Particles of finite size in the gravitational field

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I should like to speak briefly about some work that I am engaged on, although it is not yet complete. The object of this work is to set up a theory of the gravitational field interacting with particles. I want this theory to be in agreement with Einstein's theory of gravitation, and also I insist that it shall follow from an action principle. I do insist on this because I believe that Nature works according to an action principle, and if we have an action principle, we certainly have a first step towards quantization.

Now in considering a particle interacting with the gravitational field, the first thing one would think of would be a point particle, but here one runs into a difficulty, because if one keeps to physically acceptable ideas, one cannot have a particle smaller than the Schwarzschild radius, which provides a sort of natural boundary to space. The mathematicians can go beyond this Schwarzschild radius, and get inside, but I would maintain that this inside region is not physical space, because to send a signal inside and get it out again would take an infinite time, so I feel that the space inside the Schwarzschild radius must belong to a different universe and should not be taken into account in any physical theory. So from the physical point of view, the possibility of having a point singularity in the Einstein field is ruled out. Each particle must have a finite size no smaller than the Schwarzschild radius.

I tried for some time to work with a particle with radius equal to the Schwarzschild radius, but I found great difficulties, because the field at the Schwarzschild radius is so strongly singular, and it seems that a more profitable line of investigation is to take a particle bigger than the Schwarzschild radius and to try to construct a theory for such a particle interacting with the gravitational field. There we have quite a definite problem, and we can get some help by considering the analogous problem in electrodynamics. Previous speakers have called attention to the close analogies between the electromagnetic field and the gravitational field, and I am going to follow in their footsteps and start off by considering the corresponding problem in the electromagnetic field.

This is the problem of setting up a theory of an extended electron in an electromagnetic field. We must make some basic assumptions about this extended electron, and I make the simplest ones which give a reasonable physical theory.

I assume that the electron has a definite surface—a definite boundary—outside which the field is described by Maxwell's equations.

I assume the surface itself to be a perfect conductor, so that there is no field inside the surface.

I assume that the potentials are continuous at the surface, while their first derivatives may have discontinuities, so that the field conditions just outside the surface are the usual ones for a perfect conductor, namely, that the normal component of the magnetic field and the tangential components of the electric field vanish in a frame of reference in which the element of surface is instantaneously at rest.

We have now an electron with a definite surface, a surface bearing a charge which distributes itself in accordance with the assumption that the surface is a perfect conductor. We shall need some forces to prevent the electron from flying apart under the Coulomb repulsion of this charge, some non-Maxwellian forces; I take these to be the simplest possible, namely forces corresponding to a surface tension, so that we have a model of the electron as a sort of bubble in the electromagnetic field.

We can set up an action principle for this model of the electron. The action consists of two parts, a four-dimensional integral taken over the space outside the electron, and a three-dimensional integral taken over the surface of the electron, which we may look upon four-dimensionally as the surface of a tube in space-time. The four-dimensional integral is just the usual one for the Maxwell field. The three-dimensional integral I take to be just a constant times the three-dimensional 'area' of the surface. This is the simplest integral that we could take, and it does give a force like a surface tension.

One can carry out the variational procedure by standard methods and get equations of motion, and one finds that one has quite a reasonable model for the electron. The electron does not have a definite shape or size—the shape and size can vary—but it has an equilibrium position, about which it can oscillate. One finds that the oscillations are stable. The action principle gives completely the equations of motion for the electron as a whole, and for all the oscillations and changes of shape of the surface. One finds that there are no run-away solutions with this model; everything behaves in a proper physical way, and as everything follows from an action principle, we may apply quantum mechanical ideas.

One interesting result comes out. Let us consider spherically symmetrical solutions, in which the electron may change its size but not its shape. Its centre remains fixed, while it pulsates, and the field outside is always just the Coulomb field. Then conditions are very simple, and we have equations of motion for the pulsations. We can apply quantum ideas to this pulsation, and we find that the energy of the first excited state is very much greater than the energy of the ground state; a preliminary calculation gave it to be about 50 times greater. I hope that a more accurate theory will put that number up and will give us a theory of the mu-meson, in which it appears as a pulsating electron, keeping the spherical symmetry. Of course to get a satisfactory physical theory, one would have to bring in the spin somehow. I do not yet know how to do that.

There are possibilities on these lines of getting a reasonable theory in which we have a description of both the electron and the mu-meson and a basis from which one might tackle the gravitational problem.

I have been working on a corresponding gravitational problem in which there is a particle of a size greater than the Schwarzschild size, and the action consists of the usual action for the gravitational field, outside the particle, plus a surface term, which I again take to be just the surface 'area' of the tube, multiplied by a constant.

There are some difficulties in working this out, which I haven't altogether solved. I am not sure what boundary conditions to take, which would correspond to the boundary conditions in the electromagnetic case that the surface should be a perfect conductor. I hope that further investigation will yield a definite answer to that question.*

I have made a detailed study of the spherically symmetrical solution—the pulsating particle. Outside the particle we have the Schwarzschild solution, which we know all about, and the action principle again provides definite equations of motion for the pulsations.

It turns out that there is a difference from the electromagnetic case—there we needed a surface tension to hold the particle together against the Coulomb repulsion. In the gravitational case we need a surface pressure to hold it apart against the Newtonian attraction. We find that, as a result of this change in sign, the particle is unstable. To secure stability one has to add a further term to the action. There is a simple way of choosing this extra term and arranging the numerical coefficients so that the particle is stable under pulsations. I have not yet developed the theory sufficiently to be able to say whether it is stable also under distortions, but so long as the spherical symmetry is preserved, the model is a stable one, and so the situation is hopeful for developing this into a theory of particles of finite size interacting with the gravitational field according to Einstein's theory of gravitation.

Discussion

C. Møller

Professor Dirac said that in the electromagnetic case, there were some difficulties about introducing the spin. Is it not that one could never hope to get out the Dirac electron from such a model?

P. A. M. Dirac

One would hope to have some generalization of it. If you work out the Hamiltonian according to this model, a square root comes in, and somehow one has to get rid of that square root and bring in the spin matrices—a mathematical problem which needs some bright idea to solve it.

A. Lichnerowicz

I am a mathematician, and I always have the impression that the physicist in his domain has some idolatry for the Lagrangian and the Hamiltonian. These are very useful, but it is not really always the case that they are the keys to all problems. In general the key is perhaps an operator of the second order, a good operator, which, in general terms, corresponds to the Lagrangian or Hamiltonian. For the problem of interaction this is certainly necessary. But for a mathematician, many problems of interaction are without real mathematical significance.

* One gets satisfactory boundary conditions if one assumes that the $g_{\mu\nu}$ are constant inside the particle and continuous at the surface. This is closely analogous to the electromagnetic case, for which the potentials are constant inside and continuous at the surface.