A theory of the dependence of the rate of detonation of solid explosives on the diameter of the charge

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The velocity of a detonation wave, in a cylindrical charge of solid explosive, is shown to be dependent on the diameter of the charge, and the relation between the velocity and the diameter is calculated. It is shown that this effect depends upon the rate of the chemical reaction occurring in the front portions of the detonation wave, and that it is possible, therefore, to determine this rate of reaction by measuring the velocity of detonation in bare charges of different diameters. The effect of a metal case surrounding the charge is also briefly discussed.

INTRODUCTION

A successful theory of the detonation of gases contained in tubes was first given by D. L. Chapman (1899) and later was developed in greater detail by Jouguet (1906). These authors were able to predict with considerable accuracy the velocity of the detonation wave in an explosive gas in terms of the heat of the reaction, and such quantities as the specific heats of the resultant gases. An interesting feature of the theory is that, not only is the velocity of the detonation wave determined, but also the pressure and the temperature immediately behind the detonation front. In accordance with many observations, the theory is based on the assumption that the detonation process is a steady one, i.e. that the velocity of the detonation wave remains constant. It is on this account that, in this theory, the rate of the reaction occurring in the initial portion of the detonation wave does not enter into the calculation of the velocity of propagation.

It is natural that the Chapman-Jouguet theory should have been applied to the detonation of cylindrical charges of solid and liquid high explosives. Many authors have attempted this extension of the theory but with only limited success. There are two main reasons why a theory of detonation in solids is more difficult than in gases. First, because the products of the explosion are, in general, more complicated; many different molecular species being formed, and, secondly, because the pressure, temperature and molecular volume in the detonation wave have values which lie far outside the range which has been studied in other connexions. Thus very little guidance is available for setting up an equation of state applicable to the conditions of the detonation wave.

Apart, however, from the difficulty associated with the equation of state, the dynamics of the Chapman-Jouguet theory require modification when applied to solid explosives. This arises from the fact that, while the detonation of a gas in a rigid tube leads to a flow which may be regarded as strictly one-dimensional, in solid high explosives no practicable confinement can prevent a certain radial expansion of the decomposing materials. As far as the determination of the velocity of detonation is concerned, only the flow within that region of the wave where the
decomposition of the explosive takes place is of significance. The subsequent flow of the reaction products has no effect on the steady velocity of the detonation wave. The effect of radial expansion within the reaction zone can easily be understood in general terms. It is known from experience that the detonation velocity of a solid explosive generally increases with the loading density. Hence, if as a result of radial expansion the density is reduced before the reaction is complete, part of the explosive decomposes at a lower effective loading density, and a velocity of detonation may be expected less than that which would occur if no radial expansion took place, and therefore less than that given by a straightforward application of the Chapman-Jouguet theory. It is with this problem of the reduction of the detonation velocity by radial expansion within the reaction zone that this paper is concerned. The main interest in this matter lies in the fact that it provides a means of determining experimentally the rate of the reaction in a detonation wave. Clearly the degree of effective radial expansion depends upon the confinement of the charge and upon the length of the reaction zone. A knowledge of the length of the reaction zone gives at once the time required for the complete decomposition of the explosive and the release of the chemical energy. In the second and third sections of this paper formulae are given from which the time of reaction can be obtained from data giving the variation of the velocity of detonation with the diameter of the charge, or, for cased charges, with the weight of the casing.

It is often convenient to consider the detonation process as judged from axes moving with the detonation wave. The front portion of this wave is simply a powerful shock wave travelling through the unexploded material. It is the compression and the temperature in this shock-wave front which initiates the reaction. If we imagine the wave to be travelling from right to left, the shock-wave front, from the moving axes, appears to be stationary and unexploded material appears to be passing through it from left to right with the detonation velocity. It is assumed that, at points near the axis, the shock front may be regarded as a plane normal to the axis. Since the detonation process is a steady one the velocity of detonation is, therefore, by this assumption, determined entirely by the dynamical conditions within a small stream tube coaxial with the charge. For any other stream tube, off the axis, the expansion of the cross-section is, in general, greater than that of the axial tube in the same distance, and therefore leads to a smaller steady velocity. However, since the whole wave proceeds without change of form it follows that those parts of the shock front near to the cylindrical surface of the charge must be inclined to the axis. This paper, however, will not be concerned with the form of the detonation front, but only with the way in which the radius of the charge, or confinement by metal tubes, affects the velocity of detonation.

