta) = \rho(\theta) + nr^{n-1} H_n(\theta) + \ldots \quad (7·4)$$
Bernoulli’s equation is
\[ u^2(r, \theta) + v^2(r, \theta) = u^2(\theta) + v^2(\theta) - 2 \int_{\rho(\theta)}^{\rho(r, \theta)} a^2 \frac{d\rho}{\rho}. \quad (7.5) \]

On substituting the series in equation (7.5) and equating to zero the coefficient of \( r^{n-1} \), there results
\[ nuR_n + vR'_n = -\frac{a^2}{\rho} H_n, \quad (7.6) \]
where all variable quantities are functions of \( \theta \) only.

The \( r^{n-1} \) term in the continuity equation
\[ \rho u + r \frac{\partial}{\partial r} (\rho u) + \frac{\partial}{\partial \theta} (\rho v) = 0 \]
gives
\[ \rho R'_n + \rho' R'_n + n^2 \rho R_n + vH'_n + (nu + v') H_n = 0. \quad (7.7) \]

The elimination of \( H_n \) between equations (7.6) and (7.7) leads, after some reduction, to
\[ -2R'_n + R_n \left[ \frac{v'}{v} - \frac{nu}{v} + (n-1) \frac{u'}{u} - \frac{\rho'}{\rho} \right] = 0. \]

\( R_n' \) has disappeared during the elimination, due to the relation \( v = a \).

Integrating
\[ R_u(\theta) = C \frac{u_{\theta(n-1)\theta\theta}}{\rho^4} \exp \left[ -\frac{n}{2} \int_0^\theta \frac{u d\theta}{v} \right], \quad (7.8) \]
where \( C \) is a constant to be determined by the remaining boundary condition, viz. the equivalence of this expansion when \( \theta \) is small with that of § 6 when \( r \) is small.

To order of \( \theta^3 \),
\[ R_u(\theta) = \frac{Ca_0^{1n}}{\rho_0^4} \theta^3 + \ldots. \]

When the solution in § 6 is expanded for small \( r \), it is found that the first term of type \( \theta^3 r^n \), where \( n \geq 2 \), is
\[ \frac{P_{3(0)}^2}{7} A^2 \theta^3 r^n \quad (A = \frac{3(3 + \beta)}{a_0} \text{ as in equation } (5.9)). \]

It follows that \( n = 7 \),
\[ \frac{Ca_0^3}{\rho_0^4} = \frac{P_{3(0)}^2}{7} A^2, \]
and so
\[ R_u(\theta) = \frac{P_{3(0)}^2}{7} A^2 \frac{[u/a_0]^3 [v/a_0]^4}{[\rho/\rho_0]^4} \exp \left[ -\frac{7}{2} \int_0^\theta \frac{u d\theta}{v} \right]. \quad (7.9) \]

In the case considered in the present paper, when the block has a thickness of 2 cm., \( R_u(\theta) \) increases from zero on \( \theta = 0 \) to a maximum value of 0.0117 when \( \theta \) is about 48.4°, and thereafter decreases asymptotically to zero. \( R_u(\theta) \) attains a maximum positive value of 0.025 when \( \theta = 30^\circ \) and a maximum negative value of -0.020 when \( \theta = 72^\circ \) and then decreases numerically to zero. Thus the second terms of equations (7.2) and (7.3) are always very small compared with the first terms whenever \( r \) is < \( \frac{1}{2} \), say.
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Equations (7·2), (7·3) and (7·4) make it possible to assess the accuracy to which the solution for a semi-infinite block holds in the case of a finite block in a region near a corner.

Since the perturbation terms depend on \( r^8 \), the accuracy will be very good, certainly for \( r < \frac{1}{4} \). As might be expected on physical grounds alone, the accuracy is least in the region \( 30^\circ < \theta < 75^\circ \), and highest in the region round the corner away from the axis of symmetry.

8. General method of approximate solution

It is possible to give a more general treatment of the way in which the solution in the gas field for a finite block differs from the solution for a semi-infinite block. Taking \( (r, \theta) \) co-ordinates from one corner, let \( \phi(r, \theta) \) be the velocity potential in the first case, and, as in § 7, let \( u(\theta), v(\theta), \rho(\theta), a(\theta) \) denote the known quantities occurring in the second case.

Writing \[ \phi(r, \theta) = ru(\theta) + \epsilon(r, \theta) \] (8·1)
and \[ \rho(r, \theta) = \rho(\theta) + \zeta(r, \theta), \] (8·2)
\( \epsilon \) and \( \zeta \) are determined over any region in which the two solutions differ by small quantities of the first order. From Bernoulli's equation

\[ \frac{u \partial \epsilon}{\partial r} + \frac{v \partial \epsilon}{\partial \theta} = -\frac{v^2}{\rho} \zeta. \] (8·3)

From the equation of continuity

\[ (u + v') \zeta + ru \frac{\partial \zeta}{\partial r} + v \frac{\partial \zeta}{\partial \theta} + \rho \left[ \frac{\partial^2 \epsilon}{\partial r^2} + \frac{\partial \epsilon}{\partial r} + \frac{1}{r} \frac{\partial^2 \epsilon}{\partial \theta^2} + \frac{\rho'}{\rho} \frac{\partial \epsilon}{\partial \theta} \right] = 0. \] (8·4)

Eliminating \( \zeta \) from equations (8·3) and (8·4)

\[ \left[ -\frac{u}{v} + \frac{v'}{v} - \frac{\rho'}{\rho} \right] \frac{\partial \epsilon}{\partial r} + \left[ \frac{v}{u} - \frac{u}{v} \right] r \frac{\partial^2 \epsilon}{\partial r^2} - \frac{2 \partial^2 \epsilon}{\partial r \partial \theta} = 0, \] (8·5)
using the relation \( v = u' \).

Integrating equation (8·5) with respect to \( r \)

\[ \left[ -\frac{u'}{u} + \frac{v'}{v} - \frac{\rho'}{\rho} \right] \epsilon + r \left[ \frac{v}{u} - \frac{u}{v} \right] \frac{\partial \epsilon}{\partial r} - \frac{2 \partial \epsilon}{\partial \theta} = 0, \] (8·6)

since \( \epsilon, \frac{\partial \epsilon}{\partial r} \) and \( \frac{1}{r} \frac{\partial \epsilon}{\partial \theta} \to 0 \) as \( r \to 0 \) for all \( \theta \).

The general integral of equation (8·6) is

\[ \frac{\rho u}{v} e^2 = F \left[ w^2 \exp \left( -\int_0^\theta ud \theta \right) \right], \] (8·7)

where \( F \) is a function to be determined by the size of the block.

Now the curves \( r^2 u \exp \left[ -\int_0^\theta ud \theta \right] = \text{constant} \),
constitute the family of curved characteristics in the semi-infinite block solution, and equation (8·7) shows that \( \rho u e^2/v \) is constant along each member of the set. The value of this constant is determined from the intersection of the characteristic with known boundary conditions. For example, the solution in §5 could be used. It may be noted in passing that by taking \( F(z) \sim z^7 \) the solution in §7 is found.

Returning to equation (8·7) the factor \( \rho u/v \) is itself constant along the other set of characteristics in the case of a semi-infinite block, i.e. along radius vectors from the corner. Thus \( \varepsilon \) at any point of the field is determined by the product of two factors which parametrically specify the characteristics through the point. Theoretically it would be possible to use equation (8·7) to replace the individual solutions in §§5, 6 and 7, but in practice the latter were found to be more convenient. Equation (8·7), apart from indicating in a general way how the presence of an axis modifies the Meyer solution, makes it possible to assess the extent of the regions over which the individual solutions may safely be used. The curved set of characteristics in the Meyer solution bend away rapidly from the corner in a manner closely resembling that illustrated in figure 2. Since \( e^2 \) varies like \( v/\rho u \) along such a characteristic and \( v/\rho u \) does not vary much for \( \theta > 5^\circ \), the size of \( \varepsilon \) is of the same order all over the region between a given characteristic and the corner. This explains, for example, why the solution §7 holds so much further out from the corner than solution §5 does from the face, and also, in the case of a surrounding medium of air, why the shock wave is straight for a considerable distance from the corner.