Effect of the radial expansion on the velocity of detonation

Consider first, from a reference system moving with the detonation wave, a stream tube, whose axis coincides with that of the charge, and which begins at the shock-wave front. Let the radius of this stream tube at any point further along the
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jet be denoted by \( r \) in units of its initial radius. It is convenient to work entirely with dimensionless variables such as \( r \). Thus if \( p \) denotes the pressure in excess of one atmosphere, \( U \) the velocity of the detonation wave, \( \Lambda \) the loading density of the explosive, and \( \rho \) the density of the fluid at any point beyond the shock-wave front, then let

\[
y = \Lambda / \rho, \quad z = p / U^2 \Lambda.
\]

(1)

The reaction is initiated at the shock-wave front and develops over a certain distance of the jet; this region is referred to as the reaction zone. Let the fraction of the explosive which has decomposed up to a certain point in the reaction zone be \( \epsilon \), so that \( \epsilon \) varies from 0 to 1 from the shock-wave front to the point where the reaction is complete. If, at any point in the reaction zone, \( E \) denotes the internal energy per unit mass measured relative to the energy of the unexploded material at atmospheric pressure, and \( Q \) the chemical energy which has been liberated per unit mass by the reaction up to that point, then both the quantities \( E \) and \( Q \) are functions of \( \epsilon \), \( y \) and \( z \), and each has the dimensions of the square of a velocity.

The conservation laws, applied across the discontinuity of the shock-wave front, lead, as is well known, to the equation

\[
E = \frac{1}{2} p \left( \frac{1}{\Lambda} - \frac{1}{\rho_0} \right).
\]

(2)

The work done by an element of the fluid by expansion, per unit mass, as it passes through the detonation wave is given by \( \int p d(1/\rho) \) where the integration is taken from the value of the density \( \rho_0 \) at the high pressure side of the shock front to the value \( \rho \) at the particular point under consideration. Thus, if a dimensionless function \( f \) is defined by the equation

\[
f(\epsilon, y, z) = U^{-2}(E - Q),
\]

(3)

the equation expressing the conservation of energy can be written in the form

\[
f(\epsilon, y, z) = \frac{1}{2} z_0 (1 - y_0) - \int_{y_0}^{y} z dy,
\]

(4)

where \( z_0 \) and \( y_0 \) correspond to the pressure and the density at the high pressure side of the shock front, i.e. at the front of the reaction zone. In this equation it is assumed that no appreciable amount of energy is transmitted from one element of the fluid to another by heat conduction.

As has already been explained, in all cases the relative radial expansion of the stream tube which is being considered is small, and therefore it is a legitimate approximation to assume that at every point of the cross-section the pressure is the same. Also, of course, to the same approximation the density and the fluid velocity are constant over the cross-section. The fluid velocity, relative to the detonation wave at rest, is denoted by \( q \), so that the equation of continuity is

\[
q U^{-1} = y r^{-2},
\]

(5)
and Bernoulli's equation is

$$\frac{1}{2}q^2 + \int \frac{1}{\rho} dp = \text{constant.} \quad (6)$$

One cannot immediately carry out the integration in this equation, because here $p$ is not merely a function of $\rho$, as for adiabatic expansion, but depends also upon $\varepsilon$. Using (1) and (5) and the fact that $r = 1$ when $z = z_0$, i.e. at the front of the reaction zone, (6) can be expressed

$$\frac{1}{2}y^2r^{-4} + \int_{z_0}^z y \, dz = \frac{1}{2}y_0^2. \quad (7)$$

Applying the laws of the conservation of mass and momentum across the shock-wave front gives, in terms of the non-dimensional variables,

$$z_0 = 1 - y_0. \quad (8)$$

Differentiating (7) one can write

$$dz = \frac{1}{2}yd(r^{-4}) - d(yr^{-4})$$

so that integrating and using (8) gives

$$z = 1 - yr^{-4} - 2 \int_1^r yr^{-5} dr. \quad (9)$$

If the values of the variables $r, y, z, \varepsilon$, when $\varepsilon = 1$, are denoted by $r_1, y_1, z_1$, and if a mean value of $y$ throughout the reaction zone is defined by the equations

$$\int_1^{r_1} yr^{-5} dr = \bar{y} \int_1^{r_1} r^{-5} dr = \frac{1}{4} \bar{y}(1 - r_1^{-4}), \quad (10)$$

then (9) becomes, for $\varepsilon = 1$,

$$z_1 = 1 - y_1 r_1^{-4} - \frac{3}{2} \bar{y}(1 - r_1^{-4}). \quad (11)$$

In the Chapman-Jouguet theory it is assumed that the gas velocity relative to the detonation wave, at the point where the reaction is complete, is just equal to the velocity of sound in the product gases at that point. This so-called Chapman-Jouguet condition has been the subject of much discussion and it now appears to be well justified both theoretically and also experimentally in so far as it leads to correctly calculated detonation velocities. This condition will also be applied here in the case where there is a small radial expansion within the reaction zone. This should, in general, be substantially correct though in the rather special case of an explosive containing a very slowly reacting component it may need modification. Accordingly

$$y_1 \left( \frac{\partial z}{\partial y} \right)_1 + \frac{q_1}{U^2} = 0, \quad (12)$$

where $\partial z/\partial y$ denotes the adiabatic variation of pressure with volume. When $\varepsilon = 1$, equation (4) represents the adiabatic variation of $y$ with $z$ so that

$$\left( \frac{\partial z}{\partial y} \right)_{ad.} = - \left( z_1 + \left( \frac{\partial f}{\partial y} \right)_1 / \left( \frac{\partial f}{\partial z} \right)_1 \right) \quad (13)$$

and the Chapman-Jouguet condition becomes, using (5),

$$\left( \frac{\partial f}{\partial z} \right)_1 = r_1 \left( z_1 + \left( \frac{\partial f}{\partial y} \right)_1 \right). \quad (14)$$
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When \( f \) is a known function of \( y \) and \( z \), equations (11) and (14) determine \( y_1 \) and \( z_1 \) in terms of \( r_1 \) and \( \bar{y} \). If these values are substituted into equation (4) one obtains a relation which determines \( U \) as a function of \( r_1 \). To do this first simplify (4) by using (7) and so obtain

\[
f(1, y_1, z_1) = \frac{1}{2} y_1 z_1 - \frac{1}{2} y_1^2 r_1^{-4}.
\]  

(15)

From this point no further progress can be made without some knowledge of the form of \( f \) for the product gases of the detonation. It is not an unsatisfactory approximation to assume that the internal energy is proportional to the temperature, and it will be shown that it is not necessary to know the constant of proportionality, for this can be eliminated from the final result. The dependence of \( f \) upon the pressure and the density is, however, needed and this requires an equation of state applicable to the conditions existing in the explosion products. For the purpose of calculating the detonation velocity from thermochemical data a rather complicated form is needed to give good results, but for the present purpose, which is to obtain the correction called for by radial expansion, it should be sufficient to use an equation depending simply on a constant co-volume, especially as this co-volume can be eliminated from the final result. Therefore it may be assumed that

\[
f(1, y_1, z_1) = k z_1 (y_1 - b) - Q_1 U^{-2},
\]  

(16)

where \( k \) and \( b \) are two constants, the former being related to the specific heats of the gaseous products and the latter being the co-volume in units of \( 1/A \).

Substituting from (16) into (14), and solving the resultant equation together with (11) for \( y_1 \) and \( z_1 \), one obtains

\[
(1 + 2k) y_1 = (1 + k) \left[ 1 - \frac{1}{2} \bar{y} (1 - r_1^{-4}) \right] r_1^4 + bk,
\]  

(17)

\[
(1 + 2k) z_1 = k \left[ 1 - \frac{1}{2} \bar{y} (1 - r_1^{-4}) \right] - b k r_1^{-4}.
\]  

(18)

Introducing these values of \( y_1 \) and \( z_1 \) into (15) and using equation (16) one obtains

\[
2(1 + 2k) \frac{Q_1}{U^2} = k^2 (1 - b)^2 + (1 + k)^2 \left[ r_1^4 \left( 1 - \frac{1}{2} \bar{y} (1 - r_1^{-4}) \right)^2 - 1 \right] + k^2 b (\bar{y} - b) (1 - r_1^{-4}).
\]  

(19)

When there is no radial expansion in the reaction zone, i.e. \( r_1 = 1 \), the velocity of detonation may be denoted by \( U_0 \) and from (19) one obtains

\[
2(1 + 2k) \frac{Q_1}{U_0^2} = k^2 (1 - b)^2.
\]  

(20)