9. DETAILS OF THE NUMERICAL INTEGRATION

In the numerical integration of the equations it was hoped to obtain values accurate to within 1 in 500. In order to achieve this it was necessary to work to a much higher nominal accuracy, since an error in \( (u,v) \) at any point causes a much larger percentage error in the evaluation of the co-ordinates of the point, and since errors accumulate as the field is developed.

The chief difficulty in obtaining the accuracy required is caused by the gradients of the characteristics tending to infinity as the block face is approached. In substituting conditions at a distance from the face, by means of the series solutions, a new source of error is introduced, namely, the terms omitted from the expansions. A balance had to be struck between the two cases. It was decided to take the series near the axis as valid for boundary conditions on \( x = 0·1 \) from the axis of \( y = \pm 0·5 \). The rest of the boundary conditions were supplied by the expansion near the corner in powers of \( r \) over a circular arc of radius 0·5, and by the vanishing of \( v \) on the axis of symmetry. The gas field was covered by the network of characteristics from the points taken on the boundary chosen above.

The values of the velocity components at a point \( P \) of the field were obtained from nearby points on the two characteristics passing through \( P \). The two points were called 'top' or 'bottom', according as they lay on the characteristic through \( P \) emanating from the upper or lower corner of the block, respectively, and the suffixes
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$T$ and $B$ were used to distinguish them. The gradients of the characteristics were denoted by $\mu^+, \mu^-$ respectively, with the appropriate suffixes.

In the equations which follow, quantities without suffixes refer to the point $P$ for which $(u, v, x, y)$ are to be found.

In the approximate integration of the differential relations in §4, arithmetic mean values of the gradients are taken, and hence the differential relations are replaced by the following difference equations

$$
\begin{align*}
    u - u_T + \frac{1}{2}[\mu^+ + \mu_T^-] [v - v_T] &= 0, \\
    u - u_B + \frac{1}{2}[\mu^+ + \mu_B^-] [v - v_B] &= 0, \\
    y - y_T = \frac{1}{2} [\mu^- + \mu_T^+] [x - x_T], \\
    y - y_B = \frac{1}{2} [\mu^- + \mu_B^+] [x - x_B].
\end{align*}
$$

(9.1)

(9.2)

If the quantities $\Sigma = \mu^- + \mu_T^+, \Sigma = \mu^- + \mu_B^+, \Sigma' = \mu^+ + \mu_B^-, \Sigma' = \mu^+ + \mu_T^-$ are defined, then, by solving the simultaneous equations (9.1), $u$ and $v$ may be obtained in the useful computing form

$$
\begin{align*}
    (\Sigma - \Sigma^+) u &= \Sigma u_B + (-\Sigma) u_T + (-\Sigma^-) \left[ \frac{v_T - v_B}{2} \right], \\
    (\Sigma - \Sigma^-) v &= \Sigma v_T + (-\Sigma) v_B + 2(u_T - u_B),
\end{align*}
$$

(9.3)

and $(x, y)$ are obtained similarly from equations (9.2) in the form

$$
\begin{align*}
    (\Sigma - \Sigma') y &= \Sigma' y_T + (-\Sigma') y_B + (-\Sigma^+) \left[ \frac{x_T - x_B}{2} \right], \\
    (\Sigma - \Sigma') x &= \Sigma' x_B + (-\Sigma') x_T + 2(y_T - y_B).
\end{align*}
$$

(9.4)

The actual method of use of the equations involves successive approximations to $(u, v)$. Values of $u$, $v$ having been guessed, the corresponding values of $\mu^+$ and $\mu^-$ are calculated, and then used in equations (9.3) to obtain more accurate values. When this process ends, the values of $x$, $y$ are immediately obtainable from equations (9.4).