Eliminating \( Q_1 \) from (19) and (20) and rearranging the terms finally results in

\[
\left( \frac{U_0}{U} \right)^2 = 1 + (r_1^{-4} - 1) \left[ \left( \frac{1 + k}{k(1 - b)} \right)^2 \left[ 1 - \frac{1}{2} \bar{y} (1 - r_1^{-2}) \right] \left[ 1 - \frac{1}{2} \bar{y} (1 + r_1^{-2}) \right] + \frac{b}{(1 - b)^2} \left( \frac{\bar{y} - b}{r_1^4} \right) \right].
\]  

(21)

It is worth remarking that this equation remains valid even when \( (r_1 - 1) \) is not small. The form of the equations of motion and continuity, (6) and (5), only requires that the rate of expansion along the stream tube shall be small. However, in practical applications, \( (r_1 - 1) \) is usually very small compared with unity, i.e. \( U \) is, in general,
only a little less than the maximum possible velocity $U_0$. Hence, one may, to a sufficient approximation, write $r_1 = 1$ for all terms inside the square brackets. These terms are denoted by $c$ and therefore

$$c = \left(\frac{1 + k}{k(1 - b)}\right)^2 (1 - \bar{y}) + \frac{b}{(1 - b)^2} (\bar{y} - b). \quad (22)$$

For $r_1 = 1$ equation (17) becomes

$$k(1 - b) = (1 + 2k)(1 - y_1) \quad (23)$$

and, using this equation, (20) may be written

$$1 + 2k = \frac{2Q_1}{U_0^2} (1 - y_1)^{-2}. \quad (24)$$

$U_0$ is known with considerable accuracy for many explosives, and a very good idea of the magnitude of $Q_1$ can be obtained from experiments such as those of Robertson & Garner (1923). These experiments do not give $Q_1$ exactly because the end products after adiabatic expansion are, in general, rather different from the products in the detonation wave front. The most uncertain quantity on the right-hand side of (24) is $y_1$. The reciprocal of this, $1/y_1$, is the compression ratio, i.e. the ratio of the density of the explosion products in the detonation front to the loading density of the explosive. Detailed calculations according to the Chapman-Jouguet theory usually give values of this ratio about $1\cdot33$, varying somewhat from one explosive to another. It will therefore be assumed that $y_1 = \frac{3}{4}$. For $U_0$ take $6\cdot8 \times 10^5$ cm./sec., which is about the value for T.N.T., and for $Q_1$ $1000$ cal./g. which is also about right for T.N.T. These values give, according to (24), $k = 0\cdot95$ and according to (23) $b = 0\cdot237$. Thus (22) becomes

$$c = 7\cdot24(1 - \bar{y}) + 0\cdot41(\bar{y} - 0\cdot237). \quad (25)$$

At the beginning of the reaction zone $\rho_0$ must be rather greater than the true crystal density of the explosive, since in the shock front the pressure is very high, being of order $10^5$ atm. Now for cast explosives $A$ is usually not very much less than the true density, so that for such explosives $y_0$ cannot be a small fraction. As we know from the dynamics that $y \leq y_1$ throughout the reaction zone, it may be assumed that for cast explosives $\bar{y}$ is approximately equal to $y_1$. In this limiting case, in which $\bar{y} = \frac{3}{4}$, we have, from (25), $c = 2\cdot02$.

For charges made up of powdered explosives the loading density may be a half, or less, of the true density, in which case $\bar{y}$ may be somewhat less than $0\cdot75$. If the value $\bar{y} = 0\cdot6$ is taken, then (25) gives $c = 3\cdot05$. Thus it must be concluded that the numerical value of the coefficient $c$ in the formula

$$\left(\frac{U_0}{U}\right)^2 = 1 + c(r_A^4 - 1) \quad (26)$$

is approximately equal to $2\cdot0$ for cast explosives, but may be a little greater for powdered charges of low loading density.
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The radial expansion of a bare cylindrical charge

In this section, the value of \( r_1 \) is calculated for a bare cylindrical charge, of radius \( R \), detonating in a vacuum. Instead of actually finding the expansion in the reacting gases, which would be very difficult, the corresponding radial expansion in a jet of the product gases is found, assuming an adiabatic pressure volume relation. The value of \( r_1 \) is taken to be the relative expansion of the axial stream tube in a distance \( X \) equal to the length of the reaction zone. The dependence of the degree of expansion on \( X \) and the charge radius \( R \) should be very similar indeed in the two cases and, therefore, this approximation should give a satisfactory account, with equation (26), of the dependence of \( U \) upon \( X \) and \( R \).

![Diagram](attachment:Diagram.png)

**Figure 1**

The method of calculation is best explained by reference to a figure. In figure 1 let \( AB \) denote the plane of a nozzle from which gas streams into an evacuated space from left to right. The velocity at \( AB \) is equal to the local velocity of sound. If \( p_0 \) and \( \rho_0 \) are the pressure and density at \( AB \), it is assumed that elsewhere, in the stream, the relation between \( p \) and \( \rho \) is given by

\[
 p\rho^{-\gamma} = p_0\rho_0^{-\gamma}. \tag{27}
\]

The appropriate value to take for \( \gamma \) in this equation can be obtained as follows. Consider the expansion just behind the detonation front in the purely
one-dimensional case in which there is no radial expansion, then, according to the Chapman-Jouguet condition,

$$q^2 = \left(\frac{dp}{d\rho}\right)_{ad} = \frac{\gamma p}{\rho},$$  \hspace{1cm} (28)

where the form (27) is used for the adiabatic relation. According to (5), for \( r = 1 \), it is found that \( q = y U \) and therefore (28) gives \( y_1 = \gamma z_1 \), which together with (9), again for \( r = 1 \), gives

$$\gamma = y_1/(1 - y_1).$$  \hspace{1cm} (29)

As in the previous section, take \( y_1 = \frac{3}{4} \), from which results \( \gamma = 3 \). The adiabatic relation (27) with \( \gamma = 3 \) will, of course, be only approximately correct for small expansions from the initial condition at \( AB \). This, however, is all that is required in the present connexion since we are only concerned with distances along the stream less than a radius in length. The physical reason for the high value of \( \gamma \) is that the explosion products form initially a very imperfect gas.

The flow in the immediate neighbourhood of a point such as \( A \) or \( B \) is identical with the flow round a straight edge, which may be analyzed simply, as was first shown by Meyer. It will be assumed that the form of the stream lines, and the pressure along them, near the circular edge through \( A \) can be taken from Meyer’s solution. It appears, for points not more than a radius downstream and not further from the edge than is indicated by the point \( D \), that this approximation is not too unsatisfactory for the present purpose. Consider a tube generated by the rotation of a stream line of Meyer’s solution such as \( DG \) about the axis, then within this region calculate the flow and the pressure, using equations (5) and (6). The pressure along the same line \( DG \) given by the two solutions will not, of course, be quite the same. That stream line is chosen, therefore, which minimizes the integral of the magnitude of the pressure difference along a distance \( R \) of the jet.

Let the components of the velocity at \( G \) be denoted by \( u \) and \( v \), \( u \) referring to the component along the radius vector \( s \), i.e. from \( A \) to \( G \), and \( v \) the component at right angles to \( s \) counted positive in the direction of \( \theta \) increasing. \( \theta \) is the angle which \( s \) makes with the plane of the shock front denoted by \( AB \). If \( V \) denotes the velocity of the issuing gas, which in this case is equal to the local velocity of sound, Meyer’s solution is expressed by the following equations:

$$v = V \cos \lambda \theta,$$

$$u = (V/\lambda) \sin \lambda \theta,$$  \hspace{1cm} (30)

where

$$\lambda = [(\gamma - 1)/(\gamma + 1)]^{\frac{1}{2}}.$$  \hspace{1cm} (32)

\( v \) is given in terms of the density by the relation

$$v^2 = \frac{dp}{d\rho}.$$  \hspace{1cm} (33)

The equation of a stream line in polar coordinates \((s, \theta)\) is as follows:

$$s = s_0(\cos \lambda \theta)^{-1/\lambda^2}.$$  \hspace{1cm} (34)
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Expressed in terms of \( r \) and \( \theta \) this equation becomes

\[
\frac{\rho}{P} = 1 + \omega \left( 1 - (\cos \theta)^{-1/3} \cos \theta \right),
\]

where

\[
\omega = \frac{s_0}{(R - s_0)}.
\]  

(35)  

(36)

The pressure along the stream line as given by Meyer's solution is denoted by \( p_m \). Thus, according to (27), (30) and (33),

\[
\frac{p_m}{p_0} = \left( \frac{p}{P} \right)^{2\gamma/(\gamma - 1)} = (\cos \theta)^{2\gamma/(\gamma - 1)}.
\]

(37)