When the point $P$ lies on the axis of symmetry, only one integration has to be performed, and the equations reduce to the simpler form

$$
\begin{align*}
    u &= u_T + \frac{1}{2} \left[ \frac{u^2}{a^2} - 1 \right] v_T, \\
    \mu^+ &= \left[ \frac{u^2}{a^2} - 1 \right]^{-1}, \\
    x &= x_T + \frac{2y_T}{(-\Sigma')}.
\end{align*}
$$

(9.5)

Diagrams based on the results obtained by the above methods are shown in figures 3 to 7 showing the distributions of the velocity components and the pressure at given distances from the block face, and the manner in which the velocity increases and the pressure drops, outwardly along the axis of the charge. The solution was
carried out to a distance approximately equal to the thickness of the charge, where the pressure on the axis is already less than 10% of its value just behind the detonation wave.

![Figure 3](image1.png)

**Figure 3.** The pressure distribution along the axis of symmetry.

![Figure 4](image2.png)

**Figure 4.** The gas velocity along the axis.
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Figure 5 shows clearly one interesting feature of the pressure distribution. The crossing of the curves implies a local, temporary, increase in pressure behind the detonation wave in regions farther from the axis of symmetry than 1 cm. (i.e. farther from the axis than the corner of the charge). At greater distances from the detonation wave the pressure falls again. An allied phenomenon can be seen in figure 7, where the transverse-velocity curves also cross.

![Figure 5. The pressure distribution along lines perpendicular to the axis. The number attached to each curve denotes the distance from the block in centimetres.](image)

![Figure 6. Distribution of the axial component of velocity along lines perpendicular to the axis. The number attached to each curve denotes the distance from the block in centimetres. The discontinuity in gradient on the 0.5 cm. curve occurs at the boundary between the primary and secondary gas fields.](image)
10. Calculation of the position of the shock wave

The discussion in §3 shows that the initial direction of \( OC \) (figure 1) is found from the semi-infinite block solution, and that it depends on the conditions in the outer medium. The position of \( OC \) and velocity distribution along it then follow from the primary field solution. The velocity and pressure distributions in the secondary field are defined by the boundary conditions along \( OC \) and the initial conditions in the outer medium. The most convenient method to be adopted in numerical calculation seems to be that of successive approximation, in which the position of an elementary segment of \( OD \) or \( OS \) is guessed. Each guess entails solving the field equations in the secondary region. In the vortex region behind the shock wave the field equations are analogous to those derived by Crocco (1936) with the appropriate generalizations when the surrounding medium cannot be treated as a perfect gas. The leading feature of vortex flow of a compressible fluid is that each streamline is an adiabatic.

Along the streamline boundary \( OD \), the vorticity in the outer medium is zero; this follows since the vorticity vanishes at the corner and is elsewhere proportional to the product of density and absolute temperature along a streamline. When the vorticity is everywhere small, it is sufficiently accurate to proceed from \( OS \) to \( OD \) along a characteristic in one step, and the numerical integration is then greatly simplified. This case occurs when the curvature of \( OC \) is small.

When the outer medium is air (as in the present paper), the position of \( OC \) is such that it lies in the part of the primary field closely approximating to the Meyer expansion and so \( OC \) is found to be straight for two charge widths (4 cm.) from the
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corner to the nominal accuracy of the calculations (1 in 5000). To this accuracy the shock wave is therefore straight up to the point where it is met by the characteristic from the point on OC distant 4 cm. from O. A calculation shows that the shock wave is straight for nearly 12 cm. from O. The shock wave then gradually bends round towards the axis of symmetry, until at a great distance from the charge it makes an angle with the axis equal to the Mach angle corresponding to air streaming with the detonation velocity under the given initial conditions of pressure and density.

The angles made with the axis by OC, OD and OS are found (from the Meyer expansion) to be

\[ OC, \ 35\cdot45^\circ; \ OD, \ 40\cdot98^\circ; \ OS, \ 54\cdot52^\circ. \]

Relatively to the detonation wave the air streams along OD with a velocity of 4054 m./sec. and the gas with a velocity of 7021 m./sec. The pressure behind the shock wave is 306·4 atm.

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