Now consider the flow inside a tube generated by rotating the stream line (34) about the axis of symmetry. It will be assumed that inside this tube the pressure may be regarded as constant over planes normal to the axis. Thus Bernoulli’s equation may be written as follows:

\[
\frac{1}{2}q^2 + \frac{V^2}{(\gamma - 1)} \left( \frac{P}{P_0} \right)^{(\gamma - 1)/\gamma} = \frac{1}{2}V^2,
\]

and this, together with (27) and the equation of continuity

\[
\rho qr^2 = U \Delta,
\]

gives at once

\[
\rho = \left( \frac{P_0}{P} \right)^{2/(\gamma - 1)} \left( \frac{\gamma + 1}{\gamma - 1} \right) \left( \frac{P}{P_0} \right)^{(\gamma - 1)/\gamma} - 1.
\]

(38)  

(39)  

(40)

Using (35) to determine \( r \), we find from (40) the pressure along the inside of the stream tube as given by the solution based on (38) and (39). This pressure is denoted by \( p_t \). When \( \gamma = 3 \) the formulae simplify, and if non-dimensional variables are introduced as follows:

\[
\xi_m = (p_m/p_0)^{\gamma}, \quad \xi_t = (p_t/p_0)^{\gamma};
\]

(41)

one gets from (35) and (40)

\[
\xi_t(2 - \xi_t) = \left( 1 + \omega (1 - \cos^{-2}(\theta/\sqrt{2}) \cos \theta) \right)^{-4}
\]

(42)

and from (37)

\[
\xi_m = \cos^2(\theta/\sqrt{2}).
\]

(43)

The factor \( 1 - \cos^2(\theta/\sqrt{2}) \cos \theta \), which is denoted by \( g(\theta) \), is small for all values of \( \theta \) less than about 60°, in fact \( g(60) = 0.0824 \). Thus, since \( \omega \) is less than unity, expansion of the fourth power in (42) results in

\[
\xi_t = 1 - 2\omega^4 \sqrt{g(\theta)}.
\]

(44)

Now find the stream line which gives the best approximation, i.e. the best value of \( \omega \) to give the closest agreement between \( p_t \) and \( p_m \) over a distance along the axis of the order of the length of one charge radius. This is done by minimizing the integral

\[
\int_{0}^{1} (\xi_t - \xi_m)^2 d\theta
\]

(45)

with respect to \( \omega \).
The range of $\theta$ from 0 to 1·0 radian corresponds to a distance along the axis of about one charge radius. Minimizing (45) is simpler than minimizing the integral of $(p_i - p_n)^2$ and is as effective for the present purpose. In this way

$$2\omega^2 \int_0^1 g(\theta) d\theta = \int_0^1 \sqrt{|g(\theta)|} \sin^2 \left(\theta/\sqrt{2}\right) d\theta$$

(46)

is obtained, and a numerical integration gives $\omega^2 = 0.922$, and therefore $\omega = 0.85$. Denoting distance along the axis by $x$ gives, as can be seen immediately from the figure and equation (34),

$$\frac{x}{R} = \left(\frac{\omega}{1 + \omega}\right) \frac{\sin \theta}{\cos^2 \left(\theta/\sqrt{2}\right)}.$$  

(47)

Hence, substituting the numerical value of $\omega$ into (35) and (47), the following equations, which give $r$ as a function of $x/R$ in terms of the parameter $\theta$, are obtained:

$$r = 1.85 \left(1 - \frac{x}{R} \cot \theta\right),$$

(48)

$$\frac{x}{R} = 0.919 \left(\frac{\sin \theta}{1 + \cos \sqrt{2} \theta}\right).$$

(49)

The following table shows numerically the relation between $r$ and $x/R$ given by these equations.

<table>
<thead>
<tr>
<th>$x/R$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(r-1)10^2$</td>
<td>0.10</td>
<td>0.48</td>
<td>1.20</td>
<td>2.40</td>
<td>4.08</td>
<td>6.24</td>
<td>8.85</td>
<td>11.80</td>
<td>15.10</td>
</tr>
</tbody>
</table>

The percentage diminution in the detonation velocity due to radial expansion within the reaction zone may be taken as $10^2(U_0 - U)/U$, and from (26), with $c = 2\cdot0$ and remembering that $(r_1 - 1) \ll 1$,

$$10^2(U_0 - U)/U = 4(r_1 - 1)10^2$$

approximately. Thus the percentage diminution of the velocity of detonation below its maximum value is given, for small values of $(r_1 - 1)$, by multiplying the figures of the lower line of the above table by 4. For instance, if the length of the reaction zone is one-half of the radius of the charge, the velocity will be 9.6% below the maximum value as given by the one-dimensional Chapman-Jouguet theory, or as observed in charges of very large diameter.

It is useful, for some purposes, to define a reaction time $\tau$ by the relation $X = UT$, though it must be remembered, if one wants to connect $\tau$ with the actual progress of the reaction, that the velocity of an element in the reaction zone, relative to the shock-wave front, is not $U$ but $q$, so that the true time of reaction is $\int_0^X q^{-1} dx$, which is roughly $\frac{4}{3} \tau$.

**THE RADIAL EXPANSION OF A CASED CYLINDRICAL CHARGE**

In this section the effect on the velocity of detonation of a metal tube surrounding the charge is considered. The present discussion is limited to a rather special case; a fuller consideration of confined charges will be given in another paper.
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For the present, it will be supposed that the metal case can be represented by a thin flexible sheath of the same mass per unit length. It is not a serious approximation to neglect the strength of the metal, except in very special cases, since the yield stress, even of steel, is small compared with the stresses arising near the detonation front. However, the neglect of the finite thickness and, therefore, of the compression of the metal is a serious approximation, and it is with this that the later paper will deal.

It can be safely assumed that the expansion of the metal tube is entirely radial, so that the mass per unit length, denoted by \( m \), is constant throughout. Moreover, for small expansions, it may be assumed that \( \frac{d^2r}{dt^2} = \frac{U^2 \frac{d^2r}{dx^2}}{m} \) so that in this case the equation of motion for the metal case may be written

\[
\frac{d^2r}{dx^2} = \frac{2\pi \Delta}{m} r.
\] (50)

This equation could not be solved exactly without a detailed knowledge of the way in which the rate of the reaction depends upon the temperature and pressure, for only then could we determine the dependence of \( z \) upon \( x \) and \( r \) from the equations of the first section. For small expansions, however, \( z \) may be assumed to have the same form as in the one-dimensional case. It follows that, by (9), \( z = 1 - y \), where \( y \) varies between \( \Delta/\rho_0 \) at the beginning of the reaction zone and \( \Delta/\rho_1 \) at the end. It has been seen previously that for cast explosives both these ratios are approximately \( \frac{3}{4} \). Hence (50) will give a good idea of the variation of \( r \) with \( x \), if \( z \) is taken to be constant and equal to \( \frac{3}{4} \). Equation (50) may now be written

\[
\frac{d^2r}{dx^2} = \left( \frac{\Delta}{4\pi R \sigma} \right) r,
\] (51)

where \( \sigma \) is the mass per unit area of the confining tube before expansion, i.e. \( 2\pi R \sigma = m \). The solution of (51) for which \( dr/dx = 0 \) and \( r = 1 \) at \( x = 0 \) is

\[
r = \cosh\left( \frac{x}{2R} \sqrt{\frac{\Delta R}{\sigma}} \right).
\] (52)

Thus, if \( X \) denotes the length of the reaction zone,

\[
\left( \frac{U_0}{U} \right)^2 = 1 + 2c \cosh\left( \frac{X}{R} \sqrt{\frac{\Delta R}{\sigma}} \right) \sinh^2\left( \frac{X}{2R} \sqrt{\frac{\Delta R}{\sigma}} \right)
\] (53)

results from (26). When the percentage diminution of the velocity of detonation, due to radial expansion, is small, (53) may be written, assuming \( c = 2 \),

\[
\frac{U_0 - U}{U} = \frac{X^2 \Delta}{2 \sigma R}.
\] (54)

As a numerical example, consider an iron tube 1 cm. internal radius and 4 mm. thick, containing an explosive at density 1.5 g./cm.\(^3\) and for which \( X = 5 \) mm. These figures give \( X/R = 0.5 \) and \( \Delta R/\sigma = 0.5 \) from which, by (54), the detonation
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velocity is found to be 6.25% below $U_0$. It may be noted that the same charge completely bare would, according to the formulae of the preceding section, have a velocity of detonation 9.6% less than $U_0$. Thus for such a charge the confining tube increases the rate of detonation by a little over 3%.

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REFERENCES

Chapman, D. L. 1899 Phil. Mag. 47, 90.
Jouguet, E. 1906 J. Math. pures appl. 6 (Séries II), 5.